# APOLLONIUS OF PERGA

## CONICS. BOOKS ONE - SEVEN

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Apollonius of Perga (ca 250 B.C. - ca 170 B.C.) was one of the greatest mathematicians of antiquity.

During 1990 - 2002 first English translations of Apollonius' main work Conics were published. These translations [Ap5](Books 1-3), [Ap6](Book 4), [Ap7] (Books5-7) are very different. The best of these editions is [Ap6].

The editions [Ap4] and [Ap5] are very careless and often are far from the Greek original. The editors of [Ap5] have corrected many defects of [Ap4], but not all; they did not compare this text with the Greek original. Some defects remain also in the edition [Ap6].
The translation [Ap7], being the first rate work, is not a translation of Greek text because this text is lost, and is the translation of Arabic exposition by Thabit ibn Qurra (826 - 901).

Therefore we present the new English translation of this classic work written in one style more near to Greek text by Apollonius, in our translation some expressions of the translations [Ap5], [Ap6], and [Ap7] are used.

The authors of the translations [Ap5], [Ap6], and [Ap7] are linguists and in their translations many discoveries of Apollonius in affine, projective, conformal, and differential geometries in Apollonius' Conics being special cases of general theorems proved in Western Europe only in 17th -19th centuries were not mentioned.

The commentary to our translation from the standpoint of modern mathematics uses books [Ro1] and [Ro2] by the translator.

I am very grateful to my master student, now Ph.D. and the author of the thesis[Rho1] and [Rho2] Diana L. Raodes, possessing ancient Greek. This work could not be completed without the help of translator's daughter, Professor of the Pennsylvania State University, Svetlana R. Katok, and also Ph.D. Daniel Genin and Nicholas Ahlbin.

Diagrams to Books I-IV should be taken from editions [AP3] Heiberg or [AP12] of Stamatis, diagrams to Books V-VII should be taken from the edition [AP7] of Toomer.

# BOOK ONE

## Preface

Apollonius greets Eudemus[1]

If you are restored in body, and other things go with you to your mind, well; and we too fare  pretty well. At the time I was with you in Pergamum, I observed you were quite eager to be kept informed of the work I was doing in conics. And so I am sending you this first book revised. I will send you other books when I will be satisfied with them. For I  don't believe you have forgotten hearing from me how I worked out the plan for these conics at the request of Naucrates[2], the geometer, at the time he was with me in Alexandria lecturing, and how on arranging them in eight books I immediately communicated them in great haste because of his near departure, not revising them but putting down whatever came to me with the intention of a final going over. And so finding now the occasion of correcting them, one book after another, I  will publish them. And since it happened that some others among those frequenting me got acquainted with the first and second books before the revision, don't be surprised if you come upon them in a different for.

Of the eight books the first four belong to a course in the elements [3].

The  first book contains the generation of the three sections and of the opposite [sections][4], and the principal properties in them worked out more fully and universally than in the writings of others.

The second book contains the properties having to do with the diameters and axes and also the asymptotes, and other things of a general and necessary use for limits of possibility. And what I call diameters and what I call axes you will know from this book.

The third book contains many unexpected theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new. And when I had grasped these, I knew that the three-line and four-line locus[5] had not been constructed by Euclid, but only a chance part of it and that not very happily. For it was not possible for this construction to be completed without the additional things found by me.

The fourth book shows in how many ways the sections of cone intersect with each other and with the circumference of a circle, and contains other things in addition none of which has been written up by my predecessors, that is in how many points the section of a cone or the circumference of a circle and the opposite sections meet the opposite sections.

The last four books are fuller in treatment. For there is one [the fifth book] dealing more fully with maxima and minima, and one [the sixth book] with equal and similar sections of a cone, and one [the seventh book] with limiting theorems, and one [the eighth book] with determinate problems.

And so indeed, with all of them published, those happening upon them can judge them as they see fit.

Let the happiness will be to you.

## First definitions

1. If a point is joined by a straight line with a point in the circumference of a circle which is not in the same plane with the point, and the line is continued in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began, then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating straight line is continued indefinitely, I call a conic surface [6], and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle I call the axis.

2. And the figure contained by the circle and by the conic surface between the vertex and the circumference of the circle I call a cone[7], and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle I call the axis, and the circle I call the base of the cone.

3. I call right cones those having axes perpendicular to their bases, and I call oblique those not having axes perpendicular to their bases.

4. For any curved line that is in one plane, I call straight line drawn from the curved line that bisects all straight lines drawn to this curved line parallel to some straight line the diameter [8,9]. And I call the end of the diameter situated on the curved line the vertex of the curved line, and I call these parallels the ordinates drawn to the diameter [10].

5. Likewise, for any two curved lines lying in one plane, I call the straight line which cuts the two curved lines and bisects all straight lines drawn to either of the curved lines parallel to some straight line the transverse diameter. And I call the ends of the [transverse] diameter situated on the curved lines the vertices of the curved lines. And I call the straight line lying between the two curved lines, bisecting all straight lines intercepted between the curved lines and drawn parallel to some straight lines the upright diameter [11]. And I call the parallels the ordinates drawn to the [transverse or upright] diameter.

6. The two straight lines, each of which, being a diameter, bisecting the straight lines parallel to the other, I call the conjugate diameters[12] of a curved line and of two curved lines.

7. And I call that straight line which is a diameter of the curved line or lines cutting the parallel straight lines at right angles the axis of curved line and of two curved lines [13,14].

8. And I call those straight lines which are conjugate diameters cutting the straight lines parallel to each other at right angles the conjugate axes of a curved line and of two curved lines.

[Proposition] 1

*The straight lines drawn from the vertex of the conic surface to points on the surface are on that surface* [15].

Let there be a conic surface whose vertex is the point Α, and let there be taken some point Β on the conic surface, and let a straight line ΑΓΒ be joined.

I say that the straight line ΑΓΒ is on the conic surface.

[Proof]. For if possible, let it not be [and the straight line ΑΒ is not on the conic surface], and let the straight line ΔΕ be the line generating the surface, and ΕΖ be the circle along which ΕΔ is moved. Then if, the point Α remaining fixed, the straight line ΔΕ is moved along the circumference of the circle ΕΖ. This straight line [according Definition 1] will also go through the point Β, and two straight lines will have the same ends. And this is impossible. Therefore, the straight line joined from Α to Β cannot not be on the surface. Therefore, it is on the surface.

Porism

It is also evident that, if a straight line is joined from the vertex to some point among those within the surface, it will fall within the conic surface. And if it is joined to some point among those without, it will be outside the surface.

[Proposition] 2

*If on either one of the two vertically opposite surfaces two points are taken, and the straight line joining the points, when continued, does not pass through the vertex, then it will fall within the surface, and continued it will fall outside* [16].

Let there be a conic surface whose vertex is the point Α, and a circle ΒΓ along whose circumference the generating straight line is moved, and let two points Δ and Ε be taken on either one of the two vertically opposite surfaces, and let the joining straight line ΔΕ, when continued not pass through the point Α.

I say that ΔΕ will be within the surface, and continued will be without.

[Proof]. Let ΑΕ and ΑΔ be joined and continued. Then [according to PropositionI.1] they will fall on the circumference of the circle. Let them fall to B and Γ, and let ΒΓ be joined. Therefore the ΒΓ will be within the circle, and so too within the conic surface. Then let Ζ be taken at random on ΔΕ, and let ΑΖ be joined and continued. Then it will fall on ΒΓ, for the triangle ΒΓΑ is in one plane [according to Proposition XI.2 of Euclid]. Let it fall to Η. Since then H is within the conic surface, therefore [according to the porism to Proposition I.1] the straight line ΑΗ is also within the conic surface, and so too the point Ζ is within the conic surface. Then likewise it will be shown that all the points on the straight line ΔΕ are within the surface. Therefore the straight line ΔΕ is within the conic surface.

Then let ΔΕ be continued to Θ. I say that it will fall outside the conic surface. For it possible, let there be some point Θ of it not outside the conic surface, and let ΑΘ be joined and continued. Then it will fall either on the circumference of the circle or within [according to Proposition I.1 and its porism]. And this is impossible, for it falls on ΒΓ continued; as for example to the point Κ. Therefore the straight line ΕΘ is outside the surface.

Therefore the straight line ΔΕ is within the conic surface, and continued is outside.

[Proposition] 3

*If a cone is cut by a plane through the vertex, the section is a triangle* [17].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by some plane through the point Α, and let it make, as section, lines ΑΒ and ΑΓ on the surface, and the straight line ΒΓ in the base.

I say that ΑΒΓ is a triangle.

[Proof]. For since the line joined from Α to Β is the common section of the cutting plane and of the surface of the cone, therefore ΑΒ is a straight line. And likewise also ΑΓ. And ΒΓ is also a straight line. Therefore ΑΒΓ is a triangle. If then a cone is cut by some plane through the vertex, the section is a triangle.

## [Proposition] 4

*If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis, and the figure contained by the circle and the conic surface intercepted by the cutting plane on the side of the vertex will be a cone* [18].

Let there be a conic surface whose vertex is the point Α and whose circle along which the straight line generating the surface is moved is ΒΓ, and let it be cut by some plane parallel to the circle ΒΓ, and let it make on the surface as a section the line ΔΕ.

I say that the line ΔΕ is a circle having the center on the axis.

[Proof]. For let Ζ be taken as the center of the circle ΒΓ, and let ΑΖ be joined. Therefore [according to Definition 1] ΑΖ is the axis and meets the cutting plane. Let it meet it at Η, and let some plane be drawn through ΑΖ. Then [according to Proposition I.3] the section will be the triangle ΑΒΓ. And since the points Δ, Η, Ε are points in the cutting plane, and are also in the plane of the triangle ΑΒΓ, [according to Proposition XI.3 of Euclid] ΔΗΕ is a straight line.

Then let some point Θ be taken on the line ΔΕ, let ΑΘ be joined and continued. Then [according to Proposition I.1] it falls on the circumference ΒΓ. Let it meet it at Κ, and let ΗΘ and ΖΚ be joined. And since two parallel planes, ΔΕ and ΒΓ, are cut by a plane ΑΒΓ, [according to Proposition XI.16 of Euclid] their common sections are parallel. Therefore ΔΕ is parallel to ΒΓ. Then for the same reason ΗΘ is also parallel to ΚΖ. Therefore [according to Proposition VI.4 of Euclid] as ΖΑ is to ΑΗ, so ΖΒ is to ΔΗ, and ΖΓ is to ΗΕ, and ΖΚ is to ΗΘ.

Since ΒΖ is equal to ΚΖ and to ΖΓ [according to Proposition V.9 of Euclid] ΔΗ is equal to ΗΘ and to ΗΕ.

Then likewise we could show also that all the straight lines falling from the point Η on the line ΔΕ are equal to each other.

Therefore the line ΔΕ is a circle having its center on the axis.

And it is evident that the figure contained by the circle ΔΕ and the conic surface cut off by it on the side of the point Α is a cone.

And it is there with proved that the common section of the cutting plane and of the axial triangle [that is triangle through the axis] is a diameter of the circle.

[Proposition] 5

*If an oblique cone is cut by a plane through the axis at right angles to the base, and is also cut by another plane on the one hand at right angles to the axial triangle, and on the other hand cutting off on the side of the vertex a triangle similar to the axial triangle and situated antiparallel, then the section is a circle, and let such a section be called antiparallel* [19].

Let there be an oblique cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut through the axis by a plane perpendicular to the circle ΒΓ, and let it make as a section the triangle ΑΒΓ. Then let it also be cut by another plane perpendicular to the triangle ΑΒΓ and cutting off on the side of Α the triangle ΑΚΗ similar to the triangle ΑΒΓ and situated antiparallel, that is so that the angle ΑΚΗ is equal to the angle ΑΒΓ. And let it make as a section on the surface [of the cone] the line ΗΘΚ.

I say that the line ΗΘΚ is a circle.

[Proof]. For let any points Θ and Λ be taken on the lines ΗΘΚ and ΒΓ, and from Θ and ΛΛ let perpendiculars be dropped to the plane of the triangle ΑΒΓ. Then [according to Definition XI.4 of Euclid] they will fall to the common sections of the planes. Let them fall for example as ΖΘ and ΛΜ.
Therefore [according to Proposition XI.6 of Euclid] ΖΘ is parallel to ΛΜ.

Then ΔΖΕ be drawn through Ζ parallel to ΒΓ, and ΖΘ is parallel to ΛΜ. Therefore [according to Proposition XI.15 of Euclid] the plane through ΖΘ and ΔΕ is parallel to the base of the cone. Therefore [according to Proposition I.4] it is a circle whose diameter is ΔΕ. Therefore [according to Proposition II.14 of Euclid] [20] pl. ΔΖΕ is equal to sq. ΖΘ.

And since ΕΔ is parallel to ΒΓ, the angle ΑΔΕ is equal to the angle ΑΒΓ. And the angle ΑΚΗ is supposed equal to the angle ΑΒΓ. Therefore the angle ΑΚΗ is equal to the angle ΑΔΕ. And the vertical angles at Ζ are also equal. Therefore the triangle ΔΖΗ is similar to the triangle ΚΖΕ, and therefore [according to Proposition VI.4 of Euclid] as ΕΖ is to ΖΚ, so ΗΖ is to ΖΔ.
Therefore [according to Proposition VI.16 of Euclid] pl. ΕΖΔ is equal to pl.ΚΖΗ.

But it has been shown that sq.ΖΘ is equal to pl.ΕΖΔ.

Therefore pl.ΚΖΗ is equal to sq.ΖΘ.

Likewise then all the perpendiculars drawn from the line ΗΘΚ to ΗΚ could also be shown to be equal in square to the rectangular plane, in each case under the segments of ΗΚ.

Therefore the section is a circle[21] whose diameter is ΗΚ.

[Proposition] 6

*If a cone is cut by a plane through the axis, and if on the surface of the cone some point is taken which is not on a side of the axial triangle, and if from this point is drawn a straight line parallel to some straight line which is a perpendicular from the circumference of the circle to the base of the triangle, then that drawn straight line meets the axial triangle, and on being continued to the other side of the surface the drawn straight line will be bisected by the triangle*[22].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let the cone be cut by a plane through the axis, and let it make as a common section the triangle ΑΒΓ, and from some point Μ on the circumference let ΜΝ be drawn perpendicular to [the straight line] ΕΒΓ. Then let some point Δ be taken on the surface of the cone, and through Δ let ΔΕ be drawn parallel to ΜΝ.

I say that the continued ΔΕ will meet the plane of the triangle ΑΒΓ, and if further continued toward the other side of the cone until it meet its surface, will be bisected by the triangle ΑΒΓ.

[Proof]. Let ΑΔ be joined and continued. Therefore it will meet the circumference of the circle ΒΓ. Let it meet it at Κ and from Κ let ΚΘΛ be drawn perpendicular to ΒΓ.Therefore ΚΘ is parallel to ΜΝ, and therefore [according to Proposition XI.9 of Euclid] also to ΔΕ. Let ΑΘ be joined. Since then in the triangle ΑΘΚ [the straight line] ΔΕ is parallel to ΘΚ, therefore ΔΕ continued will meet ΑΘ. But ΑΘ is in the plane of the triangle ΑΒΓ; therefore ΔΕ will meet this plane.

For the same reasons it also meets ΑΘ, let it meet it at Ζ, and let ΔΖ be continued in a straight line until it meet the surface of the cone. Let it meet it at Η. I say that ΔΖ is equal to ΖΗ.

For since Α, Η, Λ are points on the surface of the cone, but also in the plane drawn through ΑΘ, ΑΚ, ΔΗ, ΚΛ, which is a triangle through the vertex of the cone, therefore Α, Η, Λ are points of the common section of the cone's surface and of the triangle. Therefore the line through Α, Η, and Λ is a straight line. Since then in the triangle ΑΛΚ [the straight line] ΔΗ has been drawn parallel in the base ΚΘΛ, and some straight line ΑΖΘ has been drawn across them from A, therefore [according to Proposition VI.4 of Euclid] as ΚΘ is to ΘΛ, so ΔΖ is to ΖΗ. But ΚΘ [according to Proposition III.3 of Euclid] is equal to ΘΛ since ΚΛ is a chord in the circle ΒΓ perpendicular to the diameter. Therefore ΔΖ is equal to ΖΗ.

[Proposition] 7

*If a cone is cut by a plane through the axis, and if the cone is also cut by another plane, so that the plane of the base of the cone is cut in a straight line perpendicular either to the base of the axial triangle or to it continued, and if from the cutting plane's resulting section on the cone's surface, straight lines are drawn parallel to the straight line perpendicular to the base of the triangle, then these straight lines will fall on the common section of the cutting plane and of the axial triangle, and further continued to the other side of the section, these straight lines will be bisected by the common section, and if the cone is right, then the straight line in the base will be perpendicular to the common section of the cutting plane and of the axial triangle, but if the cone is oblique, then the straight line in the base will be perpendicular to that common section only whenever the plane through the axis is perpendicular to the base of the cone*[23,24].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by a plane through the axis, and let it make as a common section the triangle ΑΒΓ. And let it also be cut by another plane cutting the plane of the circle ΒΓ in ΔΕ perpendicular either to ΒΓ or to it continued, and let it make as a section on the surface of the cone the line ΔΖΕ. Then ΖΗ is the common section of the cutting plane and of the triangle ΑΒΓ. And let some point Θ be taken on the section ΔΖΕ, and let ΘΚ be drawn through Θ parallel to ΔΕ.

I say that ΘΚ meets ΖΗ, and if continued to the other side of the section ΔΖΕ will be bisected by ΖΗ.

[Proof]. For since a cone whose vertex is the point Α and whose base is the circle ΒΓ has been cut by a plane through its axis, and makes as a section the triangle ΑΒΓ, and since some point Θ on the surface, not on a side of the triangle ΑΒΓ, has been taken, and since ΔΗ is perpendicular to [the straight line] ΒΓ, therefore the straight line drawn through Θ parallel to ΔΗ, that is ΘΚ, meets the triangle ΑΒΓ, and [according to Proposition I.6] if further continued to the other side of the surface, will be bisected by the triangle.

Then since the straight line drawn through Θ parallel to ΔΕ meets the triangle ΑΒΓ and is in the planes of the section ΔΖΕ, therefore it will fall on the common section of the cutting plane and of the triangle ΑΒΓ. But ΖΗ is the common section of the planes. Therefore the straight line drawn through Θ parallel to ΔΕ will fall on ΖΗ, and, if further continued to the other side of the section ΔΖΕ, will be bisected by ΖΗ.

Then either the cone is right, or the axial triangle ΑΒΓ is perpendicular to the circle ΒΓ, or neither.

First let the cone be right. Then [according to Definition 3 and according to Proposition XI.18 of Euclid] the triangle ΑΒΓ would be perpendicular to the circle ΒΓ. Since then the plane ΑΒΓ is perpendicular to the plane [of the circle] ΒΓ, and ΔΕ has been drawn in one of these two planes, [the plane of the circle] ΒΓ, perpendicular to their common section, [the straight line] ΒΓ, therefore [according to Definition XI.4 of Euclid] ΔΕ is perpendicular to the triangle ΑΒΓ, and therefore to all straight lines touching it and situated in the triangle ΑΒΓ.And so ΔΕ is also perpendicular to ΖΗ.

Then let the cone not be right. If now the axial triangle is perpendicular to the circle ΒΓ, we could likewise show that ΔΕ is perpendicular to ΖΗ.

Then let the axial triangle ΑΒΓ not be perpendicular to the circle ΒΓ.

I say that ΔΕ is not perpendicular to ΖΗ. For, if possible, let it be so. And it is also perpendicular to [the straight line] ΒΓ.Therefore ΔΕ is perpendicular to both ΒΓ and ΖΗ, and therefore it will be perpendicular to the plane through ΒΓ and ΖΗ. But the plane of through ΒΓ and HZ is the [plane of the] triangle ΑΒΓ, and therefore ΔΕ is perpendicular to the triangle ΑΒΓ. And therefore all planes through it are perpendicular to the triangle ΑΒΓ. But one of the planes through ΔΕ is the [plane of the] circle ΒΓ. Therefore the circle ΒΓ is perpendicular to the triangle ΑΒΓ. And so the triangle ΑΒΓ will also be perpendicular to the circle ΒΓ. And this is not supposed. Therefore ΔE is not perpendicular to ΖΗ.

### Porism

Then from this it is evident that ΖΗ is the diameter of the section ΔΖΕ, since it bisects the straight lines drawn parallel to some straight line ΔΕ, and it is evident that it is possible for some parallels to be bisected by the diameter ΖΗ and not be perpendicular to ΖΗ.

### [Proposition] 8

*If a cone is cut by a plane through its axis, and if the cone is cut by another plane cutting the base of the cone in a straight line perpendicular to the* *base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of the sides continued beyond the vertex of the cone, and if both surface of the cone and cutting plane are continued indefinitely, then the section will also increase indefinitely and some straight line drawn from the section of the cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line*[25].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by a plane through its axis, and let it make as a section the triangle ΑΒΓ. And let it be cut also by another plane cutting the circle ΒΓ in a straight line ΔΕ perpendicular to [the straight line] ΒΓ, and let it make as a section on the surface the line ΔΖΕ. And let the diameter ΖΗ of the section ΔΖΕ [according to Proposition I.7 and its porism] be either parallel to ΑΓ or on being continued meet it beyond the point Α.

I say that if both the surface of the cone and the cutting plane are continued indefinitely, the section ΔΖΕ also will increase indefinitely.

[Proof]. For let both the surface of the cone and the cutting plane are continued. Then it is evident that also ΑΒ, ΑΓ, ΖΗ will be therewith continued. Since ΖΗ is either parallel to ΑΓ or continued meets it beyond the point Α, therefore ΖΗ and ΑΓ on being continued in the direction of Γ and H will never meet. Then let them be continued and let some point Θ be taken at random on ΖΗ, and let ΚΘΛ be drawn through Θ parallel to ΒΓ, and ΜΘΝ parallel to ΔΕ. Therefore the plane through ΚΛ and ΜΝ [according to Proposition XI.15 of Euclid] is parallel to the plane through ΒΓ and ΔΕ. Therefore [according to Proposition I.4] the plane ΚΛΜΝ is a [plane of a circle].

And since the points Δ, Ε, Μ, Ν are in the cutting plane and also on the surface of the cone, therefore they are on the common section. Therefore the section ΔΖΕ has increased to the points Μ and Ν. Therefore, with the surface of the cone and the cutting plane increased to the circle ΚΛΜΝ, the section ΔΖΕ has also increased to the points Μ and Ν.Then likewise we could show also that if the surface of the cone and the cutting plane are continued indefinitely, the section ΜΔΖΕΝ will also increase indefinitely.

And it is evident that some straight line will cut off on straight line ΖΘ on the side of the point Ζ a straight line equal to any given straight line. For if we lay dawn ΖΞ equal to the given straight line, and draw a parallel to ΔΕ through Ξ, it will meet the section, just as the straight line through Θ was also proved to meet the section in the points Μ and Ν. And so some straight line is drawn meeting the section, parallel to ΔE, and cutting off on ZH on the side of point Z a straight line equal to the given straight line.

### [Proposition] 9

*If a cone is cut by a plane, which meets both sides of the axial triangle and is neither parallel to the base [of the cone], nor antiparallel to it, then the section will not be a circle* [26].

Let there be a cone whose vertex is the point A and whose base is the circle BΓ, and let it be cut by some plane neither parallel to the base [of the cone], nor antiparallel to it, and let it make as a section on the surface the line ΔKE.

I say that the line ΔKE will not be a circle.

[Proof]. For, if possible, let it be, and let the cutting plane meet the base, and let ZH be the common section of these planes, and let Θ be the center of the circle BΓ, and from Θ let ΘH be drawn perpendicular to ZH. And let a plane be drawn through HΘ and the axis and let [according to Proposition I.1] it make as sections on the conic surface BA and AΓ. Since then Δ, E, H are points in the plane through the line ΔKE, and also in the plane through the points A, B, Γ, therefore Δ, E, H are points on the common section of these planes. Therefore [according to Proposition XI.3 of Euclid] HEΔ is a straight line.

Then let some point K be taken on the line ΔKE, and through K let KΛ be drawn parallel to ZH. Then [according to Proposition I.7] KM will be equal to MΛ. Therefore ΔE is the diameter of the [supposed] circle ΔΞΛE. Then let NMΞ be drawn through M parallel to BΓ. But KΛ is also parallel to ZH. And so the plane through NΞ and KM [according to Proposition XI.15 of Euclid] is parallel to the plane through BΓ and ZH, which is to the base, and the section [according to Proposition I.4] will be a circle. Let it be the circle NKΞ.

And since ZH is perpendicular to BH, and KM [according to Proposition XI.10 of Euclid] is also perpendicular to NΞ. And so [according to Proposition II.14 of Euclid] pl.NMΞ is equal to sq.KM.

But pl.ΔME is equal to sq.KM for the line ΔKEΛ is supposed a circle, and ΔE is its diameter.

Therefore pl.NMΞ is equal to pl.ΔME. Therefore [according to Proposition VI.16 of Euclid] as MN is to MΔ, so EM is to MΞ.

Therefore [according to Proposition VI.6 and Definition VI.1 of Euclid] the triangle ΔMN is similar to the triangle ΞME, and the angle ΔNM is equal to the angle ΜΕΞ. But the angle ΔΝΜ is equal to the angle ΑΒΓ for ΝΞ is parallel to ΒΓ. And therefore the angle ΑΒΓ is equal to the angle ΜΕΞ. Therefore [according to Proposition I.5] the section is antiparallel to the base of the cone. And this is not supposed. Therefore the line ΔΚΕ is not a circle.

[Proposition] 10

*If two points are taken on the section of a cone, the straight line joining these two points will fall within the section, and continued in a straight line it will fall outside*[27].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by a plane through the axis, and let it make as a section the triangle ΑΒΓ. Then let it also be cut [not through the vertex] by another plane, and let it make as a section on the surface of the cone the line ΔΕΖ, and let two points Η and Θ be taken on the line ΔΕΖ. I say that the straight line joining two points Η and Θ will fall within the line ΔΕΖ, and continued in a straight line it will fall outside.

[Proof]. For since a cone, whose vertex is the point Α and whose base is the circle ΒΓ, has been cut by a plane through the axis, and some points Η and Θ have been taken on its surface which are not on a side of the axial triangle and since the straight line joining Η and Θ does not verge to the point Α, therefore [according to Proposition I.2] the straight line joining H and Θ will fall within the cone, and continued in a straight line it will fall outside, consequently also outside the section ΔΖΕ.

[Proposition] 11

*If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one [lateral] side of the axial triangle, and if any straight line is drawn from the section of the cone to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone's base, then this straight line dropped to the diameter will equal in square to [the rectangular plane] under the straight line from the section's vertex to [the point] where the straight line dropped to the diameter cuts it off and under another straight line which is to the straight line between the angle of the cone and the vertex of the section as the square on the base of the axial triangle to [the rectangular plane] under the remaining two sides of the triangle.*

*I call such a section a* parabola[28,29].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by a plane through its axis, and let it make as a section the triangle ΑΒΓ. And let it also be cut by another plane cutting the base of the cone in the straight line ΔΕ perpendicular to [the straight line], ΒΓ and let it make as a section on the surface of the cone the line ΔΖΕ, and let the diameter of the section ΖΗ be parallel to one side ΑΓ of the axial triangle. And let ΖΘ be drawn from the point Ζ perpendicular to ΖΗ, and let it be contrived that as sq. ΒΓ is to pl. ΒΑΓ, so ΖΘ is to ΖΑ.

And let some point Κ be taken at random on the section, and through Κ let ΚΛ be drawn parallel to ΔΕ.

I say that sq. ΚΛ is equal to pl. ΘΖΛ.

[Proof]. For let ΜΝ be drawn through Λ parallel to ΒΓ. And ΔΕ is also parallel to ΚΛ.Therefore [according to Proposition XI.15 of Euclid] the plane through ΚΛ and ΜΝ is parallel to the plane through ΒΓ and ΔΕ, which is to the base of the cone. Therefore [according to Proposition I.4] the plane through ΚΛ and ΜΝ is a circle whose diameter is ΜΝ. And ΚΛ is perpendicular to ΜΝ, since ΔΕ is also [according to Proposition XI.10 of Euclid] perpendicular to ΒΓ. Therefore [according to Proposition II.14 of Euclid] pl.ΜΛΝ is equal to sq.ΚΛ.

And since as sq.ΒΓ is to pl.ΒΑΓ, so ΘΖ is to ΖΑ, and [according to Proposition VI.23 of Euclid] the ratio sq.ΒΓ to pl.ΒΑΓ is compounded[30] of [the ratios] ΒΓ to ΓΑ and ΒΓ to ΒΑ. Therefore the ratio ΘΖ to ΖΑ ισ compounded of [the ratios] ΒΓ to ΒΑ and ΜΝ to ΝΑ. But [according to Proposition VI.4 of Euclid] as ΒΓ is to ΓΑ so ΜΝ is to ΝΑ, and ΜΛ is to ΛΖ and [according to Propositions VI.2 and VI.4 of Euclid] as ΒΓ is to ΒΑ, so ΜΝ is to ΜΑ, ΛΜ is to ΜΖ, and ΝΛ is to ΖΑ.

Therefore the ratio ΘΖ to ΖΑ is compounded of [the ratios] ΜΛ to ΛΖ and ΝΛ to ZA. But [according to Proposition VI.23 of Euclid] the ratio pl.ΜΛΝ to pl.ΛΖΑ is compounded of [the ratios] ΜΛ to ΛZ and ΛΝ to ΖΑ.

Therefore as ΘΖ is to ΖΑ, so pl.ΜΛΝ is to pl.ΛΖΑ.

But, with ΖΛ taken as common height [of two rectangular planes, according to Proposition VI.1 of Euclid] as ΘΖ is to ΖΑ, so pl.ΘΖΛ is to pl.ΛΖΑ.

Therefore [according to Proposition V.11 of Euclid] as pl.ΜΛΝ is to pl.ΛΖΑ, so pl.ΘΖΛ is to pl.ΛΖΑ.

Therefore [according to Proposition V.9 of Euclid] pl.ΜΛΝ is equal to pl.ΘΖΛ.

But pl.ΜΛΝ is equal to sq.ΚΛ; therefore also sq.ΚΛ is equal to pl.ΘΖΛ.

I will call such a section a parabola, and ΘZ be called the straight line of application [of rectangular planes] to which the ordinates drawn to ZH are equal in square. I will call this straight line also the *latus rectum*.

[Proposition] 12

*If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the section continued meets [continued] one [lateral] side of the axial triangle beyond the vertex of the cone, and if any straight line is drawn from the section to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone's base, then this straight line to the diameter will equal in square to some [rectangular] plane which is applied to a straight line increased by the segment added along the diameter of the section, such that this added segment subtends the exterior angle of the [vertex of the axial] triangle, and as the added segment, is to the mentioned the straight line, so the square on the straight line drawn parallel to the section's diameter from the cone's vertex to the [axial] triangle's base is to the [rectangular] plane under the segments of the triangle's base divided by the straight line drawn from the vertex [of the cone], and the applied plane has as breadth the straight line on the diameter from the section's vertex to [the point] where the diameter is cut off by the straight line drawn from the section to the diameter, this plane is [the rectangular plane under two mentioned straight lines] and increased by a figure similar and similarly situated to the plane under the mentioned straight line and the diameter.*
*I will call such a section a* hyperbola[31].

*Let there be a cone whose vertex is the point* A and whose base is the circle BΓ, and let it be cut by a plane through its axis, and let it make as a section the triangle ABΓ. And let the cone also be cut by another plane cutting the base of the cone in ΔE perpendicular to BΓ, the base of the triangle ABΓ, and let this second cutting plane make as a section on the surface of the cone the line ΔZE, and let the diameter of the section ZH [according to Proposition I.7 and Definition 4] when continued meet AΓ, one [lateral] side of the triangle ABΓ beyond the vertex of the cone at Θ. And let AK be drawn through A parallel to the diameter of the section ZH, and let it cut BΓ [at K]. And let ZΛ be drawn from Z perpendicular to ZH, and let it be contrived that as sq.KA is to pl.BKΓ, so ZΘ is to ZΛ.

And let some point M be taken at random on the section and through M let MN be drawn parallel to ΔE, and through N let NOΞ be drawn parallel to ZΛ.

And let ΘΛ be joined and continued to Ξ, and let ΛΟ and ΞΠ be drawn through Λ and Ξ parallel to ΖΝ.

I say that ΜΝ is equal in square to the rectangular plane ΖΞ, which is applied to ΖΛ having ΖΝ as breadth, and increased by a figure ΛΞ similar to pl. ΘΖΛ.

[Proof]. For let ΡΝΣ be drawn through Ν parallel to ΒΓ. And ΝΜ is also parallel to ΔΕ. Therefore [according to Proposition XI.15 of Euclid] the plane through ΜΝ and ΡΣ is parallel to the plane through ΒΓ and ΔΕ, which is to the base of the cone. Therefore if the plane is drawn through ΜΝ and ΡΣ, the section [according to Proposition I.4] will be a circle whose diameter is ΡΝΣ. And ΜΝ is perpendicular to it. Therefore pl.ΡΝΣ is equal to sq.ΜΝ.

And since as sq.ΑΚ is to pl.ΒΚΓ, so ΖΘ is to ΖΛ, and [according to Proposition VI.23 of Euclid] the ratio sq.ΑΚ to pl.ΒΚΓ is compounded of [the ratios] ΑΚ to ΚΓ and ΑΚ to ΚΒ, therefore also the ratio ΖΘ to ΖΛ is compounded of [the ratios] ΑΚ to ΚΓ and ΑΚ to ΚΒ.

But [according to Proposition VI.4 of Euclid] as ΑΚ is to ΚΓ, so ΘΗ is to ΗΓ, and ΘΝ is to ΝΣ and as ΑΚ is to ΚΒ, so ΖΗ is to ΗΒ and ΖΝ is to ΝΡ.

Therefore the ratio ΘΖ to ΖΛ is compounded of [the ratios] ΘΝ to ΝΣ and ΖΝ to ΝΡ. And [according to Proposition VI.23 of Euclid] the ratio pl.ΘΝΖ to pl.ΣΝΡ is compounded of [the ratios] ΘΝ to ΝΣ and ΖΝ to ΝΡ.

Therefore also [according to Proposition VI.4 of Euclid] as pl.ΘΝΖ is to pl.ΣΝΡ, so ΘΖ is to ΖΛ and ΘΝ is to ΝΞ.

But, with ΖΝ taken as common height [according to Proposition VI.1 of Euclid] as ΘΝ is to ΝΞ, so pl.ΘΝΖ is to pl.ΖΝΞ.

Therefore also [according to Proposition V.11 of Euclid] as pl.ΘΝΖ is to pl.ΣΝΡ, so pl.ΘΝΖ is to pl.ΞΝΖ, and [according to Proposition V.9 of Euclid] pl.ΣΝΡ is equal to pl.ΞΝΖ.

But it was shown that sq.ΜΝ is equal to pl.ΣΝΡ, therefore also sq.ΜΝ is equal to pl.ΞΝΖ.

But pl.ΞΝΖ is the parallelogram ΞΖ. Therefore ΜΝ is equal in square to ΞΖ which is applied to ΖΛ and having ΖΝ as breadth increased by the parallelogram ΛΞ similar to pl.ΘΖΛ. I will call such a section a hyperbola, and ΛΖ be called the straight line of application [of rectangular planes] to which the ordinates drawn to ΖΗ are equal in square.

I will call this straight line also the *latus rectum*, and the straight line ΖΘ the *latus transversum*.

[Proposition] 13

*If a cone is cut by a plane through its axis, and is also cut by another plane which on the one hand meets both [lateral] sides of the axial triangle, and on the other hand, when continued, is neither parallel to the base [of the cone] nor antiparallel to it, and if the plane of the base of the cone and the cutting plane meet in a straight line perpendicular either to the base of the axial triangle or to it continued, then any [straight] line drawn parallel to the common section of the [base and cutting] planes from the section of the cone to the diameter of the section will be equal in square to some [rectangular] plane applied to a straight line to which the diameter of the section is as the square on the straight line drawn parallel to the section's diameter from the cone's vertex to the [axial] triangle's base to the [rectangular] plane under the straight lines cut [on the axial triangle's base] by this straight line in the direction of the sides of the [axial] triangle, and the applied plane has as breadth the straight line on the diameter from the section's vertex to [ the point] where the diameter is cut off by the straight line drawn from the section to the diameter, this plane is [the rectangular plane under two mentioned straight lines] and decreased by a figure similar and similarly situated to the plane under the mentioned straight line and the diameter. I will call such a section an* ellipse[32].

Let there be a cone whose vertex is the point Α and whose base is the circle ΒΓ, and let it be cut by a plane through its axis, and let it make as a section the triangle ΑΒΓ. And let it also be cut by another plane on the one hand meeting both [lateral] sides of the axial triangle and on the other hand continued neither parallel to the base of the cone, nor antiparallel to it, and let it make as a section on the surface of the cone the [closed curved] line ΔΕ. And let the common section of the cutting plane and of the plane of the base of the cone be ΖΗ perpendicular to ΒΓ, and let [according to Proposition I.7 and Definition 4] the diameter of the section be [the straight line] ΕΔ. And let ΕΘ be drawn from Ε perpendicular to [the diameter] ΕΔ, and let ΑΚ be drawn through Α parallel to ΕΔ, and let it be contrived that as sq.ΑΚ is to pl.ΒΚΓ, so ΔΕ is to ΕΘ.

And let some point Λ be taken [at random] on the section, and let ΛΜ be drawn through Λ parallel to ΖΗ.

I say that ΛΜ is equal in square to the rectangular plane, which is applied to ΕΘ and having ΕΜ as breadth, and decreased by a figure similar to pl.ΔΕΘ.

[Proof]. For let ΔΘ be joined, and on the one hand let ΜΞΝ be drawn through Μ parallel to ΘΕ, and on the other hand let ΘΝ and ΞΟ be drawn through Θ and Ξ parallel to ΕΜ, and let ΠΜΡ be drawn through Μ parallel to ΒΓ

Since then ΠΡ is parallel to ΒΓ, and ΛΜ is also parallel to ΖΗ, therefore [according to Proposition XI.15 of Euclid] the plane through ΛM and ΠP is parallel to the plane through ZH and BΓ, which is to the base of the cone.

If therefore a plane is drawn through ΛM and ΠP, the section [according to Proposition I.4] will be a circle whose diameter is ΠP. And ΛM is perpendicular to it. Therefore [according to Proposition II.14 of Euclid] pl.ΠMP is equal to sq.ΛM.

And since as sq.AK is to pl.BKΓ, so EΔ is to EΘ, and [according to Proposition VI.23 of Euclid] the ratio sq.AK to pl.BKΓ is compounded of [the ratios] AK to KB and AK to KΓ.

But [according to Proposition VI.4 of Euclid] as AK is to KB, so EH is to HB and EM is to MΠ, and as AK is to KΓ, so ΔH is to HΓ and ΔM is to MP,

Therefore the ratio ΔE to EΘ is compounded of the [ratios] EM to MΠ and ΔM to MP.

But [according to Proposition VI.23 of Euclid] the ratio pl.EMΔ to pl.ΠMP is compounded of the [ratios] EM to MΠ and ΔM to MP.

Therefore [according to Proposition VI.4 of Euclid] as pl.EMΔ is to pl.ΠMP, so ΔE is to EΘ and ΔM is to MΞ.

And with the straight line ME taken as common height [according to Proposition VI.1 of Euclid] as ΔM is to MΞ, so pl.ΔME is to pl.ΞME.

Therefore also [according to Proposition V.11 of Euclid] as pl.ΔME is to pl.ΠMP, so pl.ΔME is to pl.ΞME.
Therefore [according to Proposition V.9 of Euclid] pl.ΠMP is equal to pl.ΞME.

But it was shown that pl.ΠMP is equal to sq.ΛM, therefore also pl.ΞME is equal to sq.ΛM.

Therefore ΛM is equal in square to the parallelogram MO, which is applied to ΘE and having EM as breadth and decreased by the figure ON similar to pl.ΔEΘ.

I will call such a section an ellipse, and let EΘ be called the straight line of application [of rectangular planes] to which the ordinates drawn to ΔE are equal in square. I will call this straight line also the *latus rectum*, and the straight line EΔ the *latus transversum* [33-38].

[Proposition] 14

*If the vertically opposite surfaces are cut by a plane not through the vertex, the section on each of two surfaces will be that which is called the hyperbola, and the diameter of these two hyperbolas will be the same straight line, and the straight lines, to which straight lines drawn to the diameter parallel to* *the straight line in the cone's base are applied in square, are equal, and the latus transversum of the* eidos[39] *[of these hyperbolas], that is the straight line situated between the vertices of the hyperbolas is common. I call such hyperbolas* opposite [40]*.*

Let there be the vertically opposite surfaces whose vertex is the point Α and let them be cut by a plane not through the vertex and let it make as sections on the surface the lines ΔΕΖ and ΗΘΚ.

I say that each of the two sections ΔΕΖ and ΗΘΚ is the so-called hyperbola.

[Proof]. For let there be the circle ΒΔΓΖ along which the line generating the surface moves, and let the plane ΞΗΟΚ be drawn parallel to it on the vertically opposite surfaces, and ΖΔ and ΗΚ [according to Proposition I.4] are common sections of the plane of the sections ΗΘΚ and ΖΕΔ, and of the [planes of the] circles. Then [according to Proposition XI.16 of Euclid] they will be parallel. And let the axis of the conic surface be the straight line ΛΑΥ and the centers of the circles be Λ and Υ, and let a straight line drawn from Λ perpendicular to ΖΔ be continued to the points Β and Γ, and let a plane be drawn through ΒΓ and the axis. Then [according to Proposition XI.16 of Euclid] it will make as sections in the [planes of the] circles the parallel straight lines ΞΟ and ΒΓ, and on the surface [according to Proposition I.1 and Definition1] ΒΑΟ and ΓΑΞ.

Then ΞΟ will be perpendicular to ΗΚ, since ΒΓ is also perpendicular to ΖΔ, and [according to Proposition XI.10 of Euclid] each of these two [straight lines] is parallel to the other. And since the plane through the axis meets the sections in the points Μ and Ν within the [curved] lines [ΖΔ and ΗΚ], it is clear that the plane through the axis also cuts the [curved] lines. Let it cut them at Θ and Ε. Therefore Μ, Ε, Θ and Ν are points on the plane through the axis and in the plane of the [curved] lines, therefore [according to Proposition XI.3 of Euclid] the line ΜΕΘΝ is a straight line. It is also evident both that Ξ, Θ, Α, and Γ are in a straight line and Β, Ε, Α, and Ο also for [according to Proposition I.1]; they are both on the conic surface and in the plane through the axis. Let then ΘΡ and ΕΠ be drawn from Θ and Ε perpendicular to ΘΕ, and let ΣΑΤ be drawn through Α parallel to ΜΕΘΝ, and let it be contrived that as ΘΕ is to ΕΠ, so sq.ΑΣ is to pl.ΒΣΓ, and as ΕΘ is to ΘΡ, so sq.ΑΤ is to pl.ΟΤΞ.

Since then a cone whose vertex is the point Α and whose base is the circle ΒΓ has been cut by a plane through its axis, and it has made as a section the triangle ΑΒΓ, and it has also been cut by another plane cutting the base of the cone in ΔΜΖ perpendicular to ΒΓ, and it has made as a section on the surface the line ΔΕΖ and the diameter ΜΕ continued has met one side of the axial triangle beyond the vertex of the cone, and through Α the straight line ΑΣ has been drawn parallel to the diameter of the section ΕΜ, and from Ε the straight line ΕΠ has been drawn perpendicular to ΕΜ, and as ΕΘ is to ΕΠ, so sq.ΑΣ is to pl.ΒΣΓ, therefore [according to Proposition I.12] the section ΔΕΖ is a hyperbola, and ΕΠ is the *latus rectum* of the *eidos* of this hyperbola , and ΘΕ is the *latus transversum* of this *eidos* . Likewise ΗΘΚ is also a hyperbola whose diameter is ΘΝ and the *latus rectum* of whose *eidos* is ΘΡ, and the *latus transversum* of whose *eidos* is ΘΕ.

I say that ΘΡ is equal to ΕΠ.

[Proof]. For since ΒΓ is parallel to ΞΟ, as ΑΣ is to ΣΓ, so ΑΤ is to ΤΞ, and as ΑΣ is to ΣΒ, so ΑΤ is to ΤΟ.

But [according to Proposition VI.23 of Euclid] the ratio sq. ΑΣ to pl. ΒΣΓ is compounded of [the ratios] ΑΣ to ΒΣ and ΑΣ to ΣΓ and the ratio sq. ΑΤ to pl. ΞΤΟ is compounded of [the ratios] ΑΤ to ΤΞ and ΑΤ to ΤΟ, therefore as sq.ΑΣ is to pl.ΒΣΓ, so sq.ΑΤ is to pl.ΞΤΟ. Also as sq.ΑΣ is to pl. ΒΣΓ, so ΘΕ is to ΕΠ, and sq.ΑΤ is to pl.ΞΤΟ, so ΘΕ is to ΘΡ.Therefore also [according to Proposition V.11 of Euclid] as ΘΕ is to ΕΠ, so ΕΘ is to ΘΡ. Therefore [according to Proposition V.9 of Euclid] ΕΠ is equal to ΘΡ [41].

## [Proposition] 15

*If in an ellipse a straight line drawn as an ordinate from the midpoint of the diameter is continued both ways to the section, and if it is contrived that as the continued straight line is to the diameter, so the diameter is to some straight line, then any straight line which is drawn parallel to the diameter from the section to the continued straight line will be equal in square to the plane which is applied to this third proportional and which has as breadth the continued straight line from the section to [the point] where the straight line drawn parallel to the diameter cuts it off, but such this plane is decreased by a figure similar to the rectangular plane under the continued straight line to which the straight lines are drawn and the latus rectum, [that is the third proportional] and if the straight line drawn parallel to the diameter is further continued to the other side of the section, this drawn straight line will be bisected by the continued straight line to which it has been drawn*[42].

Let there be an ellipse whose diameter is ΑΒ, and let ΑΒ be bisected at the point Γ, and through Γ let ΔΓΕ be drawn as an ordinate and continued both ways to the section, and from Δ let ΔΖ be drawn perpendicular to ΔΕ.

And let it be contrived that as ΔΕ is to ΑΒ, so ΑΒ is to ΔΖ.

And let some point Η be taken on the section, and through Η let ΗΘ be drawn parallel to ΑΒ, and let ΕΖ be joined, and through Θ let ΘΛ be drawn parallel to ΔΖ, and through Ζ and Λ let ΖΚ and ΛΜ be drawn parallel to ΘΔ.

I say that ΗΘ is equal in square to the [rectangular] plane ΔΛ which is applied to ΔΖ and having as breadth ΔΘ and decreased by a figure ΛΖ similar to pl.ΕΔΖ [that is ΔΕ is the diameter conjugate to the diameter ΑΒ, and ΔΖ is the *latus rectum* for the ordinates to ΔΕ].

[Proof]. For let ΑΝ be the *latus rectum* for the ordinates to ΑΒ and let ΒΝ be joined, and through Η let ΗΞ be drawn parallel to ΔΕ, and through Ξ and Γ let ΞΟ and ΓΠ be drawn parallel to ΑΝ, and through Ν, Ο, and Π let ΝΥ, ΟΣ, and ΤΠ be drawn parallel to ΑΒ.

Therefore sq.ΔΓ is equal to [the plane] ΑΠ, and [according to Proposition I.13] sq.ΗΞ ισ equal to [the plane] ΑΟ.

And since [according to Proposition VI.4 of Euclid] as ΒΑ is to ΑΝ, so ΒΓ is to ΓΠ, and ΠΤ is to ΤΝ and ΒΓ is equal to ΓΑ and is equal to ΤΠ, and ΓΠ is equal to ΤΑ. Therefore [the plane] ΑΠ is equal to [the plane] ΤΡ, and [the plane] ΞΤ is equal to [the plane] ΤΥ.

Since also [according to Proposition I.43 of Euclid the plane] ΟΤ is equal to [the plane] ΟΡ, and [the plane] ΝΟ is common, therefore [the plane] ΤΥ is equal to [the plane] ΝΣ.

But [the plane] ΤΥ is equal to [the plane] ΤΞ, and [the plane] ΤΣ is common. Therefore [the plane] ΝΠ is equal to [the plane] ΠΑ and is equal to [the planes] ΑΟ and ΠΟ, and so [the plane] ΠΑ without [the plane] ΑΟ is equal to [the plane] ΠΟ.

Also [the plane] ΑΠ is equal to sq.ΓΔ, [the plane] ΑΟ is equal to sq.ΞΗ and [the plane] ΟΠ is equal to pl.ΟΣΠ, therefore sq.ΓΔ without sq.ΗΞ is equal to pl.ΟΣΠ.

Since also ΔΕ has been cut into equal parts at Γ, and into unequal parts at Θ, therefore [according to Proposition II.5 of Euclid] the sum of pl.ΕΘΔ and sq.ΓΘ is equal to sq.ΓΔ, or the sum of pl.ΕΘΔ and sq.ΞΗ is equal to sq.ΓΔ.

Therefore sq.ΓΔ without sq.ΞΗ is equal to pl.ΕΘΔ, but sq.ΓΔ without sq.ΞΗ is equal to pl.ΟΣΠ, therefore pl.ΕΘΔ is equal to pl.ΟΣΠ.
And since as ΔΕ is to ΑΒ, so ΑΒ is to ΔΖ, therefore [according to the porism to Proposition VI.19 of Euclid] as ΔΕ is to ΔΖ, so sq.ΔΕ is to sq.ΑΒ, which is [according to Proposition V.15 of Euclid] as ΔΕ is to ΔΖ, so sq.ΓΔ is to sq.ΓΒ.

And [according to Proposition I.13] pl.ΠΓΑ is equal to pl.ΠΓΒ, and is equal to sq.ΓΔ, and since [according to Proposition VI.4 of Euclid] as ΔΕ is to ΔΖ, so ΕΘ is to ΘΛ, or [according to Propositions VI.1 and V.11 of Euclid] as ΔΕ is to

ΔZ, so pl.ΕΘΔ is to pl.ΔΘΛ, and since as ΔΕ is to ΔΖ, so pl.ΠΓΒ is to sq.ΓΒ, and as pl.ΠΓΒ is to sq.ΓΒ, so pl.ΟΣΠ is to sq.ΟΣ, therefore also as pl. ΕΘΔ is to pl.ΔΘΛ, so pl.ΟΣΠ is to sq.ΟΣ.

And pl.ΕΘΔ is equal to pl.ΟΣΠ, therefore pl. ΔΘΛ is equal to sq.ΟΣ and is equal to sq.ΗΘ.

Therefore ΗΘ is equal in square to [the plane] ΔΛ, which is applied to ΔΖ, decreased by a figure ΖΛ similar to pl.ΕΔΖ.

I say then that also, if continued to the other side of the section, ΗΘ will be bisected by ΔΕ.

[Proof]. For let it be continued and let it meet the section at Φ and let ΦΧ be drawn through Φ parallel to ΗΞ, and through Χ let ΧΨ be drawn parallel to ΑΝ. And since ΗΞ is equal to ΦΧ, therefore also sq.ΗΞ is equal to sq.ΦΧ.

But [according to Proposition I.13] sq.ΗΞ is equal to pl.ΑΞΟ and sq.ΦΧ is equal to pl.ΑΧΨ.

Therefore [according to Proposition VI.16 of Euclid] as ΟΞ is to ΨΧ, so ΧΑ is to ΑΞ.

And [according to Proposition VI.4 of Euclid] as ΟΞ is to ΨΧ, so ΞΒ is to ΒΧ, therefore also as ΧΑ is to ΑΞ, so ΞΒ is [according to Proposition V.17 of Euclid] as ΧΞ is to ΑΞ, so ΧΞ is to ΒΧ. Therefore ΑΞ is equal to ΧΒ. And also ΑΓ is equal to ΓΒ. Therefore also the remainders ΞΓ is equal to ΓΧ, and so also ΗΘ is equal to ΘΦ.

Therefore ΘΗ, continued to the other side of the section, is bisected by ΔΘ.

[Proposition] 16

*If through the midpoint of the latus transversum of the opposite hyperbolas a straight line be drawn parallel to an ordinate, it will be a diameter of the opposite hyperbolas conjugate to the diameter just mentioned*[43].

Let there be the opposite hyperbolas whose diameter is ΑΒ, and let ΑΒ be bisected at Γ and through Γ let ΓΔ be drawn parallel to an ordinate.

I say ΓΔ is a diameter conjugate to ΑΒ.

[Proof]. For let ΑΕ and ΒΖ be the *latera recta* for the ordinates to ΑΒ, and let ΑΖ and ΒΕ be joined and continued, and let some point Η be taken at random on either section, and through H let ΗΘ be drawn parallel to ΑΒ, and from Η and Θ let ΗΚ and ΘΛ be drawn as ordinates, and through Κ and Λ let KM and ΛΝ be drawn parallel to ΑΕ and ΒΖ. Since then [according to Proposition I.34 of Euclid] ΗΚ is equal to ΘΛ, therefore also sq.ΗΚ is equal to sq.ΘΛ.

But [according to Proposition I.12] sq.HK is equal to pl.AKM and sq.ΘΛ is equal to pl.BΛN. Therefore pl.AKM is equal to pl.BΛN.

And since [according to Proposition I.14] AE is equal to BZ, therefore [according to Proposition V.7 of Euclid] as AE is to AB, so BZ is to BA.

But [according to Proposition VI.4 of Euclid] as AE is to AB, so MK is to KB, and as BZ is to BA, so NΛ is to ΛA. Therefore as MK is to KB, so NΛ is to ΛA.

But, with KA taken as common height, as MK is to KB, so pl.MKA is to pl.BKA, and, with BΛ taken as common height, as NΛ is to ΛA, so pl.NΛB is to pl.AΛB.

And therefore as pl.MKA is to pl.BKA, so pl.NΛB is to pl.AΛB.

And alternately [according to Proposition V.16 of Euclid] as pl.MKA is to pl.NΛB, so pl.BKA is to pl.AΛB.

And above was proved that pl.AKM is equal to pl.BΛN, therefore pl.BKA is equal to pl.AΛB. Therefore AK is equal to ΛB.

But also AΓ is equal to ΓB, and therefore KΓ is equal to ΓΛ, and so also HΞ is equal to ΞΘ.

Therefore HΘ is bisected by ΞΓΔ, and is parallel to AB. Therefore ΞΓΔ is the diameter and conjugate to AB.

## Second definitions

9. Let the midpoint of the diameter of both the hyperbola and the ellipse be called the center[44] of the section, and let the straight line drawn from the center to meet the section be called the radius of the section.

10. And likewise let the midpoint of the *latus transversum* of the opposite hyperbolas be called the center.

11. And let the straight line drawn from the center [of the hyperbola or of the ellipse] parallel to an ordinate, being a mean proportional to the sides of the *eidos* and bisected by the center, be called the second diameter[45].

## [Proposition] 17

*If in a section of a cone a straight line is drawn from the vertex of the section and parallel to an ordinate it will fall outside the section*[46].

Let there be a section of a cone whose diameter is AB.

I say that the straight line drawn from the vertex, that is from the point A, parallel to an ordinate, will fall outside the section.

[Proof]. For, if possible, let it fall within as ΑΓ. Since then a point Γ has been taken at random on a section of a cone, therefore the straight line drawn from Γ within the section parallel to an ordinate will meet the diameter ΑΒ and [according to Proposition I.7] will be bisected by it. Therefore ΑΓ continued will be bisected by ΑΒ. And this is impossible for ΑΓ, if continued, [according to Proposition I.10] will fall outside the section. Therefore the straight line drawn from the point Α parallel to an ordinate will not fall within the section, therefore it will fall outside, and so it is tangent to the section.

[Proposition] 18

*If a straight line meeting a section of a cone and continued both ways, falls outside the section, and some point is taken within the section, and through it a parallel to the straight line meeting the section is drawn, the parallel so drawn, if continued both ways, will meet the section*[47].

Let there be a section of a cone and the straight line ΑΖΒ meeting it, and let it fall, when continued both ways, outside the section. And let some point Γ be taken within the section, and through Γ let ΓΔ be drawn parallel to ΑΒ.

I say that ΓΔ continued both ways will meet the section.

[Proof]. For, let some point Ε be taken on the section, and let ΕΖ be joined. And since ΑΒ is parallel to ΓΔ, and some straight line ΕΖ meets ΑΒ, therefore ΓΔ continued will also meet ΕΖ. And if it meets ΕΖ between Ε and Ζ, it is evident that it also meets the section, but if it meets it beyond Ε, that will first meet the section. Therefore, if ΓΔ is continued to the side of Δ and Ε, it meets the section. Then likewise we could show that, if it is continued to the side of Ζ and Β, it also meets it.

Therefore, ΓΔ continued both ways will meet the section.

[Proposition] 19

*In every section of a cone any straight line drawn from the diameter parallel to an ordinate will meet the section*[48].

Let there be a section of a cone whose diameter is ΑΒ, and let some point Β be taken on the diameter, and through Β let ΒΓ be drawn parallel to an ordinate.

I say that ΒΓ continued will meet the section.

[Proof]. For let some point Δ be taken on the section. But Α is also on the section; therefore the straight line joined from Α to Δ [according to Proposition I.10] will fall within the section. And since the straight line drawn from Α parallel to an ordinate [according to Proposition I.17] falls outside the section, and ΑΔ meets it, and ΒΓ is parallel to the ordinate, therefore ΒΓ will also meet ΑΔ. And if it meets ΑΔ between Α and Δ, it is evident that it will also meet the section, but, if it meets if beyond Δ as at Ε, that it will first meet the section. Therefore the straight line drawn from Β parallel to an ordinate will meet the section.

### [Proposition] 20

*If in a parabola two straight lines are dropped as ordinates to the diameter, the squares on them will be to each other as the straight lines cut off by them on the diameter beginning from the vertex are to each other*[49].

Let there be a parabola whose diameter is ΑΒ, and let some points Γ and Δ be taken on it, and from Γ and Δ let ΓΕ and ΔΖ be dropped as ordinates to ΑΒ.

I say that as sq.ΔΖ is to sq.ΓΕ, so ΖΑ is to ΑΕ.

[Proof]. For let ΑΗ be the *latus rectum* for the ordinates to the diameter. Therefore [according to the Proposition I.11] sq.ΔΖ is equal to pl.ΖΑΗ and sq.ΓΕ is equal to pl.ΕΑΗ.

Therefore as sq.ΔΖ is to sq.ΓΕ, so pl.ΖΑΗ is to pl.ΕΑΗ.

But [according to Proposition VI.1 of Euclid] as pl.ΖΑΗ is to pl.ΕΑΗ, so ΖΑ is to ΑΕ, and therefore as sq.ΔΖ is to sq.ΓΕ, so ΖΑ is to ΑΕ.

### [Proposition] 21

*If in a hyperbola or an ellipse or in the circumference of a circle*[50] *[two] straight lines are dropped as ordinates to the diameter, the squares on them will be to the [rectangular] planes under the straight lines cut off by them beginning from the [both] ends of the latus transversum of the* eidos *as the latus rectum of the* eidos *is to the latus transversum, and to each other as the planes under the straight lines cut off as we have said*[51].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ and whose *latus rectum* for the ordinates to the diameter is ΑΓ, and let the ordinates ΔΕ and ΖΗ be dropped to the diameter.

I say that as sq.ΖΗ is to pl.ΑΗΒ, so ΑΓ is to ΑΒ, and as sq.ΖΗ is to sq.ΔΕ, so pl.ΑΗΒ is to pl.ΑΕΒ.

[Proof]. For let ΒΓ determining the *eidos* be joined, and through Ε and Η let ΕΘ and ΗΚ be drawn parallel to ΑΓ. Therefore [according to Propositions I.12 and I.13] sq.ΖΗ is equal to pl.ΚΗΑ, and sq.ΔΕ is equal to pl.ΘΕΑ.

And since as KH is to HB, so ΓA is to AB, and with AH taken as common height as KH is to HB, so pl.KHA is to pl.BHA, therefore as ΓA is to AB, so pl.KHA is to pl.BHA, or as ΓA is to AB, so sq.ZH is to pl.BHA.

Then also for the same reasons as ΓA is to AB, so sq.ΔE is to pl.BEA.

And therefore as sq.ZH is to pl.BHA, so sq.ΔE is to pl.BEA, and alternately as sq.ZH is to sq.ΔE, so pl.BHA is to pl.BEA.

[Proposition] 22

*If a straight line cuts a parabola or a hyperbola at two points not meeting the diameter inside, it will, if continued, meet the diameter of the section outside the section*[52].

Let there be a parabola or a hyperbola whose diameter is AB, and let some straight line cut the section at two points Γ and Δ [and do not cut the diameter AB].

I say that ΔΓ, if continued, will meet AB outside the section.

[Proof]. For let ΓE and ΔB be dropped as ordinates from Γ and Δ, and first let the section be a parabola. Since then in the parabola [according to Proposition I.20] as sq.ΓE is to sq.ΔB, so EA is to AB and EA is greater than AB, therefore also sq.ΓE is greater than sq.ΔB.
And so also ΓE is greater than ΔB.

And they are parallel; therefore [according to Proposition I.10] ΓΔ continued will meet AB outside the section.

But then let it be a hyperbola [with the *latus transversum* AZ]. Since then in the hyperbola [according to Proposition I.21] as sq.ΓE is to sq.ΔB, so pl.ZEA is to pl.ZBA, therefore also sq.ΓE is greater than sq.ΔB.

And they are parallel; therefore ΓΔ continued will meet AB outside the section.

[Proposition] 23

*If a straight line situated between two diameters cuts the ellipse, it will, when continued, meet each of the diameters outside the section* [53].

Let there be an ellipse whose diameters are AB and ΓΔ, and let some straight line EZ is situated between the diameters AB and ΓΔ.

I say that EZ, when continued, will meet each of AB and ΓΔ outside the section.

[Proof]. For let HE and ZΘ be dropped as ordinates from E and Z to AB, and EK and ZΛ as ordinates to ΓΔ.Therefore [according to Proposition I.21] as sq.EH is to sq.ZΘ, so pl.BHA is to pl.BΘA, and as sq.ZΛ is to sq.EK, so pl.ΔΛΓ is to pl.ΔKΓ.

And pl.BHA is greater than pl.BΘA for [according to Proposition II.5 of Euclid] H is nearer to the midpoint of AB than Θ, and pl.ΔΛΓ is greater than pl.ΔKΓ [for Λ is nearer to the midpoint of ΓΔ than K].
Therefore also sq.HE is greater than sq.ZΘ, and sq.ZΛ is greater than sq.EK. Therefore also HE is greater than ZΘ, and ZΛ is greater than EK.

And HE is parallel to ZΘ, and ZΛ to EK, therefore [according to Proposition I.10 and Proposition I.33 of Euclid] EZ continued will meet each of the diameters AB and ΓΔ outside the section [54].

[Proposition] 24

*If a straight line meeting a parabola or a hyperbola at a point, when continued both ways falls outside the section, then it will meet the diameter* [55].

Let there be a parabola or a hyperbola whose diameter is AB, and let ΓΔE meet it at Δ, and when continued both ways, let it fall outside the section.

I say that it will meet the diameter AB.

[Proof]. For let some point Z be taken on the section, and let ΔZ be joined, therefore [according to Proposition I.22] ΔZ continued will meet the diameter of the section. Let it meet it at A, and ΓΔE is situated between the section and ZΔA. And therefore ΓΔE continued will meet the diameter outside the section.

[Proposition] 25

*If a straight line meeting an ellipse between two diameters and continued both ways falls outside the section, it will meet each of the diameters* [56].

Let there be an ellipse whose diameters are AB and ΓΔ, and let EZ, some straight line between two diameters, meet it at H, and continued both ways fall outside the section.

I say that EZ will meet each of AB and ΓΔ.

[Proof]. Let HΘ and HK be dropped as ordinates to AB and ΓΔ respectively. Since [according to Proposition I.15] HK is parallel to AB, and some straight line HZ has met HK, therefore it will also meet AB. Then likewise EZ will also meet ΓΔ.

[Proposition] 26

*If in a parabola or a hyperbola a straight line if drawn parallel to the diameter of the section, it will meet the section at one point only* [57].

Let there first be a parabola whose diameter is ΑΒΓ, and whose *latus rectum* is ΑΔ, and let ΕΖ be drawn parallel to ΑΒ.

I say that ΕΖ continued will meet the section [at one point only].

[Proof]. For let some point Ε be taken on ΕΖ, and from Ε let ΕΗ be drawn parallel to an ordinate, and let pl.ΔΑΓ is greater than sq.ΗΕ, and from Γ let [according to Proposition I.19] ΓΘ be erected as an ordinate.

Therefore [according to Proposition I.11] sq.ΘΓ is equal to pl.ΔΑΓ.

But pl.ΔΑΓ is greater than sq.ΕΗ, therefore sq.ΘΓ is greater than sq.ΕΗ, therefore ΘΓ is greater than ΕΗ. And they are parallel.

Therefore ΕΖ continued cuts ΘΓ, and so it will also meet the section.

Let it meet it at Κ. Then I say also that it will meet it at Κ only.

[Proof]. For, if possible, let it also meet it at Λ. Since then a straight line cuts a parabola at two points, if continued [according to Proposition I.22] it will meet the diameter of the section, and this is impossible for it is supposed parallel.

Therefore ΕΖ continued meets the section at only one point.

Next let the section be a hyperbola, and ΑΒ be the *latus transversum* of the *eidos*, and ΑΔ be the *latus rectum*, and let ΔΒ be joined and continued. Then with the same construction let ΓΜ be drawn from Γ parallel to ΑΔ. Since then pl.ΜΓΑ is greater than ΔΑΓ, sq.ΓΘ is equal to pl.ΜΓΑ, and pl.ΔΑΓ is greater than sq.ΗΕ, therefore also sq.ΓΘ is greater than sq.ΗΕ. And so also ΓΘ is greater than ΗΕ, and the same reasons as in the first case will come to pass.

[Proposition] 27

*If a straight line [within the section] cuts the diameter of a parabola, then continued both ways it will meet the section* [58].

Let there be a parabola whose diameter is ΑΒ, and let some straight line ΓΔ cut it within the section.

I say that ΓΔ continued both ways will meet the section.

[Proof]. For let some straight line ΑΕ be drawn from Α parallel to an ordinate, therefore [according to Proposition I.17] ΑΕ will fall outside the section.

Then either ΓΔ is parallel to ΑΕ or not.

If it is parallel to it, it has been dropped as an ordinate, so that continued both ways [according to Proposition I.18] it will meet the section.

Next let it not be parallel to ΑΕ, but continued let it meet ΑΕ at Ε.

Then it is evident that it meets the section in the side of E for if it meets AE, and a fortiori it cuts the section.

I say that if continued the other way, it also meets the section.

[Proof]. For let MA be the *latus rectum* for the ordinates to the diameter, and HZ be an ordinate, and let [according to Propositions VI.11 and VI.17 of Euclid] sq.AΔ is equal to pl.BAZ, and let BK parallel to an ordinate meet ΔΓ at Γ. Since pl.BAZ is equal to sq.AΔ, hence as AB is to AΔ, so AΔ is to AZ, and therefore [according to Proposition V.10 of Euclid] as BΔ is to ΔZ, so AB is to AΔ.Therefore also as sq.BΔ is to sq.ΔZ, so sq.AB is to sq.AΔ.

But since sq.AΔ is equal to pl.BAZ, hence as AB is to AZ, so sq.AB is to sq.AΔ, and sq.BΔ is to sq.ZΔ.

But as sq.BΔ is to sq.ΔZ, so sq.BΓ is to sq.ZA, and as AB is to AZ, so pl.BAM is to pl.ZAM.

Therefore as sq.BΓ is to sq.ZH, so pl.BAM is to pl.ZAM, and correspondingly as sq.BΓ is to pl.BAM, so sq.ZH is to pl.ZAM.
But because of the section [according to Proposition I.11] sq.ZH is equal to pl.ZAM. Therefore also sq.BΓ is equal to pl.BAM.

But AM is the *latus rectum*, and BΓ is parallel to an ordinate. Therefore [according to the Proposition I.11] the section passes through Γ, and ΓΔ meets the section at Γ.

[Proposition] 28

*If a straight line touches one of the opposite hyperbolas, and some point is taken within the other hyperbola, and through it a straight line is drawn parallel to the tangent, than continued both ways, it will meet the section* [59].

Let there be opposite hyperbolas whose diameter is AB, and let some straight line ΓΔ touch the hyperbola A, and let some point E be taken within the other hyperbola, and through E let EZ be parallel to ΓΔ.

I say that EZ continued both ways will meet the section.

[Proof]. Since then it has been proved [in Proposition I.24] that ΓΔ continued will meet the diameter AB, and EZ is parallel to it, therefore EZ continued will meet the diameter. Let it meet it at H, and let AΘ be made equal to HB, and through Θ let ΘK be drawn parallel to EZ, and let KΛ be dropped as an ordinate, and let HM be made equal to ΛΘ, and let MN be drawn parallel to an ordinate, and let HN be further continued in the same straight line. And since KΛ is parallel to MN, and KΘ to HN, and ΛM is one straight line [with the diameter AB] the triangle ΚΘΛ is similar to the triangle ΗΜΝ. And ΛΘ is equal to ΗΜ; therefore ΚΛ is equal to ΜΝ.and so also sq.ΚΛ is equal to sq.ΜΝ.

And since ΛΘ is equal to ΗΜ and ΑΘ is equal to ΒΗ, and ΑΒ is common, therefore ΒΛ is equal to ΑΜ, and therefore pl.ΒΛΑ is equal to pl.ΑΜΒ.

Therefore as pl.ΒΛΑ is to sq.ΛΚ, so pl.ΑΜΒ is to sq.ΜΝ.

And [according to Proposition I.21] as pl.ΒΛΑ is to sq.ΛΚ, so the *latus transversum* is to the *latus rectum*.

Therefore also as pl.ΑΜΒ is to sq.ΜΝ, so *latus transversum* is to the *latus rectum*.

Therefore Ν is on the section. Therefore [according to Proposition I.21] ΕΖ continued will meet the section at Ν.

Likewise then it could be shown that continued to the other side it will meet the section.

[Proposition] 29

*If in opposite hyperbolas a straight line is drawn through the center to meet either of the hyperbolas, then continued it will cut the other hyperbola* [60].

Let there be opposite hyperbolas whose transverse diameter is ΑΒ, and whose center is Γ, and let ΓΔ cut the hyperbola ΑΔ.

I say that it will also cut the other hyperbola.

[Proof]. For let ΕΔ be dropped as an ordinate, and let ΒΖ be made equal to ΑΕ, and let ΖΗ be drawn as an ordinate. And since ΕΑ is equal to ΒΖ, and ΑΒ is common, therefore pl.ΒΕΑ is equal to pl.ΒΖΑ.

And since [according to Proposition I.21] as pl.ΒΕΑ is to sq.ΔΕ, so the *latus transversum* is to the *latus rectum*, but also pl. ΒΖΑ is to sq.ΖΗ, so the *latus transversum* is to the *latus rectum*, therefore also [according to Proposition I.14] as pl.ΒΕΑ is to sq.ΔΕ, so pl.ΒΖΑ is to sq.ΖΗ.

But pl.ΒΕΑ is equal to pl.ΒΖΑ; therefore sq.ΔΕ is equal to sq.ΖΗ.

Since then ΕΓ is equal to ΓΖ and ΔΕ is equal to ΖΗ, and ΕΖ is a straight line, and ΕΔ is parallel to ΖΗ, therefore [according to Proposition VI.32 of Euclid] ΔΗ is also a straight line. And therefore [continued] ΓΔ will also cut the other hyperbola.

[Proposition] 30

*If in an ellipse or in opposite hyperbolas a straight line is drawn in both directions from the center, meeting the section, it will be bisected at the center*[61].

Let there be an ellipse or opposite hyperbolas, and their diameter ΑΒ, and their center Γ, and through Γ let some straight line ΔΓΕ be drawn.

I say that ΓΔ is equal to ΓΕ.

[Proof]. For let ΔΖ and ΕΗ be drawn as ordinates. And since [according to Proposition I.21] as pl.ΒΖΑ is to sq.ΖΔ, so the *latus transversum* is to the *latus rectum*, but also as pl.ΑΗΒ is to sq.ΗΕ, so the *latus transversum* is to the *latus rectum*, therefore also [according to Proposition V.11 of Euclid] as pl.ΒΖΑ is to sq.ΖΔ, so pl.ΑΗΒ is to sq.ΗΕ.

And alternately as pl.ΒΖΑ is to pl.ΑΗΒ, so sq.ΖΔ is to sq.ΗΕ.

But [according to Propositions V.16, VI.4 and VI.22 of Euclid] as sq.ΖΔ is to sq.ΗΕ, so sq.ΖΓ is to sq.ΓΗ, therefore alternately as pl.ΒΖΑ is to sq.ΖΓ, so pl.ΑΗΒ is to sq.ΓΗ.

Therefore also [according to Propositions II 5 and II.6 of Euclid] *componendo* in the case of the ellipse and inversely and *convertendo* [62] in the case of the opposite hyperbolas, as sq.ΑΓ is to sq.ΓΖ, so sq.ΒΓ is to sq.ΓΗ, and alternately [as sq.ΑΓ is to sq.ΒΓ, so sq.ΓΖ is to sq.ΓΗ]. But sq.ΓΒ is equal to sq.ΑΓ, therefore also sq.ΓΗ is equal to sq.ΓΖ, therefore ΓΗ is equal to ΓΖ.

And ΔΖ and ΗΕ are parallel; therefore also ΔΓ is equal to ΓΕ.

[Proposition] 31

*If on the latus transversum of the* eidos *of a hyperbola some point be taken cutting off from the vertex of the section not less than half of the latus transversum of the* eidos*, and a straight line be drawn from it to meet to section, then when further continued it will fall within the section on the near side of the section* [63].

Let there be a hyperbola whose diameter is ΑΒ, and let some point Γ on the diameter be taken cutting off ΓΒ not less than half of ΑΒ, and let some straight line ΓΔ be drawn to meet the section.

I say that ΓΔ continued will fall within the section.

[Proof]. For, if possible, let it fall outside the section as ΓΔΕ, and from Ε, a point at random, let ΕΗ be dropped as an ordinate, also ΔΘ [let be dropped as an ordinate]; and first let ΑΒ be equal to ΓΒ.

And since [according to Propositions V.8 and VI.22 of Euclid] the ratio sq.ΕΗ to sq.ΔΘ is greater than the ratio sq.ΖΗ to sq.ΔΘ, but as sq.ΕΗ is to sq.ΔΘ, so sq.ΓΗ is to sq.ΓΘ because ΕΗ is parallel to ΔΘ, and as sq.ΖΗ is to sq.ΔΘ, so pl.ΑΗΒ is to pl.ΑΘΒ because for the section [according to Proposition I.21], therefore the ratio sq.ΓΗ to sq.ΓΘ is greater than the ratio pl.ΑΗΒ to pl.AΘB. Therefore alternately the ratio sq.ΓΘ to pl.AHB is greater than the ratio sq.ΓΘ to pl.AΘB.

Therefore *separando* [according to Propositions II.6 and V.17 of Euclid] the ratio sq.ΓB to pl.AHB is greater than the ratio sq.ΓB to pl.AΘB, and this is impossible [according to Proposition V.8 of Euclid]. Therefore ΓΔE will not fall outside the section, and it falls inside.

And for this reason the straight line from some of the points on AΓ will a fortiori fall inside, since it will also fall inside ΓΔ.

[Proposition] 32

*If a straight line is drawn through the vertex of a section of a cone parallel to an ordinate, then it touches the section, and another straight line will not fall into the space between the conic section and this straight line* [64].

Let there be a section of a cone, first the so-called parabola whose diameter is AB [and whose vertex is A], and from A let AΓ be drawn parallel to an ordinate.

Now [in the Proposition I.17] it has been shown that it falls outside the section.

Then I say that also another straight line will not fall into the space between AΓ and the section.

[Proof]. For, if possible, let it fall inside as AΔ, and let some point Δ be taken on it at random, and let ΔE be dropped as the ordinate, and let AZ be the *latus rectum* for the ordinates to AB. And since [according to Propositions V.8 and VI.22 of Euclid] the ratio sq.ΔE to sq.EA is greater than the ratio sq.HE to sq.EA, and [according to Proposition I.11] sq.HE is equal to pl.ZAE, therefore also the ratio sq.ΔE to sq.EA is greater than the ratio pl.ZAE to sq.EA, or is greater than the ratio ZA to EA.

Let then it be contrived that as sq.ΔE is to sq.EA, so ZA is to ΘA, and through Θ let ΘΛK be drawn parallel to EΔ.

Since then as sq.ΔE is to sq.EA, so ZA is to AΘ, and pl.ZAΘ is to sq.AΘ and [according to Propositions VI.4 and Vi.22 of Euclid] as sq.ΔE is to sq.EA, so sq.KΘ is to sq.ΘA, and [according to Proposition I.11] sq. ΘΛ is equal to pl.ZAΘ, therefore also as sq.KΘ is to sq.ΘA, so sq.ΛΘ is to sq.ΘA.

Therefore KΘ is equal to ΘΛ, and this is impossible. Therefore another straight line will not fall into the space between AΓ and the section.

Next let the section be a hyperbola or an ellipse or the circumference of a circle whose diameter is AB, and whose *latus rectum* is AZ, and let BZ be joined and continued, and from A let AΓ be drawn parallel to an ordinate.

Now [in Proposition I.17] it has been shown that it falls outside the section.

Then I say that also another straight line will not fall into the space between AΓ and the section.

[Proof], For, if possible, let it fall inside as AΔ, and let some point Δ be taken on it at random, and let ΔE be dropped as an ordinate, and let EM be drawn parallel to AZ.

And since [according to Propositions I.12 and I.13] sq.HE is equal to pl.AEM, let it be contrived that pl.AEN is equal to sq.ΔE, and let AN cut ZM at Ξ, and through Ξ let ΞΘ be drawn parallel to ZA, and through Θ λετ ΘΛK parallel to AΓ. Since then sq.ΔE is equal to pl.AEN, hence as NE is to EΔ, so ΔE is to EA, and therefore [according to Propositions V.11 and VI.22 and the porism to Proposition VI.19 of Euclid] as NE is to EA, so sq.ΔE is to sq.EA.

But as NE is to EA, so ΞΘ is to ΘA, and as sq.ΔE is to sq.EA, so sq.KΘ is to sq.ΘA. Therefore as ΞΘ is to ΘA, so sq.KΘ is to sq.ΘA, therefore [according to the porism to Proposition VI.19 of Euclid] as ΞΘ is to ΘK, so KΘ is to ΘA.

Therefore sq.KΘ is equal to pl.AΘΞ, but also because for the section [according to Propositions I.12 and I.13] sq.ΛΘ is equal to pl.AΘΞ, therefore sq.KΘ is equal to sq.ΘΛ, and this is impossible. Therefore another straight line will not fall into the space between AΓ and the section.

## [Proposition] 33

*If on a parabola some point is taken, and from it an ordinate is drawn to the diameter, and to the straight line cut off by it on the diameter from the vertex a straight line in the same straight line from its extremity is made equal, then the straight line joined from the point thus resulting to the point taken will touch the section* [65].

Let there be a parabola whose diameter is AB, [and whose vertex is E], and let ΓΔ be dropped as an ordinate, and let AE be made equal to EΔ, and let AΓ be joined.

I say that AΓ continued will fall outside the section.

[Proof]. For, if possible, let it fall within as ΓZ, and let HB be dropped as an ordinate. And since the ratio sq.BH to sq.ΓΔ is greater than sq.ZB to sq.ΓΔ, but as sq.ZB is to sq.ΓΔ, so sq.BA is to sq. AΔ, and [according to Proposition

I.20] as sq.BH is to sq.ΓΔ, so BE is to ΔE, therefore the ratio BE to ΔE is greater than sq.BA to sq.AΔ.

But as BE is to ΔE, so quadruple pl.BEA is to quadruple pl.ΔEA, therefore also the ratio quadruple pl.BEA to quadruple pl.ΔEA is greater than sq.AB to sq.AΔ.

Therefore, alternately the ratio quadruple pl.BEA to sq.AB is greater than the ratio quadruple pl.ΔEA to sq.AΔ, and this is impossible for since AE is equal to ΔE, hence quadruple pl.BEA is less than sq.AB for [according to Proposition II.5 of Euclid], E is not the midpoint of AB.Therefore tAΓ does not fall within the section, therefore it touches it.

[Proposition] 34

*If on a hyperbola or an ellipse or the circumference of a circle some point is taken, and if from it a straight line is dropped as an ordinate to the diameter, and if the straight lines which the ordinate cuts off from the ends of the latus transversum of the* eidos *have to each other a ratio which other segments of the latus transversum have to each other, so that the segments from the vertex are homologous* [66]*, then the straight line joining the point taken on the latus transversum and that taken on the section will touch the section* [67].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is AB, and let some point Γ be taken on the section, and from Γ let ΓΔ be drawn as an ordinate, and let it be contrived that as BΔ is to ΔA, so BE is to EA, and let EΓ be joined.

I say that ΓE touches the section.

[Proof]. For, if possible, let it cut it, as EΓZ, and let some point Z be taken on it, and let HZΘ be dropped as an ordinate, and let AΛ and BK be drawn through A and B parallel to EΓ, and let ΔΓ, BΓ, and HΓ be joined and continued to K, Ξ, and M. And since as BΔ it to ΔA, so BE is to EA, but [according to Proposition VI.4 of Euclid] as BΔ is to ΔA, so BK is to AN, and as BE is to AE, so BΓ is to ΓK, and BK is to ΞN, therefore as BK is to AN, so BK is to ΞN, therefore AN is equal to NΞ.

Therefore [according to Propositions II.5 and VI.27 of Euclid] pl.ANΞ is greater than pl.AOΞ.

Therefore the ratio NΞ to ΞO is greater than the ratio OA to AN.

But [according to Proposition VI.4 of Euclid] as NΞ to ΞO, so KB is to BM, therefore the ratio KB to BM is greater than the ratio OA to AN.

Therefore pl.KB, AN is greater than pl.BM,OA.

And so [according to Proposition V.8 of Euclid] the ratio pl.KB,AN to sq.ΓE is greater than the ratio pl.BM,OA to sq.ΓE.
But as pl.KB,AN is to sq.ΓE, so pl.BΔA is to sq.ΔE because the triangles BKΔ, EΓΔ, and NAΔ are similar, and as pl.BM,OA is to sq.ΓE, so pl.BHA is to sq.HE, therefore the ratio pl.BΔA to sq.ΔE is greater than the ratio pl.BHA to sq.HE, therefore alternately the ratio BΔA to pl.BHA is greater than the ratio sq.ΔE to sq.HE.

But [according to Proposition I.21] as pl.BΔA is to pl.AHB, so sq.ΓΔ is to sq.HΘ and [according to Propositions VI.4 and VI.22 of Euclid] as sq.ΔE is to sq.EA, so sq.ΓΔ is to sq.ZH, therefore also the ratio sq.ΓΔ to sq.ΘH is greater than the ratio sq.ΓΔ to sq.ZH.

Therefore [according to Proposition V.10 of Euclid] ΘH is less than ZH, and this is impossible. Therefore EΓ does not cut the section. Therefore, it touches it [68-69].

[Proposition] 35

*If a straight line touching a parabola, meets the diameter outside the section, the straight line drawn from the point of contact as an ordinate to the diameter will cut off on the diameter beginning from the vertex of the section a straight line equal to the straight line between the vertex and the [diameter's intersection with the] tangent, and not straight line will fall into the space between the tangent and the section* [70].

Let there be a parabola whose diameter is AB, [whose vertex is H], and let BΓ be erected as an ordinate, and let AΓ be tangent to the section.

I say that AH is equal to HB.

[Proof]. For, if possible, let it be unequal to it, and let HE be made equal to AH, and let EZ be upright as an ordinate, and let AZ be joined. Therefore [according to Proposition I.33] AZ continued will meet AΓ, and this is impossible for two straight lines will have the same ends. Therefore AH is not unequal to HB; therefore it is equal to it.

Then I say that no straight line will fall into the space between AΓ and the section.

[Proof]. For, if possible, let ΓΔ fall between, and let HE be made equal to HΔ, and let EZ be erected as an ordinate. Therefore [according to Proposition I.33] the straight line joined from Δ to Z touches the section, therefore continued it will fall outside it. And so it will meet ΔΓ, and two straight lines will have the same ends, and this is impossible. Therefore a straight line will not fall into the space between the section and ΑΓ.

[Proposition] 36

*If some straight line meeting the latus transversum of the* eidos *touches a hyperbola or an ellipse or the circumference of a circle, and if a straight line dropped from the point of contact as an ordinate to the diameter, then as the straight line cut off by the tangent from the end of the latus transversum is to the straight line cut off by the tangent from the other end of the latus transversum, so the straight line will cut off by the ordinate from the end of the latus transversum be to the straight line cut off by the ordinate from the other end of the latus transversum in such a way that the homologous straight lines are in continuous correspondence, and another straight line will not fall into the space between the tangent and the section of the cone* [71].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ, and let ΓΔ be tangent, and let ΓΕ be dropped as an ordinate.

I say that as BE is to ΕΑ, so ΒΔ is to ΔΑ.

[Proof]. For if it is not, let it be as ΒΔ is to ΔΑ, so ΒΗ is to ΗΑ, and let ΗΖ be erected as an ordinate, therefore the straight line joined from Δ to Ζ [according to Proposition I.34] will touch the section, therefore continued it will meet ΓΔ. Therefore two straight lines will have the same ends, and this is impossible.

I say that no straight line will fall between the section and ΓΔ.

[Proof]. For, if possible, let it fall between, as ΓΘ, and let it be contrived that as ΒΘ is to ΘΑ, so ΒΑ to ΗΑ, and let ΗΖ be erected as an ordinate, therefore the straight line joined from Θ to Ζ, when continued [according to Proposition I.34] will meet ΘΓ. Therefore two straight lines will have the same ends, and this is impossible. Therefore a straight line will not fall into the space between the section and ΓΔ.

[Proposition] 37

If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the diameter, and from the point of contact to the diameter a straight line is dropped as an ordinate, then the straight line cut off by the ordinate from the center of the section with the straight line cut off by the tangent from the center of the section will contain an area equal to the square on the radius of the section, and with the straight line between the ordinate and the tangent will contain an area having the ratio to the square on the ordinate which the *latus transversum* has to the *latus rectum* [72].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is AB and let ΓΔ be drawn tangent, and let ΓE be dropped as an ordinate, and let Z be the center.

I say that pl.ΔZE is equal to sq.ZB, and as pl.ΔEZ is to sq.EΓ. so the *latus transversum* is to the *latus rectum*.

[Proof]. For since ΓΔ touches the section, and ΓE has been dropped as an ordinate, hence [according to Proposition I.36] as AΔ is to ΔB, so AE is to EB. Therefore *componendo* as the sum of AΔ and ΔB is to ΔB, so the sum of AE and EB is to EB.

And [according to Proposition V.15 of Euclid] let the halves of the antecedents be taken. In the case of the hyperbola we shall say: but half of the sum of AE and EB is equal to ZE, and half of AB is equal to ZB, therefore as ZE is to EB, so ZB is to BΔ. Therefore *convertendo* as ZE is to ZB, so ZB is to ZΔ, therefore pl.EZΔ is equal to sq.ZB.

And since as ZE is to EB, so EB is to BΔ, and AZ is to BΔ, and alternately as AZ is to ZE, so ΔB is to BE, and *componendo* as AE is to EZ, so ΔE is to EB and so, pl.AEB is equal to pl.ZEΔ.

But [according to Proposition I.21] as pl.AEB is to sq.ΓE, so the *latus transversum* is to the *latus rectum*, therefore also pl.ZEΔ is to sq.ΓE, so the *latus transversum* is to the *latus rectum*.

And in the case of the ellipse and of the circle we shall say: but half of the sum of AD and ΔB is equal to ΔZ and half of AB is equal to ZB, therefore as ZΔ is to ΔB, so ZB is to BE. Therefore convertendo as ΔZ is to ZB, so BZ is to ZE. Therefore pl.ΔZE is equal to sq.BZ.

But [according to Proposition II.3 of Euclid] pl.ΔZE is equal to the sum of pl.ΔEZ and sq.ZE and [according to Proposition II.5 of Euclid] sq.BZ is equal to the sum pl.AEB and sq.ZE.

Let the common sq.EZ be subtracted, therefore pl.ΔEZ is equal to pl.AEB. Therefore as pl.ΔEZ is to sq.ΓE, so pl.AEB is to sq.ΓE.

But [according to Proposition I.21] as pl.AEB is to sq.ΓE, so the *latus transversum* is to the *latus rectum*. Therefore as pl.ΔEZ is to sq.ΓE, so the *latus transversum* is to the *latus rectum* [73-80].

[Proposition] 38

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the second diameter and if from the point of contact a straight line is dropped to the same diameter parallel to the other diameter then the straight line cut off from the center of the section by the dropped straight line, together with the straight line cut off [on the second diameter] by the tangent from the center of the section will contain an area equal to the square on the half of the second diameter and together with the straight line [on the second diameter] between the dropped straight line and the tangent will contain an area having a ratio to the square on the dropped straight line which the latus rectum of the* eidos *has to the latus transversum* [81].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is AHB, and whose second diameter is ΓΗΔ, and let ΕΛΖ meeting ΓΔ at Z be a tangent to the section, and let the ΘΕ be parallel to AB.

I say that pl.ΖΗΘ is equal to sq.ΗΓ and as pl.ΗΘΖ is to sq.ΘΕ, so the *latus rectum* is to the *latus transversum*.

[Proof]. Let ME be drawn as an ordinate, therefore [according to Proposition I.37] as pl.ΗΜΛ is to sq.ΜΕ, so the *latus transversum* is to the *latus rectum*.

But [according to Definition 11] as the *latus transversum* BA is to ΓΔ, σο ΓΔ is to the *latus rectum* and therefore [according to the porism to Proposition VI.19 of Euclid] as the *latus transversum* is to the *latus rectum*, so sq. BA is to sq.ΓΔ, and as the quarters of them, that is as the *latus transversum* is to the *latus rectum*, so sq.ΗΑ, is to sq.ΗΓ, therefore also as pl.ΗΜΛ is to sq.ΜΕ, so sq.ΗΑ is to sq.ΗΓ.

But the ratio pl.ΗΜΛ to sq.ΜΕ is compounded of [the ratios] ΗΜ to ΜΕ and ΛΜ to ΜΕ or the ratio pl.ΗΜΛ to sq.ΜΕ is compounded of [the ratios] ΗΜ to ΗΘ and ΛΜ to ΜΕ. Therefore inversely as sq.ΓΗ is to sq.ΗΑ, so ΕΜ is to ΜΗ or the ratio compounded of [the ratios] ΘΗ to ΗΜ and ΕΜ to ΜΛ or the ratio ΖΗ to ΗΛ.

Therefore, the ratio sq.ΗΓ to sq.ΗΑ is compounded of [the ratios] ΘΗ to ΗΜ and ΖΓ to ΗΛ which is the same as the ratio pl.ΖΗΘ to pl.ΜΗΛ. Therefore as pl.ΖΗΘ is to pl.ΜΗΛ, so sq.ΓΗ is to sq.ΗΑ. And alternately [as pl.ΖΗΘ is to sq.ΓΗ, so pl.ΜΗΛ is to sq.ΗΑ.].

But [according to Proposition I.37] pl.ΜΗΛ is equal to sq.ΗΑ, therefore also pl.ΖΗΘ is equal to sq.ΓΗ.

Again since [according to Proposition I.37] as the *latus rectum* is to the *latus transversum*, so sq.ΕΜ is to pl.ΗΜΛ, and the ratio sq.ΕΜ to pl.ΗΜΛ is compounded of [the ratios] ΕΜ to ΗΜ and ΕΜ to ΜΛ, or the ratio sq.ΕΜ to pl.ΗΜΛ is compounded of [the ratios] ΘΗ to ΘΕ and ΖΗ to ΗΛ or ΖΘ to ΘΕ, ωηιχη is the same as pl.ΖΘΗ to sq.ΘΕ. Therefore as pl.ΖΘΗ is to sq.ΘΕ, so the *latus rectum* is to the *latus transversum*.

[Porism] 1

Under the same suppositions [on the hyperbola] we shall prove that as each straight line situated [on the second diameter] between the tangent and the end of the [second] diameter from the ordinate is to the straight line situated between the tangent and the other end of the [second] diameter, so the straight line situated between the other end of the [second] diameter and the ordinate to the straight line situated between the first end and the ordinate [82].

Since pl.ΖΗΘ is equal to sq.ΗΓ, that is pl.ΓΗΔ because ΓΗ is equal to ΗΔ, pl.ΖΗΘ is equal to pl.ΓΗΔ. Therefore as ΗΓ is to ΗΘ, so ΖΗ is to ΗΔ, and separando and convertendo as ΗΓ is to ΓΘ ,so ΗΖ is to ΖΔ. If the antecedents are doubled and separando we obtain that as ΔΘ is to ΓΘ, so ΓΖ is to ΖΔ, what was to prove [83].

[Porism] 2

From the said it is evident that the straight line ΕΖ is tangent to the section because pl.ΖΗΘ is equal to sq.ΗΓ. Hence we can prove that as pl.ΗΘΖ is to sq.ΘΕ , so the ratio [of the *latus rectum* to the *latus transversum*] that was proved [in Proposition I.38].

[Proposition] 39

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the diameter and if from the point of contact a straight line is dropped as an ordinate to the diameter, then whichever of the two straight lines is taken, of which one is the straight line between the [intersection of the] ordinate [with the diameter] and the center of the section, and the other is between [the intersection of] the ordinate and the tangent [with the diameter] the ordinate will have to it the ratio compounded of the ratio of the other of the two straight lines to the ordinate and of the ratio of the latus rectum of the* eidos *to the latus transversum*[84].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ, and let the center of it be Ζ, and let ΓΔ be drawn tangent to the section, and ΓΕ be dropped as an ordinate.

I say that the ratio ΓΕ to ΖΕ is compounded of [the ratios] the latus rectum to the *latus transversum* and EΔ to EΓ and the ratio ΓE to EΔ is compounded of [the ratios] the *latus rectum* to the *latus transversum* and ZE to EΓ.

[Proof]. For let pl.ZEΔ is equal to pl.EΓ,H and since [according to Proposition I.37] as pl.ZEΔ is to sq.ΓE, so the *latus transversum* is to the *latus rectum* and pl.ZEΔ is equal to pl.ΓE,H, therefore as pl.ΓE,H is to sq.ΓE, so H is to ΓE and the *latus transversum* is to the *latus rectum*.

And since pl.ZEΔ is equal to pl.ΓE,H, hence as ZE is to EΓ, so H is to EΔ. And since the ratio ΓE to EΔ is compounded of [the ratios] ΓE to H and H to EΔ, but as ΓE is to H, so the *latus rectum* is to the *latus transversum*, therefore the ratio ΓE to EΔ is compounded of [the ratios] the *latus rectum* to the *latus transversum* and ZE to EΓ.

[Proposition] 40

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the second diameter, and if from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then whichever of two straight lines is taken [along the second diameter], of which one is the straight line between the dropped straight line and the center of the section, and the other is between the dropped straight line and the tangent, then the dropped straight line will have to one of two straight lines the ratio compounded of the ratio of the latus transversum to the latus rectum and of the ratio of the other of two straight lines to the dropped straight line*[85].

Let there be a hyperbola or an ellipse or the circumference of a circle AB, and its diameter BZΓ, and its second diameter ΔZE, and let ΘΛA be drawn tangent, and AH be drawn parallel to BΓ.

I say that the ratio AH to one of ZH,ΘH is compounded of the ratio the *latus transversum* to the *latus rectum* and the ratio the other of ZH, HΘ to HA

[Proof] . Let pl.HA,K is equal to pl.ΘH,HZ. And since [according to Proposition I.38] as the *latus rectum* is to the *latus transversum*, so pl.ΘH,HZ is to sq.HA and pl.HA,K is equal to pl.ΘH,HZ, therefore also as pl.HA,K is to sq.HA, so K is to AH and the *latus rectum* is to the *latus transversum*.

And since the ratio AH to HZ is compounded of [the ratios] AH to K and K to HZ , but as AH is to K, so the *latus transversum* is to the *latus rectum*, and as K is to HZ, so ΘH is to HA because pl.ΘHZ is equal to pl.AH,K, therefore the ratio AH to HZ is compounded of [the ratios] the *latus transversum* to the *latus rectum* and HΘ to HA.

[Proposition] 41

*If in a hyperbola or an ellipse or the circumference of a circle a straight line is dropped as an ordinate to the diameter, and if equiangular parallelogrammic figures are described both on the ordinate and on the radius, and if the ordinate side has to the remaining side of the figure the ratio compounded of the ratio of the radius to the remaining side of its figure, and of the ratio of the latus rectum of the* eidos *of the section to the latus transversum, then the figure on the straight line between the center and the ordinate, similar to the figure on the radius, is in the case of the hyperbola greater than the figure on the ordinate by the figure on the radius, and in the case of the ellipse and the circumference of a circle together with the figure on the ordinate is equal to the figure on the radius* [86].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ, and center Ε, and let ΓΔ be dropped as an ordinate, and on ΕΑ and ΓΔ let the equiangular figures ΑΖ and ΔΗ be described, and let the ratio ΓΔ to ΓΗ is compounded of [the ratios] ΑΕ to ΕΖ and the *latus rectum* to the *latus transversum*.

I say that with the figure on ΕΔ similar to [the plane]ΑΖ in the case on the hyperbola the figure on ΕΔ is equal to the sum of [the planes] ΑΖ and ΗΔ, and in the case of the ellipse and the circle the sum of the figure on ΕΔ and [the plane] ΗΔ is equal to [the plane] ΑΖ.

[Proof]. For let it be contrived that as the *latus rectum* is to the *latus transversum*, so ΔΓ is to ΓΘ.

And since as ΔΓ is to ΓΘ , so the *latus rectum* is to the *latus transversum* , but as ΔΓ is to ΓΘ , so sq.ΔΓ is to pl.ΔΓΘ , and [according to Proposition I.21] as the *latus rectum* is to the *latus transversum*, so sq.ΔΓ is to pl.ΒΔΑ, therefore pl.ΒΔΑ is equal to pl.ΔΓΘ .

And since the ratio ΔΓ to ΓΗ is compounded of [the ratios]ΑΕ to ΕΖ and the *latus rectum* to the *latus transversum* , or the ratio ΔΓ to ΓΗ is compounded of [the ratios] ΑΕ to ΕΖ and ΔΓ to ΓΘ, and further the ratio ΔΓ to ΓΗ is compounded of [the ratios] ΔΓ to ΓΘ and ΓΘ to ΓΗ, therefore the ratio compounded of[the ratios] ΑΕ to ΕΖ and ΔΓ to ΓΘ is the same, as the ratio compounded of [the ratios] ΔΓ to ΓΘ and ΓΘ to ΓΗ.

Let the common ratio ΔΓ to ΓΘ be taken away, therefore as ΑΕ is to ΕΖ, so ΓΘ is to ΓΗ.

But as ΘΓ is to ΓΗ, so pl.ΘΓΔ is to pl.ΗΓΔ ,and as ΑΕ is to ΕΖ, so sq.ΑΕ is to pl.ΑΕΖ, therefore as pl.ΘΓΔ is to pl.ΗΓΔ , so sq.ΑΕ is to pl.ΑΕΖ.

And it has been shown that pl.ΘΓΔ is equal to pl.ΒΔΑ, therefore as pl.ΒΔΑ is to pl.ΗΓΔ, so sq.ΑΕ is to pl.ΑΕΖ, and alternately as pl.ΒΔΑ is to sq.ΑΕ, so pl.ΗΓΔ is to pl.ΑΕΖ.

And as pl.ΗΓΔ is to pl.ΑΕΖ, so [the plane] ΔΗ is to [the plane] ΖΑ for they are equiangular and [according to Proposition VI.23 of Euclid] have to one another the ratio compounded of their sides, ΗΓ to ΑΕ and ΓΔ to ΕΖ, and therefore as pl.ΒΔΑ is to sq.ΕΑ, so [the plane] ΔΗ is to [the plane] ΖΑ.

Moreover in the case of the hyperbola we are to say : *componendo* as the sum of pl.ΒΔΑ and sq.ΑΕ is to sq.ΑΕ, so the sum of [the planes] ΗΔ and ΑΖ is to [the plane] ΑΖ or [according to Proposition II.6 of Euclid] as sq.ΔΕ is to sq.ΕΑ, so the sum of [the planes] ΗΔ and ΑΖ is to [the plane] ΑΖ. And as sq.ΔΕ is to sq.ΕΑ, so [according to the porism to Proposition VI,29 of Euclid] the figure described on ΕΔ is similar and similarly situated to [the plane] ΑΖ, to [the plane] ΑΖ, therefore with the figure on ΕΔ similar to [the plane] ΑΖ, as the sum of [the planes] ΗΔ and ΑΖ is to [the plane] ΑΖ, so the figure on ΕΔ is to [the plane] ΑΖ. Therefore the figure on ΕΔ is equal to the sum of [the planes] ΗΔ and ΑΖ, the figure on ΕΔ being similar to [the plane] ΑΖ. And in the case of the ellipse and of the circumference of a circle we shall say : since then [according to Proposition V.19 of Euclid] as whole sq.ΑΕ is to whole [the plane] ΑΖ, so pl.ΑΔΒ subtracted is to [the plane] ΔΗ subtracted, also remainder is to remainder as whole to whole.

And [according to Proposition II.5 of Euclid] sq.ΑΕ without pl.ΒΔΑ is equal to sq.ΔΕ, therefore as sq.ΔΕ is to [the plane] ΑΖ without [the plane] ΔΗ, so sq.ΑΕ is to [the plane] ΑΖ. But [according to the porism to Proposition VI,20 of Euclid] as sq.ΑΕ is to [the plane] ΑΖ, so sq.ΔΕ is to the figure on ΔΕ, the figure on ΔΕ being similar to [the plane] ΑΖ. Therefore as sq.ΔΕ is to [the plane] ΑΖ without [the plane] ΔΗ, so sq.ΔΕ is to the figure on the Ε. Therefore the figure on ΔΕ being similar to [the plane] ΑΖ, the figure on ΔΕ is equal to [the plane] ΑΖ without [the plane] ΔΗ.

Therefore the sum of the figure on ΔΕ and [the plane] ΔΗ is equal to [the plane] ΑΖ.

[Proposition] 42

*If a straight line touching a parabola meets the diameter, and if from the point of contact a straight line is dropped as an ordinate to the diameter, and if some point is taken on the section, two straight lines are dropped to the diameter, one of them parallel to the tangent, and the other parallel to the straight line dropped from the point of contact, then the triangle resulting from them* *[that is from the diameter and the two straight lines dropped from the point at random] is equal to the parallelogram under the straight line dropped of the point of contact and the straight line cut off by the parallel from the vertex of the section* [87] .

Let there be a parabola whose diameter is ΑΒ, and let ΑΓ be drawn tangent to the section, and let ΓΘ be dropped as an ordinate and from some point at random let ΔΖ be dropped as an ordinate and through Δ let ΔΕ be drawn parallel to ΑΓ, and through Γ let ΓΗ be drawn parallel to ΒΖ and through B let ΒΗ be drawn parallel to ΘΓ.

I say that the triangle ΔΕΖ is equal to the parallelogram ΗΖ.

[Proof]. For, since ΑΓ touches the section, and ΓΘ has been dropped as an ordinate [according to Proposition I.35] ΑΒ is equal to ΒΘ, therefore ΑΘ is equal to double ΒΘ. Therefore [according to Proposition I.41 of Euclid] the triangle ΑΘΓ is equal to the parallelogram ΒΓ.

And since as sq.ΓΘ is to sq.ΔΖ, so ΘΒ is to ΒΖ because of the section [according to Proposition I.20], but [according to the porism to Proposition VI.20 of Euclid] as sq.ΓΘ is to sq.ΔΖ, so the triangle ΑΓΘ is to the triangle ΕΔΖ and [according to Proposition VI.1 of Euclid] as ΘΒ is to ΒΖ, so the parallelogram ΗΘ is to the parallelogram ΗΖ, therefore the triangle ΑΓΘ is to the triangle ΕΔΖ, so the parallelogram ΘΗ is to the parallelogram ΖΗ.

Therefore alternately as the triangle ΑΘΓ is to the parallelogram ΒΓ, so the triangle ΕΔΖ is to the parallelogram ΗΖ.

But the triangle ΑΓΘ is equal to the parallelogram ΗΘ ,therefore the triangle ΕΔΖ is equal to the parallelogram ΗΖ.

## [Proposition] 43

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the diameter, and if from the point of contact a straight line is dropped as an ordinate to the diameter, and if through the vertex a parallel [to an ordinate] is drawn meeting the straight line drawn through the point of contact and the center, and if some point [at random] is taken on the section, two straight lines are drawn to the diameter, one of which is parallel to the tangent and the other parallel to straight line dropped [as an ordinate] from the point of contact, then in the case of the hyperbola the triangle resulting from them that is the diameter and two lines drawn through the point taken at random to the diameter] will be less than the triangle cut off by the straight line through the center to the point of contact [by the ordinate through the point at random] by the triangle on the radius similar to the triangle cut off, and in the case of the* *ellipse and the circumference of a circle [the triangle resulting from the diameter and two lines through the point taken at random to the diameter] together with the triangle cut off [by the line] from the center [to the point of contact and by the ordinate through the point at random] will be equal to the triangle on the radius similar to the triangle cut off* [88].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ, and center Γ, and let ΔΕ be drawn tangent to the section, and let ΓΕ be joined, and let ΕΖ be dropped as an ordinate, and let some point Η be taken on the section, and let ΗΘ be drawn parallel to the tangent, and let ΗΚ be dropped as an ordinate [and continued to meet ΓΕ at Μ], and through Β let ΒΛ be erected as an ordinate.

I say that the triangle ΚΜΓ differs from the triangle ΓΛΒ by the triangle ΗΚΘ .

[Proof]. For since ΕΔ touches and ΕΖ has been dropped, hence [according to Proposition I.39] the ratio ΕΖ to ΖΔ is compounded of [the ratios] ΓΖ to ΖΕ and the *latus rectum* to the *latus transversum*.

But as ΕΖ to ΖΔ, so ΗΚ is to ΚΘ, and [according to Proposition VI.4 of Euclid] as ΓΖ is to ΖΕ, so ΓΒ is to ΒΛ, therefore the ratio ΗΚ to ΚΘ is compounded of [the ratios] ΒΓ to ΒΛ and the *latus rectum* to the *latus transversum*.

And through those reasons it has been shown in the theorem 41[that is Proposition I.41] the triangle ΓΚΜ differs from the triangle ΒΓΛ by the triangle ΗΘΚ for the same reasons have also been shown in the case of the parallelograms, their doubles.

[Proposition] 44

*If a straight line touching one of the opposite hyperbolas meets the diameter, and if from the point of contact some straight line is dropped as an ordinate to the diameter, and if a parallel to it is drawn through the vertex of the other hyperbola meeting the straight line drawn through the point of contact and the center, and if some point is taken at random on the section and [from it] two straight lines are dropped to the diameter, one of which is parallel to the tangent and the other parallel to the straight line dropped as an ordinate from the point of contact, then the triangle resulting from them will be less than the triangle cut off by the dropped straight line from the center of the section by the triangle on the radius similar to the triangle cut off* [89].

Let there be the opposite hyperbolas ΑΖ and ΒΕ and let their diameter be ΑΒ and center Γ, and from some point Ζ on the hyperbola ΖΑ let ΖΗ be drawn tangent to the section, and ΖΟ as an ordinate, and let ΓΖ be joined and continued as ΓE, and through B let BΛ be drawn parallel to ZO, and let some point N be taken on the hyperbola BE, and from N let NΘ be dropped as an ordinate, and let NK be drawn parallel to ZH.

I say that the sum of the triangles ΘKN and ΓBΛ is equal to the triangle ΓMΘ.

[Proof]. For through E let EΔ be drawn tangent to the hyperbola BE, and let EΞ be drawn as an ordinate. Since then ZA and BE are opposite hyperbolas whose diameter is AB, and the straight line through whose center is ZΓE, and ZH and EΔ are tangents to the section, hence ΔE is parallel to ZH. And NK is parallel to ZH, therefore NK is also parallel to EΔ, and MΘ to BΛ. Since then BE is a hyperbola whose diameter is AB and whose center is Γ, and ΔE is tangent to the section, and EΞ drawn as an ordinate, and BΛ is parallel to EΞ, and N has been taken on the section as the point from which NΘ has been dropped as an ordinate, and KN has been drawn parallel to ΔE, therefore the sum of the triangles NΘK and BΓΛ is equal to the triangle ΘMΓ for this has been shown in the theorem 43 [that is Proposition I.43].

[Proposition] 45

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the second diameter, and if from the point of contact some straight line is dropped to same diameter parallel to the other diameter, and if through the point of contact and the center a straight line is drawn, and if some point is taken as random on the section, and [from it] two straight lines are drawn to the second diameter, one of which is parallel to the tangent and the other parallel to the dropped straight line, then in the case of the hyperbola the triangle resulting from them is greater than the triangle cut off by the dropped straight line from the center by the triangle whose base is the tangent and vertex is the center of the section, and in the case of the ellipse and the circle [resulting from the second diameter and two straight lines drawn to the second diameter] together with the triangle cut off will be equal to the triangle whose base is the tangent and whose vertex is the center of the section* [90] .

Let there be a hyperbola or an ellipse or the circumference of a circle ABΓ, whose diameter is AΘ, and second diameter ΘΔ, and center Θ, and let ΓMΛ touch it at Γ, and let ΓΔ be drawn parallel to AΘ, and let ΘΓ be joined and continued, and let some point B be taken at random on the section, and from B let BE and BZ be drawn parallel to ΛΓ and ΓΔ.

I say that in the case of the hyperbola the triangle ΒΕΖ is equal to the sum of the triangles ΗΘΖ and ΛΓΘ, and in the case of the ellipse and the circle the sum of the triangles ΒΕΖ and ΖΗΘ is equal to the triangle ΓΛΘ.

[Proof]. For let ΓΚ and ΒΝ be drawn parallel to ΔΘ. Since then ΓΜ is tangent, and ΓΚ has been dropped as an ordinate, hence [according to Proposition I.39] the ratio ΓΚ to ΚΘ is compounded of [the ratios] ΜΚ to ΚΓ and the *latus rectum* to the *latus transversum*, and [according to Proposition VI.4 of Euclid] as ΜΚ is to ΚΓ, so ΓΔ is to ΔΛ, therefore the ratio ΓΚ to ΚΘ is compounded of [the ratios] ΓΔ to ΔΛ and the *latus rectum* is to the *latus transversum*.

And the triangle ΓΔΛ is the figure on ΚΘ, and the triangle ΓΚΘ, that is the triangle ΓΔΘ, is the figure on ΓΚ, that is on ΔΘ, therefore in the case of the hyperbola the triangle ΓΔΛ is equal to the sum of the triangle ΓΚΘ and the triangle on ΑΘ similar to the triangle ΓΔΛ, and in the case of the ellipse and the circle the sum of the triangles ΓΔΘ and ΓΔΛ is equal to the triangle on ΑΘ similar to the triangle ΓΔΛ for this was also shown in the case of their doubles in the theorem 41 [that is Proposition I.41].

Since then the triangle ΓΔΛ differs either from the triangle ΓΚΘ or from the triangle ΓΔΘ by the triangle on ΑΘ similar to the triangle ΓΔΛ, and it also differs by the triangle ΓΘΛ, therefore the triangle ΓΘΛ is equal to the triangle on ΑΘ similar to the triangle ΓΔΛ. Since then the triangle ΒΖΕ is similar to the triangle ΓΔΛ, and the triangle ΗΖΘ [is similar] to the triangle ΓΔΘ, therefore they have the same ratio. And the triangle ΒΖΕ is described on ΝΘ between the ordinate and the center, and the triangle ΗΖΘ on the ordinate ΒΝ, which is on ΖΘ, and by already shown [in Proposition I.41] the triangle ΒΖΕ differs from the triangle ΗΘΖ by the triangle on ΑΘ similar to the triangle ΓΔΛ, and so also by the triangle ΓΘΛ.

## [Proposition] 46

*If a straight line touching a parabola meets the diameter, then the straight line drawn through the point of contact parallel to the diameter in the direction of the section bisects the straight lines drawn in the section parallel to the tangent* [91] ·

Let there be a parabola whose diameter is ΑΒΔ, and let ΑΓ touch the section, and through Γ let ΗΓΜ be drawn parallel to ΑΔ, and let some point Λ be taken at random on the section and let ΛΝΖΕ be drawn parallel to ΑΓ.

I say that ΛΝ is equal to ΝΖ.

[Proof] . Let ΒΘ, ΚΖΗ, and ΛΜΔ be drawn as ordinates. Since then by the already shown in the theorem 42 [that is Proposition I.42] the triangle ΕΛΔ is equal to the parallelogram BM and [the triangle] ΕΖΗ is equal to the [parallelogram] ΒΚ, therefore the remainders the parallelogram ΗΜ is equal to the quadrangle[92] ΛΖΗΔ.

Let the common the quinquangle[93] ΜΔΗΖΝ be subtracted, therefore the remainders the triangle ΚΖΝ is equal to [the triangle] ΛΜΝ, therefore [according to Proposition VI.22 of Euclid] ΖΝ is equal to ΛΝ[94].

[Proposition] 47

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the diameter, and if through the point of contact and the center a straight line is drawn in the direction of the section, then it bisects the straight lines drawn in the section parallel to the tangent*[95].

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ and center Γ, and let ΔΕ be drawn tangent to the section, and let ΓΕ joined and continued, and let a point Ν be taken at random on the section, and through Ν let [the straight] line ΘΝΟΗ be drawn parallel to ΝΗ.

I say that ΝΟ is equal to ΟΗ.

[Proof]. For let ΞΝΖ, ΒΛ, and ΗΜΚ be dropped as ordinates. Therefore by reasons already shown in the theorem 43 [that is Proposition I.43] the triangle ΘΝΖ is equal to the quadrangle ΛΒΖΞ, and the triangle ΗΘΚ is equal to the quadrangle ΛΒΚΜ. Therefore the remainders quadrangle ΝΗΚΖ is equal to the quadrangle ΜΚΖΞ.

Let the common quinquangle ΟΝΖΚΜ be subtracted, therefore the remainder triangle ΟΜΗ is equal to triangle ΝΞΟ .

And ΜΗ is parallel to ΝΞ, therefore [according to Proposition VI.22 of Euclid] ΝΟ is equal to ΟΗ[96] .

[Proposition] 48

*If a straight line touching one of opposite hyperbolas meets the diameter, and if through the point of contact and the center a straight line drawn cuts the other hyperbola, then whatever line is drawn in the other hyperbola parallel to the tangent, will be bisected by the drawn straight line*[97] .

Let there be opposite hyperbolas whose diameter is ΑΒ and center Γ, and let ΚΛ touch the hyperbola A and let ΛΓ be joined and continued, and let some point N be taken on the hyperbola B, and through N let NH be drawn parallel to ΛK.

I say that NO is equal to OH.

[Proof]. For let EΔ be drawn through E tangent to the section, therefore [according to Proposition i.44] EΔ is parallel to ΛK. And so also to NH since then BNH is a hyperbola whose center is Γ and tangent ΔE, and since ΓE has been joined and a point N has been taken on the section and through it NH has been drawn parallel to ΔE, by a theorem already shown [in Proposition I.47] for the hyperbola NO is equal to OH.

[Proposition] 49

*If a straight line touching a parabola meets the diameter and if through the point of contact a parallel to the diameter is drawn, and if from the vertex a straight line is drawn parallel to an ordinate, and if it is contrived that as the segment of the tangent between the straight line erected [as an ordinate] and the point of contact is to the segment of the parallel between the point of contact and the straight line erected [as an ordinate], so is some straight line to the double of the tangent, then whatever straight line is drawn [parallel to the tangent] from the section to the straight line drawn through the point of contact parallel to the diameter, will equal in square to the rectangular plane under the straight line found [that is the* latus rectum*] and the straight line cut off by it [that is the line parallel to the tangent] from the point of contact* [98].

Let there be a parabola whose diameter is MBΓ, and ΓΔ its tangent, and through Δ let ZΔN be drawn parallel to BΓ, and let ZB be erected as an ordinate, and let it be contrived that as EΔ is to ΔZ, so some straight line H is to double ΓΔ, and let some point K be taken on the section, and let KΛΠ be drawn through K parallel to ΓΔ.

I say that sq.KΛ is equal to pl.H, ΔΛ, that is that with ΔΛ as diameter, H is the *latus rectum*.

[Proof]. For let ΔΞ and KNM be dropped as ordinates. And since ΓΔ touches the section, and ΔΞ has been dropped as an ordinate, then [according to Proposition I.35] ΓB is equal to BΞ.

But BΞ is equal to ZΔ. And therefore ΓB is equal to ZΔ. And so also the triangle EΓB is equal to the triangle EZΔ.

Let the common figure ΔEBMN be added, therefore [according to Proposition I.42] the quadrangle ΔΓMN is equal to the parallelogram ZM and is equal to the triangle KΠM.

Let the common quadrangle ΛΠΜΝ be subtracted therefore the remainders triangle ΚΛΝ is equal to parallelogram ΛΓ. And the angle ΔΛΠ is equal to the angle ΚΛΝ, therefore pl.ΚΛΝ is equal to double pl.ΛΔΓ. And since as ΕΔ is to ΔΖ, so Η is to double ΓΔ, and as ΕΔ is to ΔΖ, so ΚΛ is to ΛΝ, therefore also as Η is to double ΓΔ, so ΚΛ is to ΚΝ.

But as ΚΛ is to ΛΝ, so sq.ΚΛ is to pl.ΚΛΝ, and as Η is to double ΓΔ, so pl.Η,ΔΛ is to double pl.ΛΔΓ, therefore as sq.ΚΛ is to pl.ΚΛΝ, so pl.Η, ΔΛ is to double pl.ΓΔΛ, and corresponding [as sq.ΚΛ is to pl.Η,ΔΛ , so pl.ΚΛΝ is to double pl.ΓΔΛ]. But pl.ΚΛΝ is equal to double pl.ΓΔΛ, therefore also sq.ΚΛ is equal to pl.Η,ΔΛ.

[Proposition] 50

*If a straight line touching a hyperbola or an ellipse or the circumference of a circle meets the diameter, and if a straight line is drawn through the point of contact and the center, and if from the vertex a straight line erected parallel to an ordinate meets the straight line drawn through the point of contact and the center, and if it is contrived that as the segment of the tangent between the point of contact and the straight line erected [as an ordinate from the vertex] is to the segment of the straight line drawn through the point of contact and the center between the point of contact and the straight line erected [as an ordinate from the vertex], so some straight line is to the double tangent, then any straight line parallel to the tangent and drawn from the section to the straight line drawn through the point of contact and the center will equal in square to a rectangular plane applied to the found straight line having as breadth the straight line cut off [of the diameter] by the ordinate from the point of contact, and in the case of the hyperbola increased by a figure similar to the rectangular plane under the double straight line between the center and the point of contact and the found straight line, but in the case of the ellipse and the circle decreased by the same figure* [99] .

Let there be a hyperbola or an ellipse or the circumference of a circle whose diameter is ΑΒ and center Γ, and let ΔΕ be a tangent, and let ΓΕ be joined and continued both ways, and let ΓΚ be made equal to ΕΓ, and through Β let ΒΖΗ be erected as an ordinate, and through Ε let ΕΘ be drawn perpendicular to ΕΓ, and let it be that as ΖΕ is to ΕΗ, so ΕΘ is to double ΕΔ, and let ΘΚ be joined and continued, and let some point Λ be taken on the section, and through it let ΛΜΞ be drawn parallel to ΕΔ, and ΛΡΝ parallel to ΒΗ, and let ΜΠ [be drawn] parallel to ΕΘ.

I say that sq.ΛΜ is equal to pl.ΕΜΠ.

[Proof]. For let ΓΣΟ be drawn through Γ parallel to ΚΠ. And since ΕΓ is equal to ΓΚ,and as ΕΓ is to ΚΓ,so ΕΣ is to ΣΘ, therefore also ΕΣ is equal to ΣΘ..

And since as ΖΕ is to ΕΗ, so ΘΕ is to double ΕΔ, and double ΕΣ is equal to ΕΘ, therefore also as ΖΕ is to ΕΗ, so ΣΕ is to ΕΔ, and [according to Proposition VI.4 of Euclid] as ΖΕ is to ΕΗ, so ΛΜ is to ΜΡ, therefore as ΛΜ is to ΜΡ, so ΣΕ is to ΕΔ.

And since it was shown [in Proposition I.43] that in the case of the hyperbola the triangle ΡΝΓ is equal to the sum of the triangles ΛΝΞ and ΗΒΓ, and is equal to the sum of the triangles ΛΝΞ and ΓΔΕ, and in the case of the ellipse and the circle the sum of the triangles ΡΝΓ and ΛΝΞ is equal to the triangle ΗΒΓ, and is equal to the triangle ΓΔΕ.

Therefore in the case of the hyperbola with the common triangle ΕΓΔ and common quadrangle ΝΡΜΞ subtracted, and in the case of the ellipse and the circle with the common triangle ΜΞΓ subtracted the triangle ΛΜΡ is equal to the quadrangle ΜΕΔΞ. And ΜΞ is parallel to ΔΕ, and the angle ΛΜΡ is equal to the angle ΕΜΞ. Therefore [according to Proposition I.49] pl.ΛΜΡ is equal to pl.ΕΜ, the sum of ΕΔ and ΜΞ. And since as ΜΓ is to ΓΕ, so ΜΞ is to ΕΔ, and as ΜΓ is to ΓΕ, so ΜΟ is to ΕΣ, therefore as ΜΟ is to ΕΣ, so ΜΞ is to ΕΔ. And *componendo* as the sum of ΜΟ and ΕΣ is to ΕΣ, so the sum of ΜΞ and ΕΔ is to ΕΔ, and alternately as the sum of ΜΟ and ΕΣ is to the sum of ΜΞ and ΕΔ, so ΕΣ is to ΕΔ. But as the sum of ΜΟ and ΕΣ is to the sum of ΜΞ and ΕΔ, so pl.ΕΜ, the sum of ΜΟ and ΕΣ is to pl.ΕΜ, the sum of ΜΞ and ΕΔ, and as ΕΣ is to ΕΔ, so ΛΜ is to ΜΡ, and so ΖΕ is to ΕΗ, or as ΕΣ is to ΕΔ, so sq.ΛΜ is to pl.ΛΜΡ, therefore as pl. ΜΕ,the sum of ΜΟ and ΕΣ, is to pl. ΕΜ,the sum of ΜΞ and ΕΔ, so sq.ΛΜ is to pl.ΛΜΡ, and alternately as pl. ΜΕ, the sum of ΜΟ and ΕΣ is to sq.ΛΜ, so pl. ΕΜ, the sum of ΜΞ and ΕΔ is to pl.ΛΜΡ.

But pl.ΛΜΡ is equal to pl.ΜΕ, the sum of ΜΞ and ΕΔ, therefore sq.ΛΜ is equal to pl.ΕΜ, the sum of ΜΟ and ΕΣ, and ΣΕ is equal to ΑΘ , and ΣΘ is equal to ΟΠ. Therefore sq.ΛΜ is equal to ΕΜΠ.

[Proposition] 51

*If a straight line touching either of the opposite hyperbolas meets the diameter, and if through the point of contact and the center some straight line is drawn to the other hyperbola, and if from the vertex a straight line is erected parallel to an ordinate and meets the straight line drawn through the point of contact and the center, and if it is contrived that as the segment of the tangent between the erected straight line and the point of contact is to the segment of* *the straight line drawn through the point of contact between the point of contact and the erected straight line, so some straight line is to the double tangent, then whatever straight line in the other hyperbola is drawn to the straight line through the point of contact and the center parallel to the tangent, will be equal in square to the rectangular plane applied to the found straight line and having as breadth the straight line cut off by it from the point of contact and increased by a figure similar to the rectangular plane under the straight line between the opposite hyperbolas and the found straight line* [100] .

Let there be opposite hyperbolas whose diameter is AB and center E, and let ΓΔ be drawn tangent to the hyperbola B and ΓE be joined and continued, and let BΛH be drawn as an ordinate, and let it be contrived that as ΛΓ is to ΓH, so some straight line K is to double ΓΔ..

Now it is evident that the straight lines in the hyperbola BΓ parallel to ΓΔ and drawn to EΓ continued are equal in square to the planes applied to K and having as breadths the straight line cut off by them from the point of contact, and projecting by a figure similar to pl.ΓZ,K for ZΓ is equal to double ΓE.

I say then that in the hyperbola ZA the same reason will come about.

[Proof]. For let MZ be drawn through Z tangent to the hyperbola AZ, and let AΞN be erected as an ordinate. And since BΓ and AZ are opposite hyperbolas, and ΓΔ and MZ are tangents to them, therefore [according to Proposition I.44] ΓΔ is equal and parallel to MZ. But also ΓE is equal to EZ, therefore also EΔ is equal to EM. And since as ΛΓ is to ΓH, so K is to double ΓΔ or double MZ, therefore also as ΞZ is to ZN, so K is to double MZ.

Since then AZ is a hyperbola whose diameter is AB and tangent MZ, and AN has been drawn as an ordinate, and as ΞZ is to ZN, so K is to double ZM, hence any lines drawn from the section to EZ continued, parallel to ZM, will be equal in square to the rectangular plane under K and the line cut off by them from Z increased by a figure [according to Proposition I.50] similar to pl.ΓZ,K.

[Porism]

And with these reasons shown, it is at once evident that in the parabola each of the straight lines drawn parallel to the original diameter is a diameter [according to Proposition I.46] but in the hyperbolas and the ellipse and the opposite hyperbolas each of the straight lines drawn through the center is a diameter [according to Propositions I.47 and I.48], and that in the parabola the straight line dropped to each of the diameters parallel to the tangents will be equal in square to the rectangular planes applied to it [according to Proposition I.49], but in the hyperbola and the opposite hyperbolas they will equal in square to the planes applied to the diameter increased by the same figure [according to Propositions I.50 and I.51], but in the ellipse the planes applied to the diameter and decreased by the same figure [according to Proposition I.50], and that all which has been already proved about the sections as following when the principal diameters are used, will also those same reasons follow when the other diameters are taken.

[Proposition] 52 [Problem]

*Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola whose diameter is the given straight line and whose vertex is the end of the straight line, and where whatever straight line dropped from the section to the diameter at given angle will be equal in square to the rectangular plane under the straight line cut off by it from the vertex of the section and by some other given straight line* [101].

Let there be the straight line AB given in position and bounded at A, and another [straight line] ΓΔ given in magnitude, and first let the given angle be right, it is required then to find a parabola in the considered plane whose diameter is AB, whose vertex is A, and whose *latus rectum* is ΓΔ and there the straight lines dropped as ordinates will be dropped at a right angle, that is so that AB [according to Definition 7] is the axis.

[Solution]. Let AB be continued [beyond A] to E, and let ΓH be taken as quarter of ΓΔ, and let EA is greater than ΓH, and let as ΓΔ is to Θ, so Θ is to EA. Therefore as ΓΔ is to EA, so sq.Θ is to sq.EA, and ΓΔ is less than quadruple EA, therefore also sq.Θ is less than quadruple sq.EA, and Θ is less than double EA. And so double EA is greater than Θ. Therefore it is possible for a triangle to be constructed from Θ and two EA. Then let the triangle EAZ be constructed on EA at right angles to the considered plane, so that EA is equal to AZ, and Θ is equal to ZE, and let AK be drawn parallel to ZE, and ZK to EA,and let a cone be conceived whose vertex is Z and whose base is the circle about the diameter KA at right angles to the plane through [the triangle] AZK. Then the cone [according to Definition 3] will be right for AZ is equal to ZK.

And let the cone be cut [through B] by a plane parallel to the circle KA, and let it make as a section [according to Proposition I.4] the circle MNΞ at right angles clearly to the plane through [the triangle] MZN, and let MN be the common section of the circle MNΞ and of the triangle MZN, therefore it is the diameter of the circle and let ΞΛ be the common section of the considered plane and of the circle. Since then the circle MNΞ is at right angles to the triangle MZN, and the considered plane also is at right angles to the triangle MZN, therefore ΛΞ, their common section, is at right angles to the triangle MZN, that is to the triangle KZA [according to Proposition XI.19 of Euclid], and therefore it is perpendicular to all straight lines touching it in the triangle, and so it is perpendicular to both MN and AB.

Again since a cone whose base is the circle MNΞ and whose vertex is Z has been cut by a plane at right angles to the triangle MZN and makes as a section the circle MNΞ, and since it has also been cut by another plane cutting the base of the cone in ΞΛ at right angles to MN which is the common section of the circle MNΞ and the triangle MZN, and the common section of the considered plane and of the triangle MZN, [the straight line] AB, is parallel to the side of the cone ZKM, therefore the resulting section of the cone in the considered plane is a parabola, and its diameter is AB, and the straight lines dropped as ordinates from the section to AB will be dropped at right angles for they are parallel to ΞΛ which is perpendicular to AB. And since as ΓΔ is to Θ, so Θ is to EA, and EA is equal to AZ ,and is equal to ZK, and Θ is equal to EZ and is equal to AK, therefore as ΓΔ is to AK, so AK is to AZ. And therefore as ΓΔ is to AZ, so sq.AK is to sq.AZ or pl.AZK. Therefore ΓΔ is the *latus rectum* of the section for this has been shown in the theorem11 [that is Proposition I.11][102].

[Proposition] 53 [Problem]

With the same supposition let the given angle not be right, and let the angle ΘAZ be made equal to it, and let AΘ is equal to half of ΓΔ, and from Θ let ΘE be drawn parallel to BΘ, and from A let AΛ be drawn perpendicular to EΛ, and let EΛ be bisected at K, and from K let KM be drawn perpendicular to EΛ and continued to Z and H, and let pl.ΛKM is equal to sq.AΛ. And the given two straight lines ΛK and KM, KΛ in position and bounded at K, and KM in magnitude, and let a parabola be described with a right angle whose diameter is KΛ, and whose vertex is K, and whose *latus rectum* is KM, as has been shown before [in Proposition I.52] , and it will pass through A because [according to Proposition I.11] sq.AΛ is equal to pl.ΛKM, and EA will touch the section [according to Proposition I.33] because EK is equal to KΛ. And ΘA is parallel to EKΛ, therefore ΘAB is the diameter of the section, and the straight lines dropped to it parallel to AE will be bisected by AB [according to Proposition I.46],and they will be dropped at the angle ΘAE. And since the angle AEΘ is equal to the angle AHZ, and the angle at A is common, therefore the triangle

AΘE is similar to the triangle AHZ. Therefore as ΘA is to EA, so ZA is to AH, therefore as double AΘ is to double AE, so ZA is to AH.

But ΓΔ is equal to double AΘ, therefore as ZA is to AH, so ΓΔ is to double AE.. Than by already shown in the theorem 49 [Proposition I.49] ΓΔ is the *latus rectum*.

[Proposition] 54 [Problem]

*Given two bounded straight lines perpendicular to each other, one of them being drawn on the side of the right angle, to find on the continued straight line the section of a cone called hyperbola in the same plane with the straight lines, so that the continued straight line is a diameter of the section, and the point at the angle is the vertex, and where whatever straight line is dropped from the section to the diameter making an angle equal to a given angle will equal in square to the rectangular plane applied to the other straight line having as breadth the straight line cut off by the dropped straight line beginning of the vertex and increased by a figure similar and similarly situated to the plane under the original straight lines* [103] .

Let there be two bounded straight lines AB and BΓ perpendicular to each other, and let AB be continued to Δ. It is required then to find in the plane through AB and BΓ a hyperbola whose diameter will be ABΔ and vertex B, and the *latus rectum* BΓ, and where the straight lines dropped from the section to BΔ at the given angle will equal in square to the rectangular planes applied to BΓ and having as breadths the straight lines cut off by them from B and increased by a figure similar and similarly situated to pl.ABΓ.

[Solution]. First let the given angle be right, and on AB let a plane be erected at right angles to the considered plane, and let the circle AEBZ be described in it about AB, so that the segment of the diameter of the circle within the arc AEB has to the segment of the diameter within the arc AZB a ratio not greater than that of AB to BΓ, and let [the arc] AEB be bisected at E, and let EK be drawn perpendicular from E to AB and let it be continued to Λ, therefore [according to Proposition III.1 of Euclid] EΛ is a diameter. If then as AB is to BΓ, so EK is to KΛ, we use Λ, but if not, let it be contrived [according to Proposition VI.12 of Euclid] that as AB is to BΓ, so EK is to KM where KM is less than KΛ, and through M let MZ be drawn parallel to AB, and let AZ , EZ, and ZB be joined, and through B let BΞ be drawn parallel to ZE. Since then the angle AZE is equal to the angle EZB, but the angle AZE is equal to the angle AΞB, and the angle EZB is equal to the angle ΞBZ, therefore also the angle ΞBZ is equal to the angle ZΞB, therefore also ZB is equal to ZΞ.

Let a cone be conceived whose vertex is Z and whose base is the circle about diameter BΞ at right angles to the triangle BZΞ. Then the cone will be right for ZB is equal to ZΞ.

Then let BZ, ZΞ, MZ be continued, and let the cone be cut by a plane parallel to the circle BΞ, then the section [according to Proposition I.4] will be a circle. Let it be the circle HΠP, and so HΘ will be the diameter of the circle. And let ΠΔP be the common section of the circle HΘ and of the considered plane, then ΠΔP will be perpendicular to both HΘ and ΔB for both circles ΞB and ΘH are perpendicular to the triangle ZHΘ, and the considered plane is perpendicular to the triangle ZHΘ, and therefore their common section ΠΔP is perpendicular to the triangle ZHΘ, therefore it makes right angles also with all straight lines touching it and situated in the same plane.

And since a cone whose base is the circle HΘ and vertex Z has been cut by a plane perpendicular to the triangle ZHΘ, and has also been cut by another plane, the considered plane, in ΠΔP perpendicular to HΔΘ, and the common section of the considered plane and the triangle HZΘ, that is ΔB continued in the direction of B, meets HZ at A, therefore, as it was already shown before [in Proposition I.12] the section ΠBP will be a hyperbola whose vertex is B, and where the straight lines dropped as ordinates to BΔ will be dropped at a right angles for they are parallel to ΠΔP. And since as AB is to BΓ, so EK is to KM, and as EK is to KM, so EN is to NZ, and pl.ENZ is to sq.NZ, therefore as AB is to BΓ, so pl.ENZ is to sq.NZ. And [according to Proposition III.35 of Euclid] pl.ENZ is equal to pl.ANB, therefore as AB is to BΓ, so pl.ANB is to sq.NZ.

But the ratio pl.ANB to sq.NZ is compounded of [the ratios] AN to NZ and BN to NZ, but as AN is to NZ, so AΔ is to ΔH, and ZO is to OH, and as BN is to NZ, so ZO is to OΘ, therefore the ratio AB to BΓ is compounded of [the ratio] ZO to OH and ZO to OΘ, that is sq.ZO to pl.HOΘ. Therefore as AB is to BΓ, so sq.ZO is to pl.HOΘ.

And ZO is parallel to AΔ, therefore AB is the *latus transversum* and BΓ is the *latus rectum* for it has been shown in the theorem 12 [that is Proposition I.12].

[Proposition] 55 [Problem]

Then let the given angle not be right, and let there be two given straight lines AB and AΓ, and let the given angle be equal to the angle BAΘ, then it is required to describe a hyperbola whose diameter will be AB, and the *latus rectum* ΑΓ, and where the ordinates will be dropped at the angle ΘΑΒ.

Let AB be bisected at Δ, and let the semicircle ΑΖΔ be described on ΑΔ, and let some straight line ΖΗ parallel to ΑΘ be drawn to the semicircle where as sq.ΖΗ is to pl.ΔΗΑ, so ΑΓ is to ΑΒ, and let ΖΘΔ be joined and continued to Δ, and let as ΖΔ is to ΔΛ, so ΔΛ is to ΔΘ, and let ΔΚ be made equal to ΔΛ, and let pl.ΛΖΜ is equal to sq.ΑΖ, and let ΚΜ be joined, and through Λ let ΛΝ be drawn perpendicular to ΚΖ and let it be continued towards Ξ. And with two given bounded ΚΛ and ΛΝ perpendicular to each other, let a hyperbola be described whose *latus transversum* is ΚΛ and *latus rectum* ΛΝ, and where the straight lines dropped from the section to the diameter will be dropped at a right angles and will be equal in square to the rectangular plane [according to Proposition I.54] applied to ΛΝ and having as breadths the straight lines cut off by them from Λ and increased by a figure similar to pl.ΚΛΝ, and the section will pass through Α for [according to Proposition I.12] sq.ΑΖ is equal to pl.ΛΖΜ.

And ΑΘ will touch it for [according to Proposition I.37] pl.ΖΔΘ is equal to sq.ΔΛ, and so ΑΒ [according to Proposition I.47 and Definition 4] is a diameter of the section. And since as ΓΑ is to double ΑΔ or ΑΒ, so sq.ΖΗ is to pl.ΔΗΑ, but the ratio ΓΑ to double ΑΔ is compounded of [the ratios] ΓΑ to double ΑΘ and double ΑΘ to double ΑΔ, or the ratio ΓΑ to double ΑΔ is compounded of [the ratios] ΓΑ to double ΑΘ and ΑΘ to ΑΔ, and as ΑΘ is to ΑΔ, so ΖΗ is to ΗΔ, therefore the ratio ΓΑ to ΑΒ is compounded of [the ratios] ΓΑ to double ΑΘ and ΖΗ to ΗΔ.

But also the ratio sq.ΖΗ to pl.ΔΗΑ is compounded of [the ratios] ΖΗ to ΗΔ and ΖΗ to ΗΑ, therefore the ratio compounded of [the ratios] ΓΑ to double ΑΘ and ΖΗ to ΗΔ is the same, as the ratio compounded of [the ratios] ΖΗ to ΗΑ and ΖΗ to ΗΔ.

Let the common ratio ΖΗ to ΗΔ be taken away, therefore as ΓΑ is to double ΑΘ, so ΖΗ is to ΗΑ.

But as ΖΗ is to ΗΑ, so ΟΑ is to ΑΞ, therefore as ΓΑ is to double ΑΘ, so ΟΑ is to ΑΞ.

But whenever this is so, ΑΓ is the *latus rectum* for the ordinates to the diameter for this has been shown in the theorem 50 [that is Proposition I.50].

[Proposition] 56 [Problem]

*Given two bounded straight lines perpendicular to each other, to find one* *of them as diameter in the same plane with the [mentioned] two straight lines the section of a cone called ellipse whose vertex will be the point at the right angle, and where the straight lines dropped as ordinates from the section to the diameter at a given angle will be equal in square to the rectangular planes applied to the other straight line having as breadth the straight line cut off by them from the vertex of the section and decreased by a figure similar and similarly situated to the plane under the given straight lines* [104] .

Let there be two given straight lines AB and AΓ perpendicular to each other, of which the greater is AB, then it is required to describe in the considered plane an ellipse whose diameter will be AB and vertex A and the *latus rectum* AΓ, and where the ordinates will be dropped from the section to the diameter at a given angle and will be equal in square to the rectangular plane applied to AΓ and having as breadths the straight lines cut off by them from A and decreased by a figure similar and similarly situated to pl.BAΓ.

[Solution]. First let the given angle be right, and let a plane be erected from AB at right angles to the considered plane, and in it on AB let the arc of a circle AΔB be described, and its midpoint be Δ, and let ΔA and ΔB be joined, and let AΞ be made equal to AΓ, and through Ξ let ΞO be drawn parallel to ΔB, and through O let OZ be drawn parallel to AB, and let ΔZ be joined and let it meet continued AB at E, then we will have as AB is to AΓ, so AB is to AΞ, and ΔA is to AO, and ΔE is to EZ.

And let AZ and ZB be joined and continued, and let some point H be taken at random on ZA, and through it let HΛ be drawn parallel to ΔE and let it meet continued AB at K, then let ZO be continued and let it meet HK at Λ. Since then the arc AΔ is equal to the arc ΔB, [according to Proposition III.27 of Euclid] the angle ABΔ is equal to the angle ΔZB.

And since the angle EZA is equal to the sum of the angles ZΔA and ZAΔ, but the angle ZAΔ is equal to the angle ZBΔ, and the angle ZΔA is equal to the angle ZBA, therefore also the angle EZA is equal to the angle ΔBA and is equal to the angle ΔZB.

And also ΔE is parallel to ΛH, therefore the angle EZA is equal to the angle ZHΘ, and the angle ΔZB is equal to the angle ZΘH.

And also the angle ZHΘ is equal to the angle ZΘH, and ZH is equal to ZΘ.

Then let the circle HΘN be described about ΘH at right angles to the triangle ΘHZ, let a cone be conceived whose base is the circle HΘN, and whose vertex is Z, then the cone will be right because ZH is equal to ZΘ.

And since the circle HΘN is at right angles to the plane ΘHZ, and the considered plane is also at right angles to the plane through HΘ and ΘZ, therefore their common section will be at right angles to the plane through HΘ and ΘZ. Then let their common section be KM, therefore KM is perpendicular to both AK and KH.

And since a cone whose base is the circle HΘN and whose vertex is Z, has been cut by a plane through the axis and makes as a section the triangle HΘZ, and has been cut also by another plane through AK and KM, which is the considered plane, in KM which is perpendicular to HK, and the plane meets the sides of the cone ZH and ZΘ, therefore the resulting section [according to Proposition i.13] is an ellipse whose diameter AB and where the ordinates will be dropped at a right angle for they are parallel to KM. And since as ΔE is to EZ, so pl.ΔEZ or pl.BEA is to sq.EZ ,and the ratio pl.BEA to sq.EZ is compounded of [the ratios] BE to EZ and AE to EZ, but as BE is to EZ, so BK is to KΘ, and as AE is to EZ, so AK is to KH, and ZΛ is to ΛH, therefore the ratio BA to AΓ is compounded of [the ratios] ZΛ to ΛH and ZΛ to ΛΘ which is the same as the ratio sq.ZΛ to pl.HΛΘ, therefore as BA is to AΓ, so ZΛ is to pl.HΛΘ. Whenever this is so, AΓ is the *latus rectum* of the *eidos*, as it has been shown in the theorem 13 [that is Proposition I.13].

[Proposition] 57 [Problem]

With the same supposition let AB be less than AΓ, and let it be required to the scribe an ellipse about diameter AB so that AΓ is the *latus rectum*.

Let AB bisected at Δ, and from Δ let [the straight line] EΔZ be drawn perpendicular to AB, and let sq.ZE is equal to BAΓ so that ZΔ is equal to ΔE, and let ZH be drawn parallel to AB, and let it be contrived that as AΓ is to AB, so EZ is to ZH, therefore also EZ is greater than ZH. And since pl.ΓAB is equal to sq.EZ, hence as ΓA is to AB, so sq.ZE is to sq.AB, and sq.ΔZ is to sq.ΔA. But as ΓA is to AB, so EZ is to ZH, therefore as EZ is to ZH, so sq.ZΔ is to sq.ΔA. But sq.ZΔ is equal to pl.ZΔE, therefore as EZ is to ZH, so pl.EΔZ is to sq.AΔ.

Then with two bounded straight lines situated at right angles to each other and with EZ greater, let an ellipse be described whose diameter is EZ and *latus rectum* ZH [according to Proposition I.56], then the section will pass through A because [according to Proposition I.21] as pl.ZΔE is to sq.ΔA, so EZ is to ZH. And AΔ is equal to ΔB, then it will also pass through B. Then an ellipse has been described about AB.

And since as ΓA is to AB, so sq.ZΔ is to sq.ΔA,and sq.ΔA is equal to pl.AΔB, therefore as ΓA is to AB, so sq.ΔZ is to pl.AΔB. And so AΓ [according to Proposition I.21] is the *latus rectum*.

[Proposition] 58 [Problem]

But then let the given angle not be right, and let the angle ΒΑΔ be equal to it, and let ΑΒ be bisected at Ε, and let the semicircle ΑΖΕ be described on ΑΕ, and in it let ΖΗ be drawn parallel to ΑΔ making as sq.ΖΗ is to pl.ΑΗΕ, so ΓΑ is to ΑΒ, and let ΑΖ and ΕΖ be joined and continued, and let at ΔΕ is to ΕΘ, so ΕΘ is to ΕΖ, and let ΕΚ is to ΕΘ, and let it be contrived that pl.ΘΖΛ is equal to sq.ΑΖ, and let ΚΛ be joined and from Θ let ΘΜΞ be drawn perpendicular to ΘΖ and so parallel to ΑΖΛ for the angle at Ζ is right. And with given bounded ΚΘ and ΘΜ perpendicular to each other, let an ellipse be described whose the transverse diameter is ΚΘ, and the *latus rectum* of whose *eidos* is ΘΜ, and where the ordinate to ΘΚ [according to Propositions I.56 and I.57] will be dropped at right angles, then the section will pass through Α because [according to Proposition I.13] sq. ΖΑ is equal to pl.ΘΖΛ. And since ΘΕ is equal to ΕΚ, and ΑΕ is equal to ΕΒ, the section will also pass through Β, and Ε will be the center, and ΑΕΒ will be the diameter. And ΔΑ will touch the section because pl.ΔΕΖ is equal to sq.ΕΘ. And since as ΓΑ is to ΑΒ , so sq.ΖΗ is to pl.ΑΗΕ, but the ratio ΓΑ to ΑΒ is compounded of [the ratios] ΓΑ to double ΑΔ and double ΑΔ to ΑΒ or ΔΑ to ΑΕ, and the ratio sq.ΖΗ to pl.ΑΗΕ is compounded of [the ratios] ΖΗ to ΗΕ and ΖΗ to ΗΑ, therefore the ratio compounded of [the ratios] ΓΑ to double ΑΔ and ΔΑ to ΑΕ is the same, as the ratio compounded of [the ratios] ΖΗ to ΗΕ and ΖΗ to ΗΑ.

But as ΔΑ is to ΑΕ, so ΖΗ is to ΗΕ, and common ratio being taken away, we will have as ΓΑ is to double ΑΔ, so ΖΗ is to ΗΑ or as ΓΑ is to double ΑΔ, so ΞΑ is to ΑΝ.

And whenever this is so [according to Proposition I.50] ΑΓ is the *latus rectum* of the *eidos*.

[Proposition] 59 [Problem]

*Given two bounded straight lines perpendicular to each other, to find opposite hyperbolas whose diameter is one of the given straight lines and whose vertices are the ends of this straight line, and where the straight lines dropped in each of the hyperbolas at a given angle will equal in square to the rectangular planes applied to the other of the straight lines and increased by a figure similar to the rectangular plane under the given straight lines* [105].

Let there be two given bounded straight lines ΒΕ and ΒΘ perpendicular to each other, and let the given angle be Η, then it is required to describe opposite hyperbolas about one of the straight lines ΒΕ and ΒΘ, so that the ordinates are dropped at an angle Η.

[Solution]. For let ΒΕ and ΒΘ be given, and let a hyperbola be described whose transverse diameter will be ΒΕ, and the *latus rectum* of whose *eidos* will be ΘΒ, and where the ordinates to continued ΒΕ will be at an angle Η, and let it be the line ΑΒΓ for we have already described how this must be done [in Proposition I.55]. Then let ΕΚ be drawn through Ε perpendicular to ΒΕ and equal to ΒΘ, and let another hyperbola ΔΕΖ be likewise described whose diameter is ΒΕ and the *latus rectum* of whose *eidos* is ΕΚ, and where the ordinates from the hyperbola will be dropped at a same angle Η. Then it is evident that Β and Ε are opposite hyperbolas, and there is one diameter for them, their *latera recta* are equal.

[Proposition ] 60 [Problem]

*Given two straight lines bisecting each other, to describe about each of them opposite hyperbolas, so that the straight lines are their conjugate diameters, and the diameter of one pair of opposite hyperbolas is equal in square to the* eidos *of the other pair, and likewise the diameter of the second pair of opposite hyperbolas is equal in square to the* eidos *of the first pair* [106].

Let there be two given straight lines ΑΓ and ΔΕ bisecting each other, then it is required to describe opposite hyperbolas about each of them as the diameters, so that ΑΓ and ΔΕ are conjugate in them, and ΔΕ is equal in square to the *eidos* [of the hyperbola] about ΑΓ, and ΑΓ is equal in square to the *eidos* [of the hyperbola] about ΔΕ.

[Solution]. Let pl.ΑΓΛ is equal to sq.ΔΕ, and let ΛΓ be perpendicular to ΓΑ. And given ΑΓ and ΓΛ are perpendicular to each other, let the opposite hyperbolas ΡΑΗ and ΘΓΚ be described whose transverse diameter will be ΓΑ , and whose *latus rectum* will be ΓΛ, and where the ordinates from the hyperbolas to ΓΑ will be dropped at the given angle [according to Proposition I.59], then ΔΕ will be a second diameter of the opposite hyperbolas [according to Definition 11] for it is the mean proportional between sides of the *eidos*, and parallel to an ordinate it has been bisected at Β. Then again let pl.ΔΕΖ be equal to sq.ΑΓ, and let ΔΖ be perpendicular to ΔΕ.

And given ΕΔ and ΔΖ situated perpendicular to each other, let the opposite hyperbolas ΜΔΝ and ΟΕΞ be described whose transverse diameter will be ΔΕ , and the *latus rectum* of whose *eidos* will be ΔΖ. And where the ordinates from the hyperbolas will be dropped to ΔΕ at the given angle [according to Proposition I.59], then ΑΓ will also be a second diameter of the hyperbolas ΜΔΝ and ΞΕΟ, and so ΑΓ bisects the parallels to ΔΕ between the hyperbolas ΡΑΗ and ΘΓΚ, and ΔΕ bisects the parallels to ΑΓ, and this is what was to make[107].

And let such hyperbolas be called conjugate [108] .

# BOOK TWO

## Preface

Apollonius greets Eudemius[1].

If you are well, well good, and I, too fare pretty well.

I have sent you my son Apollonius[2] bringing you the second book of the Conic as was arranged by us. Go through it then carefully and acquaint those with it worthy of sharing in such things. And Philonides[3], the geometer. I introduced to you Fphesus, if ever he happen about Pergamum, acquaint him with it too.

## [Proposition] 1

*If a straight line touch a hyperbola at its vertex, and from it on both sides of the diameter a straight line is cut off equal in square to the quarter of the* eidos, *then the straight lines drawn from the center of the section to the ends thus taken on the tangent will not meet the section* [4].

There be let there be a hyperbola whose diameter ΑΒ, vertex Γ, and the *latus rectum* ΒΖ, and let ΔΕ touch the section at Β, and let the square on ΒΔ and

ΒΕ each be equal to the quarter of the [*eidos*] pl.ΑΒΖ, and let ΓΔ and ΓΕ be joined and continued.

I say that they will not meet the section,

[Proof]. For, if possible, let ΓΔ meet the section at Η, and from Η let ΗΘ be dropped as an ordinate, therefore [according to Proposition I.17] it is parallel to ΔΒ. Since then as ΑΒ is to ΒΖ, so sq.ΑΒ is to pl.ΑΒΖ, but sq.ΓΒ is equal to the quarter of sq.ΑΒ, and sq.BD is equal to the quarter of pl.ΑΒΖ, therefore as ΑΒ is to ΒΖ, so ΓΒ is to sq.ΔΒ, and sq.ΓΘ is to sq.ΘΗ.

And also [according to Proposition I.21] as ΑΒ is to ΒΖ, so pl.ΑΘΒ is to sq.ΘΗ, therefore as sq.ΓΘ is to sq.ΘΗ, so pl.ΑΘΒ is to sq.ΘΗ.

Therefore pl.ΑΘΒ is equal to sq.ΓΘ, and this [according to Proposition II.6 of Euclid] is impossible. Therefore ΓΔ will not meet the section. Then likewise we could show that neither does ΓΕ, therefore ΓΔ and ΓΕ are asymptote of the section.

[Proposition] 2

With the same suppositions it is to be shown that a strait line cutting the angle under the strait line ΔΓ and ΓΕ is not another asymptote[5].

[Proof]. For, if possible, let ΓΘ be it, and let ΒΘ be drawn through Β parallel to ΓΔ and let it meet ΓΘ as Θ, and let ΔΗ be made equal to ΒΘ and let ΗΘ be joined and continued to the points Κ, Λ, and Μ [of intersection with the hyperbola, its diameter ΓΒ and the line ΓΕ, respectively].Since then ΒΘ and ΔΗ are equal and parallel, ΔΒ and ΘΗ are also equal and parallel.
Since ΑΒ is bisected at Γ and ΒΛ added to it, [according to Proposition II.6 of Euclid] the sum of pl.ΑΛΒ and sq.ΓΒ is equal to sq.ΓΛ.

Likewise then since ΗΜ is parallel ΔΕ, and ΔΒ is equal to ΒΕ, therefore also ΗΛ is equal to ΛΜ.

And since ΗΘ is equal to ΔΒ, therefore ΗΚ is greater than ΔΒ. And also ΚΜ is greater than ΒΕ, since also ΛΜ greater than ΒΕ, therefore pl.ΜΚΗ is greater than pl.ΔΒΕ, which is greater than sq.ΔΒ.

Since then [according to Proposition II.1] as ΑΒ is to ΒΖ, so sq.ΓΒ is to sq.ΒΔ, but [according to Proposition I,21] as ΑΒ is to ΒΖ, so pl.ΑΛΒ is to sq.ΛΚ, and as sq.ΓΒ is to sq.ΒΔ, so sq. ΓΛ is to sq.ΛΗ, therefore also as sq.ΓΛ is to sq.ΛΗ, so pl.ΑΛΒ is to sq.ΛΚ.

Since then as whole sq.ΛΓ is to whole sq.ΛΗ, so subtracted part pl.ΑΛΒ is to subtracted part sq.ΛΚ, therefore also as sq.ΛΓ is to sq.ΛΗ, so remainder sq.ΓΒ is to remainder pl.ΜΚΗ, that is as sq.ΓΒ is to pl.ΜΚΗ, so sq.ΓΒ is to sq.ΔΒ.

Therefore sq.ΔB is equal to pl.MKH, and this is impossible for it has been shown to be greater than it. Therefore ΓΘ is not an asymptote to the section.

[Preposition] 3

*If a straight line touches a hyperbola it will meet both asymptotes and it will be bisected at the point of contact, and the square on each of its segments will be equal to the quarter of the* eidos *corresponding to the diameter drawn through the point of contact* [6].

Let there be the hyperbola ABΓ, and its center E, and asymptotes ZE and EH, and some straight line ΘK touch it at B.

I say that ΘK continued will meet ZE and EH.

[Proof]. For, if possible, let it not meet them, and let EB is joined and continued, and let EΔ be made equal to EB, therefore BΔ is a diameter. Then let sq.ΘB and sq.BK each be made equal to the quarter of the *eidos* corresponding to BΔ, and let EΘ and EK be joined. Therefore [according to Proposition II.1] they are asymptotes, and this is [according to Proposition II.2] is impossible for ZE and EH are supposed asymptotes. Therefore KΘ continued will meet the asymptotesEZ and EH.

I say then also that sq.BZ and sq.BH will each be equal to the quarter of the *eidos* corresponding to BΔ.

[Proof]. For let it not be, but if possible, let sq.BΘ and sq. BK each be equal to the quarter of the *eidos*. Therefore [according to Proposition II.1] ΘE and EK are asymptotes, and [according to Proposition II.2] this is impossible. Therefore sq.ZB and sq.BH will each equal to the quarter of the *eidos* corresponding to BΔ.

[Proposition] 4 [Problem]

*Given two straight lines containing an angle and a point within the angle, to describe through the point the section of a cone called hyperbola, so that the given straight lines are its asymptotes*[7].

Let there be two straight lines AΓ and AB containing a chance angle at A, and some point Δ be given, and let it be required to describe through Δ a hyperbola with the asymptote ΓA and AB.

[Solution]. Let AΔ be joined and continued to E, and let AE be made equal to ΔA, and let ΔZ be drawn through Δ parallel to AB, and let ZΓ be made equal to AZ, and let ΓΔ be joined and continued to B, and let be contrived that pl.ΔE,H is equal to sq.ΓB, and with AΔ continued let a hyperbola be described about it through Δ, so that the ordinate equal in square to the [rectangular] planes applied to H and increased by a figure similar to pl.ΔE,H . Since then ΔZ is parallel to BA, and ΓZ is equal to ΓA, therefore ΓΔ is equal to ΔB, and sq.GB is equal to quadruple sq.ΓΔ. And sq.ΓB is equal to pl.ΔE,H ,therefore sq.ΓΔ and sq.ΔB are each equal to the quarter of the *eidos* pl.ΔE,H . Therefore AB and A are asymptote of the described hyperbola.

[Proposition] 5

*If the diameter of a parabola or a hyperbola bisect some straight line [within the section], the tangent to the section at the end of the diameter will be parallel to the bisected straight line* [8].

Let there be the parabola or the hyperbola ABΓ whose diameter is ΔBE,and let ZBH touch the section, and let some straight line AEΓ be drawn in the section making AE equal to EΓ.

I say that AΓ is parallel to ZH.

[Proof]. For, if not let ΓΘ be drawn through parallel to ZH and let ΘΛ be joined. Since then ABΓ is a parabola or a hyperbola whose diameter is ΔE, and tangent ZH, and ΓΘ is parallel to it, therefore [according to Propositions I.46 and I.47] ΓK is equal to KΘ. But also ΣE is equal to EA.

Therefore AΘ is parallel to KE, and this is impossible for [according to Proposition I.22] continued it BΔ.

[Proposition] 6

*If the diameter of an ellipse or the circumference of a circle is bisects some straight line not through the center, the tangent to the section at the end of the diameter will be parallel to the bisected straight line* [9]

Let there be an ellipse or the circumference of a circle whose diameter is AB, and let AB bisect ΓΔ, a straight line not through the center, at E.

I say that the tangent to the section at A is parallel to ΓΔ.

[Proof]. For let it not be, but, if possible, let ΔZ be parallel to the tangent at A, therefore [according to Proposition I.47] ΔH is equal to ZH.

But also ΔE is equal to EΓ, therefore ΓZ is parallel to HE, and this is possible for if H is the center of the section AB, and ΓZ [according to Proposition I.23 will meet [the straight line] AB, and if it is not, suppose it to be K, and let ΔK be joined and continued to Θ, and let ΓΘ be joined. Since then ΔK is equal to

ΚΘ and also ΔΕ is equal to ΕΓ, therefore ΓΘ is parallel to ΑΒ. But also ΓΖ, and this is impossible. Therefore the tangent at Α is parallel to ΓΔ.

[Proposition] 7

*If a straight line touches a section of a cone or the circumference of a circle, and a parallel to it is drawn in the section and bisected, the straight line joined the point of contact with the midpoint will be a diameter of the section* [10].

There be a section of a cone the circumference of a circle ΑΒΓ, and ΖΗ tangent to it, and ΑΓ parallel to ΖΗ and bisected at Ε, and let ΒΕ be joined.

I say that ΒΕ is a diameter of the section.

[Proof] . For let it not be, but, if possible, let ΒΘ be a diameter of the section. Therefore [according to Definition 4] ΑΘ is equal to ΘΓ, and this is not impossible for ΑΕ is equal to ΕΓ.

Therefore ΒΘ will not be a diameter of the section. Then likewise we could show that there is no other [diameter] than ΒΕ.

[Proposition] 8

*If a straight line meets a hyperbola at two point, continued both ways it will meet the asymptotes, the straight lines cut off on it by the section from the asymptotes will be equal* [11].

Let there be the hyperbola ΑΒΓ and the asymptotes ΕΔ and ΔΖ, and let some straight line ΑΓ meet ΑΒΓ.

I say that continued both ways it will meet the asymptotes.

[Proof]. Let ΑΓ be bisected at Η and let ΔΗ be joined. Therefore [according to Proposition I.47] it is a diameter of the section, therefore the tangent at Β [according to Proposition II.5] is parallel to ΑΓ. Then let ΘΒΚ be the tangent, then it will [according to Proposition II,3] meet ΕΔ and ΔΖ. Since then ΑΓ is parallel to ΚΘ, and ΚΘ meets ΔΚ and ΔΘ, therefore also ΑΓ will meet ΔΕ and ΔΖ.

Let it meet them at Ε and Ζ, and [according to Proposition II.3] ΘΒ is equal to ΒΚ, therefore also ΖΗ is equal to ΗΕ. And so also ΓΖ is equal to ΑΕ.

[Proposition] 9

*If a straight line meeting the asymptote is bisected is by the hyperbola, it will touch the section one point only* [12].

For let ΓΔ meeting the asymptotes ΓΑ, ΑΔ be bisected by the hyperbola at Ε.

I say that it touches the hyperbola at no other point.

[Proof]. For, if possible, let meet touch it at as Β. Therefore [according to Proposition II.8] ΓΕ is equal to ΒΔ, and this is impossible for ΓΕ is supposed equal to ΕΔ. Therefore it will not touch the section as another point.

[Proposition] 10

*If some straight line cutting the hyperbola meet both asymptotes, the rectangular plane under the straight lines cut off between the asymptotes and the section is equal to the quarter of the* eidos *corresponding to the diameter bisecting the straight lines drawn parallel to the drawn straight line* [13].

Let there be the hyperbola ΑΒΓ and let ΔΕ, ΕΖ be its asymptotes, and let some straight line ΔΖ be drawn cutting the section and the asymptotes, and let ΑΓ be bisected at Η and let ΗΕ be joined, and let ΕΘ be made equal to ΒΕ, and let ΒΜ be drawn from Β perpendicular to ΘΕΒ,therefore [according to the porism to Proposition I.51] ΒΘ is a diameter and ΒΜ is the *latus rectum*.

I say that pl.ΔΑΖ is equal to the quarter of pl.ΘΒΜ, then likewise also pl.ΔΓΖ is equal to the quarter of pl.ΘΒΜ.

[Proof]. For let ΚΛ be drawn through Β tangent to the section, therefore [according to Proposition II.5] it is parallel to ΔΖ. And since it has been shown [in Proposition II.1] that as ΘΒ is to ΒΜ, so sq.ΕΒ is to sq.ΒΚ, and sq.ΕΗ is to sq.ΗΔ, and [according to Proposition I.21] as ΘΒ is to ΒΜ, so pl.ΘΗΒ is to sq.ΗΑ, therefore as sq.ΕΗ is to sq.ΗΔ, so pl.ΘΗΒ is to sq.ΗΑ.

Since then as whole sq.ΕΗ is to whole sq.ΗΔ, so subtracted part of pl.ΘΗΒ is to subtracted part of sq.ΑΗ, therefore also [according to Proposition II.5, II.6, and V.19 of Euclid] as remainder sq.ΕΒ is to remainder pl.ΔΑΖ, so sq.ΕΗ is to sq.ΗΔ or as remainder sq.ΕΒ is to remainder pl.ΔΑΖ, so sq.ΕΒ is to sq.ΒΚ.

Therefore pl.ΖΑΔ is equal to sq.ΒΚ.

Then likewise it could be shown also that pl.ΔΓΖ is equal to sq.ΒΛ, therefore also pl.ΖΑΔ is equal to pl.ΔΓΖ.

[Proposition] 11

*If some straight line cut each of the straight lines containing the angle that is adjacent to the angle which contains the hyperbola, then this straight line will meet the section at one point only, and the rectangular plane under the* *straight lines cut off [on this straight line] between the containing straight lines and the section will be equal to the quarter of the* eidos *corresponding to the diameter drawn parallel to the cutting straight line* [14].

Let there be a hyperbola whose asymptotes are ΓΑ, ΑΔ, and let ΔΑ be continued to Ε, and through some point Ε let ΕΖ be drawn cutting ΕΑ and ΑΓ [continued as necessary].

Now it is evident that it meets the section at one point only for the straight line drawn through Α parallel to ΕΖ as ΑΒ will cut the angle ΓΑΔ and [according to Proposition II.2] will meet the section and [according to the porism to Proposition I.51] be its diameter, therefore [according to Proposition I.26] ΕΖ will meet the section as one point only. Let it meet it as Η.

I say then also that pl.ΕΗΖ is equal to sq.ΑΒ .

[Proof]. For let ΘΗΛΚ be drawn as an ordinate through Η, therefore the tangent through Β [according to Proposition II.5] is parallel ΗΘ. Let it be ΓΔ. Since then [according to Proposition II.3] ΓΒ is equal to ΒΔ, therefore the ratio sq.ΓΒ or pl.ΓΒΔ to sq.ΒΑ is compounded of [the ratios] ΓΒ to ΒΑ and ΔΒ to ΒΑ. But as ΓΒ is to ΒΑ, so ΘΗ is to ΗΖ, and as ΔΒ is to ΒΑ, so ΗΚ is to ΗΕ, therefore the ratio sq.ΓΒ to sq.ΒΑ is compounded of [the ratios] ΘΗ to ΗΖ and ΚΗ to ΗΕ.

But also the ratio pl.ΚΗΘ to pl.ΕΗΖ is compounded of [the ratios] ΘΗ to ΗΖ and sq.KH to HE, therefore as pl.ΚΗΘ is to pl.ΕΗΖ, sq.ΓΒ is to sq.ΒΑ. Alternately as pl.ΚΗΘ is to sq.ΓΒ, so pl.ΕΗΖ is to sq.ΒΑ.

But it was shown [in Proposition II.10] that pl.ΚΗΘ is equal to sq.ΓΒ, therefore also pl.ΕΗΖ is equal to sq.ΑΒ.

[Proposition] 12

*If two straight lines at chance angles are drawn to the asymptotes from some point of those on the section, and parallels are drawn to two straight lines from some point of those on the section, then the rectangular plane contained by the parallels will be equal to that contained by those straight lines to which they were drawn parallel*[15].

Let there be a hyperbola whose asymptotes are ΑΒ and ΒΓ, and let some point Δ be taken on the section, and from it let ΔΕ and ΔΖ be dropped [at chance angles] to ΑΒ and ΒΓ, and let some other point Η on the section be taken, and through Η let ΗΘ and ΗΚ be drawn parallel to ΕΔ and ΔΖ.

I say that pl.ΕΔΖ is equal to pl.ΘΗΚ.

[Proof]. For let ΔΗ be joined and continued to Α and Γ. Since then [according to Proposition II.8] pl.ΑΔΓ is equal to pl.ΑΗΓ, therefore as ΑΓ is to ΑΔ, so ΔΓ is to ΓΗ.

But as ΑΗ is to ΑΔ, so ΗΘ is to ΕΔ, and as ΔΓ is to ΓΗ, so ΔΖ is to ΗΚ, therefore as ΗΘ is to ΔΕ, so ΔΖ is to ΗΚ.

Therefore pl.ΕΔΖ is equal to pl.ΘΗΚ.

## [Proposition] 13

*If in the place bounded by the asymptotes and the section some straight line is drawn parallel to one of the asymptote, it will meet the section at one point only*[16] .

Let there be a hyperbola whose asymptote are ΓΑ and ΑΒ, and let some point Ε be taken [in the place bounded by asymptotes and the section], and through it let ΕΖ be drawn parallel to ΑΒ.

I say that it will meet the section.

[Proof]. For, if possible, let it not meet it, and let some point Η on the section be taken, and through Η let ΗΓ and ΗΘ be drawn parallel to ΓΑ and ΑΒ, and let pl.ΓΗΘ is equal to pl.ΑΕΖ, and let ΑΖ be joined and continued, then [according to Proposition II.2] it will meet the section. Let it meet it as Κ, and through K parallel to ΓΑ and ΑΒ let ΚΛ and ΚΔ be drawn, therefore [according to Proposition II.12] pl.ΓΗΘ is equal to pl.ΛΚΔ.

And it is supposed that also pl.ΓΗΘ is equal to pl.ΑΕΖ, therefore pl.ΛΚΔ or pl.ΚΛΑ is equal to pl.ΑΕΖ, and this is impossible for both ΚΛ is greater than ΕΖ, and ΛΑ is greater than ΑΕ.

Therefore ΕΖ will meet the section. Let it meet it at Μ.

I say then that it will not meet it at any other point.

[Proof]. For, if possible, let it also meet it at Ν, and through Μ and Ν let ΜΞ and ΝΒ be drawn parallel to ΓΑ. Therefore [according to Proposition II.12] pl.ΕΜΞ is equal to pl.ΕΝΒ, and this is impossible. Therefore it will not meet the section at another point.

## [Proposition] 14

*The asymptote and the section, if continued indefinitely, draw nearer to each other, and they reach a distance less than any given distance* [17].

Let there be a hyperbola whose asymptotes are ΑΒ and ΑΓ, and a given distance Κ.

I say that ΑΒ and ΑΓ and the section, if continued, draw nearer to each other and will reach a distance less than Κ.

[Proof]. For let ΕΘΖ and ΓΗΔ be drawn parallel to the tangent, and let ΑΘ be joined and continued to Ξ. Since then [according to Proposition II.10] pl.ΓΗΔ is equal to pl.ΖΘΕ, therefore as ΔΗ is to ΖΘ, so ΘΕ is to ΓΗ.

But [according to Proposition VI.4 of Euclid] ΔΑ is greater than ΖΘ, therefore also ΘΕ is greater than ΓΗ.

Then likewise we could show that the succeeding straight lines are less.

Then let the distance ΖΛ be taken less than Κ, and through Λ let ΛΝ be drawn parallel to ΑΓ, therefore it [according to Proposition II.12 ] will meet the section. Let it meet it at Ν, and through Ν let ΜΝΒ be drawn parallel to ΕΖ therefore ΜΝ is equal to ΕΛ, and so ΜΝ is less than Κ.

Porism

Then from this if is evident that ΑΒ and ΑΓ are nearer than all asymptotes to the section, and the angle under ΒΑ, ΑΓ is clearly less than that under other asymptote to the section [18].

## [Proposition] 15

*The asymptotes of opposite hyperbolas are common*[19].

Let there be opposite hyperbolas whose diameter is ΑΒ and center Γ.

I say the asymptote of the hyperbolas Α and Β are common.

[Proof]. Let ΔΑΕ and ΖΒΗ be drawn tangent to the hyperbola through Α and Β, they [according to Proposition I.44] are therefore parallel. Then let each of [the straight lines] ΔΑ, ΑΕ, ΕΒ, and ΒΗ be cut off equal in square to the quarter of the *eidos* applied to ΑΒ, therefore ΔΑ is equal to ΑΕ, is equal to ΖΒ, and is equal to ΒΗ.

Then let ΓΔ, ΓΕ, ΓΖ, and ΓΗ be joined. Then it is evident that ΔΓ is in a straight line with ΓΗ, and ΓΕ with ΓΖ because of the parallel. Since then it is a hyperbola whose diameter is ΑΒ and tangent ΔΕ, and ΔΑ and ΑΕ are each equal in square to the quarter of the *eidos* applied to ΑΒ, therefore ΔΓ and ΓΕ are asymptotes. For the same reasons ΖΓ and ΓΗ are also asymptotes to hyperbola Β. Therefore the asymptote of opposite hyperbola are common.

## [Proposition] 16

*If in opposite hyperbola some straight line is drawn cutting in the straight lines containing the angle adjacent to the angles containing the sections, it will* *meet each of the opposite hyperbola in one point only, and the straight lines cut off on it by the hyperbola from the asymptotes will be equal* [20] .

Let there be the opposite hyperbolas Α and Β whose center is Γ and asymptotes ΔΓΗ and ΕΓΖ, and let some straight line Θ be drawn trough, cutting each of ΔΓ and ΓΖ.

I say that continued it will meet each of the hyperbolas in one point only.

[Proof]. For since ΔΓ and ΓΕ are asymptotes of the hyperbola Α, and some straight line ΘΚ has been drawn across cutting both of straight lines containing the adjacent angle ΔΓΖ, therefore [according Proposition II.11] ΘΚ continued will meet the section. Then likewise also Β. Let it meet them at Λ and Μ. Let ΑΓΒ be drawn through Γ parallel to ΛΜ, therefore [according to Proposition II.11] pl.ΚΛΘ is equal to sq.ΑΓ, and pl.ΘΜΚ is equal to sq.ΓΒ.

And so also pl.ΚΛΘ is equal to pl.ΘΜΚ, and ΛΘ is equal to ΚΜ.

[Proposition] 17

*The asymptotes of conjugate opposite hyperbolas are common* [21].

Let there be conjugate opposite hyperbolas whose conjugate diameters are ΑΒ and ΓΔ, and whose center is Ε.

I say that their asymptotes are common.

[Proof]. For let ΖΑΗ, ΗΔΘ, ΘΒΚ, and ΚΓΖ be drawn through [the points] Α, Β, Γ, and Δ touching the hyperbolas, therefore ΖΗΘΚ [according to Proposition I.44] is a parallelogram. Then let ΖΕΘ and ΚΕΗ be joined, therefore they are diagonals of the parallelogram, and they are all bisected at Ε. And since the figure on ΑΒ [according to Proposition I.60] is equal to sq.ΓΔ, and ΓΕ is equal to ΖΔ, therefore each of sq.ΖΑ, sq.ΑΗ, sq.ΚΒ, and sq.ΒΘ is equal to the quarter of the *eidos* corresponding to ΑΒ. Therefore ΖΕΘ and ΚΕΗ [according to Proposition II.1] are asymptotes of hyperbolas Α and Β. Then likewise we could show that same straight lines are also asymptotes of the hyperbolas Γ and Δ. Therefore the asymptotes of conjugate opposite hyperbolas are common.

[Proposition] 18

*If a straight line meeting one of the conjugate opposite hyperbolas when continued both ways, falls outside the section, it will meet both of the adjacent hyperbolas at one point only* [22].

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let some straight line ΕΖ meet the hyperbola Γ and continued both ways fall outside the section.

I say that it will meet both hyperbolas Α and Β at one point only.

[Proof]. For let ΗΘ and ΚΛ be asymptotes of the hyperbolas. Therefore [according to Proposition II.3] ΕΖ meets both ΗΘ and ΚΛ. Then it is evident that it will {according to Proposition II.16] also meet the hyperbolas Α and Β at one point only.

[Proposition] 19

*If some straight line is drawn touching one of the conjugate opposite hyperbolas at random, it will meet the adjacent hyperbolas and will be bisected at the point of contact* [23].

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let some straight line ΕΓΖ touch it at Γ.

I say that continued it will meet the hyperbolas Α and Β and will be bisected at Γ.

It is evident now that it will [according to Proposition II.18] meet the hyperbolas Α and Β, let it meet them at Η and Θ.

I say that ΓΗ is equal to ΓΘ.

[Proof]. For let the asymptotes of the hyperbolas ΚΛ and ΜΝ be drawn. Therefore [according to Proposition II.16] ΕΗ is equal to ΖΘ, and [according to Proposition II.3] ΓΕ is equal to ΓΖ, and ΓΗ is equal to ΓΘ.

[Proposition] 20

*If a straight line touches one of conjugate opposite hyperbolas, and two straight lines are drawn through their center, one through the point of contact, and one parallel to the tangent until it meet one of the adjacent hyperbolas, then the straight line touching the section at the point of meeting will be parallel to the straight line drawn through the point of contact and the center, and those through the point of contact and the center will be conjugate diameters of the opposite hyperbolas* [24].

Let there be conjugate opposite hyperbolas whose conjugate diameters are ΑΒ and ΓΔ, and center ΧΧ, and let ΕΖ be drawn touching the hyperbola Α, and continued let it meet ΓΧ at Τ, and let ΕΧ be joined and continued to Ξ, and through Χ let ΧΗ be drawn parallel to ΕΖ, and through Η let ΘΗ be drawn touching the section.

I say that ΘH is parallel to XE, and HO and EΞ are conjugate diameters.

[Proof]. For let KE, HΛ, and ΓPΠ be drawn as ordinates, and let AM and ΓN be the *latera recta*. Since then [according to Proposition I.60] as BA is to AM, so NΓ is to ΓΔ, but [according to Proposition I.37] as BA is to AN, so pl.XKZ is to sq.KE, and as NΓ is to ΓΔ, so sq.HΛ is to pl.XΛΘ, therefore also as pl.XKZ is to sq.EK, so sq.HΛ is to pl.XΛΘ.

But the ratio pl.XKZ to sq.EK is compounded of [the ratios] XK to KE and ZK to KE, and the ratio sq.HΛ to pl.XΛΘ is compounded of [the ratios] HΛ to ΛX and HΛ to ΛΘ, therefore the ratio compounded of [the ratios] XK to KE and ZK to KE is the same ratio compounded of [the ratios] HΛ to ΛX and HΛ to ΛΘ, and of these as ZK is to KE, so HL is to ΛX, for each of EK, KZ, and ZE is parallel to each of XΛ, ΛH , and HX, respectively.

Therefore as remainder XK is to KE, so HΛ is to ΛΘ.

Also the sides of equal angles at K and L are proportional, therefore the triangle EKX is similar to the triangle HΘΛ, and will have equal angles corresponding to the subtend sides.

Therefore the angle EXK is equal to the angle ΛHΘ.

But also the angle KXH is equal to the angle ΛHX, and therefore the angle EXH is equal to the angle ΘHX. Therefore EX is parallel to HΘ.

Then let it be contrived that as ΠH is to HP, so ΘH is to Σ, therefore Σ is the half of the *latus rectum* of the ordinates to the diameter HO in hyperbolas Γ and Δ [according to Proposition I.51]. Since ΓΔ is the second diameter of the hyperbolas A and B, and ET meets it, therefore pl.TX,EK is equal to sq.ΓX for if we draw from E a parallel to KX, the rectangular plane under TX and the straight line cut off by the parallel will [according to Proposition I.38] be equal to sq.ΓX.

And therefore [according to Proposition VI.20 of Euclid] as TX is to EK, so sq.TX is to sq.XΓ.

But as TX is to EK, so TZ is to ZE or [according to Proposition VI.1 of Euclid] as TX is to EK, so the triangle TXZ is to the triangle EZX, and [according to Proposition VI.19 of Euclid] as sq.TX is to sq.ΓX, so the triangle XTZ is to the triangle XΓΠ or [according to Proposition II.1] as sq.TX is to sq.ΓX, so the triangle XTZ is to the triangle HΘX. Therefore as the triangle TXZ is to the triangle EZX, so the triangle TZX is to the triangle XHΘ.

Therefore the triangle HΘX is equal to the triangle XEZ. But they also have the angle ΘHX is equal to the angle XEZ for EX is parallel to HΘ, and EZ to HX. Therefore the sides of the equal angles [according to Proposition VI.15 of Euclid] are reciprocally proportional. Therefore as HΘ is to EX, so EZ is to HX, therefore pl.ΘΗΧ is equal to pl.ΧΕΖ. And since as Σ is to ΘΗ, so ΡΗ is to ΗΠ, and as ΡΗ is to ΗΠ, so ΧΕ is to ΕΖ for they are parallel, therefore also as Σ is to ΘΗ, so ΧΕ is to ΕΖ.

But with ΧΗ taken as common height, as Σ is to ΘΗ , so pl.Σ,ΧΗ is to pl.ΘΗΧ, and as ΧΕ is to ΕΖ, so sq.ΧΕ is to pl.ΧΕΖ. And therefore as pl.Σ,ΧΗ is to pl.ΘΗΧ, so sq.ΧΕ is to pl.ΧΕΖ.

Alternately as pl.Σ,ΗΧ is to sq.ΕΧ, so pl.ΘΗΧ is to pl.ΖΕΧ.

But pl.ΘΗΧ is equal to pl.ΧΕΖ, therefore also pl.Σ,ΗΧ is equal to sq.ΕΧ.

And pl.Σ,ΗΧ is the quarter of the *eidos* corresponding to ΗΟ for ΗΧ is equal to the half of ΗΟ, and Σ is the *latus rectum*, sq.ΕΧ is equal to the quarter of sq.ΕΞ for ΕΧ is equal to ΧΞ.

Therefore sq.ΕΧ is equal to the *eidos* corresponding to ΗΟ. Then likewise we could show also that ΗΟ is equal in square to the *eidos* corresponding to ΕΞ. Therefore ΕΞ and ΗΟ are conjugate diameters of the opposite hyperbolas Α, Β, Γ, and Δ.

## [Proposition] 21

*Under the same supposition it is to be shown that the point of meeting of the tangents is on one of the asymptotes* [25].

Let there be conjugate opposite hyperbolas, whose diameters are ΑΒ and ΧΔ, and let ΑΕ and ΕΓ be drawn tangent.

I say that Ε is on the asymptote.

[Proof]. For since sq.ΓΧ is equal to the quarter of the *eidos* corresponding to ΑΒ [according to Proposition I.60], and [according to Proposition II.17] sq.ΑΕ is equal to ΓΧ,therefore also sq.ΑΕ is equal to the quarter of the *eidos* corresponding to ΑΒ. Let ΕΧ be joined, therefore [according to Proposition II.1] ΕΧ is an asymptote, therefore [the point] Ε is on the asymptote.

## [Proposition] 22

If in conjugate opposite hyperbolas a radius is drawn to any of the hyperbolas, and a parallel is drawn to it meeting one of adjacent hyperbolas and meeting the asymptotes, then the rectangular plane under the segments continued between the section and the asymptotes on the straight line drawn is equal to the square on the radius[26].

Let there be conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let there be the asymptotes of these hyperbola ΧΕΖ and ΧΗΘ, and from the center Χ let some straight line ΧΓΔ be drawn across, and let ΘΕ be drawn parallel to it cutting both adjacent hyperbolas and the asymptotes.

I say that pl.ΕΚΘ is equal to sq.ΓΧ.

[Proof]. Let ΚΛ be bisected at Μ, and let ΜΧ be joined and continued therefore ΑΒ is the diameter of the hyperbolas Α and Β [according to the porism to Proposition I.51]. And since the tangent at Α [according to Proposition II.5] is parallel to ΕΘ, therefore ΕΘ [according to Proposition I.17] has been dropped as an ordinate to ΑΒ. And center is Χ, therefore ΑΒ and ΓΔ are conjugate diameter [according to Definition 6] .Therefore sq.ΓΧ [according to Proposition I.60] is equal to the quarter of the *eidos* corresponding to ΑΒ. And pl.ΘΚΕ [according to Proposition II.10] is equal to the quarter of the *eidos* corresponding to ΑΒ, therefore also pl.ΘΚΕ is equal to sq.ΓΧ.

[Proposition] 23

*If in conjugate opposite hyperbolas some radius is drawn to any of the hyperbola, and a parallel is drawn to it meeting three adjacent hyperbolas, then the rectangular plane under the segments continued between the three hyperbolas on the straight line drawn is twice the square on the radius*[27].

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let the center of the section be Χ, and from Χ let some straight line ΓΧ be drawn to meet any one of the hyperbolas, and let ΚΛ be drawn parallel to ΓΧ cutting three adjacent hyperbolas.

I say that pl.ΚΜΛ is equal to double sq.ΓΧ.

[Proof]. Let the asymptotes to the hyperbolas, ΕΖ and ΗΘ, be drawn, therefore [according to Proposition II.22] sq.ΓΧ is equal to pl.ΘΜΕ and [according to Proposition II.11] is equal to pl.ΘΚΕ. And the sum of pl.ΘΜΕ and pl.ΘΚΕ is equal to pl.ΛΜΚ because of the straight lines on the ends[according to Propositions II.8 and II.16] being equal. Therefore also pl.ΛΜΚ is equal to double sq.ΓΧ.

[Proposition] 24

*If two straight lines meet a parabola each at two points, and if a point of meeting of neither one of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section* [28].

Let there be the parabola ΑΒΓΔ, and let ΑΒ and ΓΔ meet ΑΒΓΔ, and let a point of meeting of neither of them be contained by the points of meeting of the other.

I say that the straight lines continued will meet each other.

[Proof]. Let the diameters of the section, ΕΒΖ and ΗΓΘ, be drawn through Β and Γ,therefore [according to the porism to Proposition I.51] they are parallel and each one cut the section [according to Proposition I.26] at one point only. Then let ΒΓ be joined, therefore the sum of the angle ΕΒΓ and ΒΓΗ is equal to two right angles, and ΔΓ and ΒΑ continued make the angles less than two right angles. Therefore [according to Proposition I,10, and Euclid's Postulate 5] they will meet each other outside the section.

## [Proposition] 25

*If two straight lines meet a hyperbola each at two points, and if a point of meeting of neither of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section, but within the angle containing the section* [29].

Let there be a hyperbola whose asymptotes are ΑΒ and ΑΓ, and let ΕΖ and ΗΘ cut the section, and let a point of meeting of neither of them be contained by the points of meeting of the other.

I say that ΕΖ and ΗΘ continued will meet outside the section, but within the angle ΓΑΒ.

[Proof]. For let ΑΖ and ΑΘ be joined and continued and let ΖΘ be joined. And since ΕΖ and ΗΘ continued cut the angles ΑΖΘ and ΑΘΖ, and mentioned angles [according to Proposition I.17 of Euclid] are less than two right angles, and ΕΖ and ΗΘ continued will meet each other outside the section but within the angle ΒΑΓ.

Then we could likewise show it, even if ΕΖ and ΗΘ are tangents to the sections.

## [Proposition] 26

*If in an ellipse and in the circumference of a circle two straight lines not through the center cut each other, then they do not bisect each other* [30].

[Proof]. For, if possible, in the ellipse for in the circumference of a circle let ΓΔ and ΕΖ not through the center bisect each other at Η and let Θ be the center of the section, and let ΗΘ be joined and continued to Α and Β.

Since then AB is a diameter bisecting EZ, therefore [according to Proposition II.6] the tangent at A is parallel to EZ. We could then likewise show that it also parallel to ΓΔ. And so also EZ is parallel to ΓΔ. And this is impossible. Therefore ΓΔ and EZ do not bisected each other.

[Proposition] 27

*If two straight lines touch an ellipse or circumference of a circle, and if the straight line joining the points of contact is through the center of the section, the tangents will be parallel, but if not, they will meet on the same side of the center* [31].

Let there be the ellipse or the circumference of a circle AB, and let ΓAΔ and EBZ touch it, and let AB be joined, and first let it be through the center.

I say that ΓΔ is parallel to EZ.

[Proof]. For since AB is a diameter of the section, and ΓΔ touches it at A, therefore [according to Proposition I.17] ΓΔ is parallel to the ordinates to AB. Then or the same reasons BZ is also parallel to same ordinate. Therefore ΓΔ is also parallel to EZ.Then let AB not be through the center as in the second diagram, and let the diameter AΘ be drawn, and let KΘΛ be drawn tangent through Θ, therefore KΛ is parallel to ΓΔ. Therefore EZ continued will meet ΓΔ on the same side of the center as AB.

[Proposition] 28

*If in a section of a cone or in the circumference of a circle some straight line bisects two parallel straight lines, then it will a diameter of the section* [32].

Let AB and ΓΔ, two parallel straight lines in a conic section, bisected at E and Z, and let EZ be joined and continued.

I say that it is a diameter of the section.

[Proof]. For if not, let HZΘ be so if possible. Therefore the tangent at H [according to Proposition II.5 and II,6] is parallel to AB. And so the same straight line is parallel to ΓΔ. And HΘ is a diameter, therefore [according to Definition 4] ΓΘ is equal to ΘΔ, and this is impossible for it is supposed that ΓE is equal to EΔ. Therefore HΘ is not a diameter. Then likewise we could show that there is no other except EZ. Therefore EZ will be a diameter of the section.

[Proposition] 29

If in a section of a cone or in the circumference of a circle two tangents meet, the straight line, drawn from their t point of meeting to the midpoint of the straight line joining the points of contact is a diameter of the section [33] .

Let there be a section of a cone or the circumference of a circle to which let ΑΒ and ΑΓ , meeting at Α, be drawn tangent, and let ΒΓ be joined and bisected at Δ, and let ΑΔ be joined.

I say that it is a diameter of the section.

[Proof]. For, if possible, let ΔΕ be a diameter, and let ΕΓ be joined, then it will cut the section [according to Propositions I.5 and I.36]. Let it cut it at Ζ, and through Ζ let ZKH be drawn parallel to ΓΔΒ. Since then ΓΔ is equal to ΔΒ, also ΖΘ is equal to ΘΗ.

And since the tangent at Λ is parallel to ΒΓ [according to Propositions II.5 and II.6], and ΖΗ is also parallel to ΒΓ, therefore also ΖΗ is parallel to the tangent at Λ. Therefore [according to Propositions I.46 and I.47] ΖΘ is equal to ΘΚ, and this is impossible. Therefore ΔΕ is not a diameter. Then likewise we could show that there is no other except ΑΔ.

[Proposition] 30

*If two straight lines tangent to a section of a cone or to the circumference of a circle meet, the diameter drawn from the point of meeting will bisect the straight line joining the points of contact* [34].

Let there be the section of a cone or the circumference of a circle ΒΓ, and let two tangents ΒΑ and ΑΓ be drawn to their meeting at Α, and let ΒΓ be joined, and let ΑΔ be drawn through Α as a diameter of the section.

I say that ΔΒ is equal to ΔΓ.

[Proof]. For let it not be, but, if possible, let ΒΕ be equal to ΕΓ, and let ΑΕ be joined, therefore [according to Proposition II.29] ΑΕ is a diameter of the section. But ΑΔ it also the diameter, and this is impossible.

For if the section is an ellipse, Α at which the diameters meet each other, will be a center outside the section, and this is impossible, and if the section is a parabola the diameters [according to the porism to Proposition I.51] meet each other, and if is a hyperbola, and ΒΑ and ΑΓ meet the section without containing one another' points of meeting, then the center is within the angle containing the hyperbola [according to Proposition II.25], but it is also on it for it has been supposed a center since ΔΑ and ΑΕ are diameter [according to the porism to Proposition I.51] and this is impossible. Therefore ΒΕ is not equal to ΕΓ.

[Proposition] 31

*If two straight line touch each of the opposite hyperbolas, then if the straight line joining the points of contact falls through the center, the tangents will be parallel, but if not, they will meet on the same side as the center* [35] .

Let there be the opposite hyperbolas Α and Β, and let ΓΑΔ and ΕΒΖ be tangent to them at Α and Β, and let the straight line joined from Α to Β fall first through the center of the hyperbola.

I say that ΓΔ is parallel to ΕΖ.

[Proof]. For since they are opposite hyperbolas for which ΑΒ is a diameter, and ΓΔ touches one of them at Α, therefore the straight line drawn through Β parallel to ΓΔ [according to Proposition I.44] touches the section. But ΕΖ also touches it, therefore ΓΔ is parallel ΕΖ.

Then let the straight line from Α to Β not be through the center of the hyperbolas, and let ΑΗ be drawn as a diameter of the hyperbolas, and let ΘΚ be tangent to the section, therefore ΘΚ is parallel to ΓΔ, and since ΕΖ and ΘΚ touch a hyperbola, therefore they [according to Proposition II.25] will meet. And ΘΚ is parallel to ΓΔ, therefore also ΓΔ and ΕΖ continued will meet. And it is evident that they are on the same side as the center.

[Proposition] 32

*If straight lines meet each of the opposite hyperbolas, at one point when touching or at two points when cutting, and, when continued, the straight lines meet, then their point of meeting will be in the angle adjacent to the angle containing the hyperbola*[36].

Let there be opposite hyperbolas and ΑΒ and ΓΔ either touching the opposite hyperbolas at one point or cutting them at two points, and let them meet when continued.

I say that their point of meeting will be in the angle adjacent to the angle containing the section.

[Proof]. Let ΖΗ and ΘΚ be asymptotes to the hyperbolas, therefore ΑΒ continued [according to Proposition II.8] will meet the asymptotes. Let it meet them at Θ and Η. And since ΖΚ and ΘΗ are supposed as meeting, it is evident that either they will meet in the place under the angle ΘΛΖ or in that under the angle ΚΛΗ. Likewise also if they touch [according to Proposition II.3].

[Proposition] 33

*Let them be the opposite hyperbolas A and B, and let some straight line ΓΔ cut A, and, when continued both ways, let it fall outside the section* [37].

I say that ΓΔ does not meet the hyperbola B.

[Proof]. For let EZ and HΘ be drawn as asymptote to the hyperbolas, therefore ΓΔ continued will meet [according to Proposition ii.8] the asymptotes. And it only meets them at E and Θ. And so it will not meet the hyperbola B.

And it is evident that it will fall through three places. For if some straight line meets both of opposite hyperbolas it will meet neither of opposite hyperbolas at two points. For it meets it at two points, by what has just been proved it will not meet the other hyperbola.

## [Proposition] 34

*If some straight line touch one of opposite hyperbolas and a parallel to it be drawn in the other hyperbola, then the straight line drawn from the point of contact to the midpoint of the parallel will be a diameter of the opposite hyperbolas*[38].

Let there be the opposite hyperbolas A and B,and let some straight line ΓΔ touch one of them A at A, and let EZ be drawn parallel to ΓΔ in the other hyperbola, and let it be bisected at H, and let AH be joined.

I say that AH is a diameter of the opposite hyperbolas.

[Proof]. For, if possible, let AΘK be [a diameter] therefore the tangent at Θ is parallel to ΓΔ [according to Proposition II.31]. But ΓΔ is also parallel to EZ, and therefore the tangent at Θ is parallel to EZ. Therefore [according to Proposition I.47] EK is equal to KZ, and this is impossible for EH is equal to HZ. Therefore AΘ is not a diameter of the opposite hyperbolas. Therefore AB is [a diameter].

## [Proposition] 35

*If a diameter in one of opposite hyperbola bisects some straight line, the straight line touching the other hyperbola at the end of the diameter will be parallel to the bisected straight line* [39] .

Let there be the opposite hyperbolas A and B, and let their diameter AB bisect ΓΔ in hyperbola B at E.

I say that the tangent the hyperbola [A] at A is parallel to ΓΔ.

[Proof]. For, if possible, let ΔZ be parallel to the tangent to the hyperbola at A, therefore [according to Proposition I.48] ΔH is equal to HZ.

But also ΔE is equal to EΓ. Therefore ΓZ is parallel to EH, and this is impossible for continued it [according to Proposition I.22] meets it. Therefore ΔZ is not parallel to the tangent to the hyperbola at A, nor is any other straight line except ΓΔ.

## [Proposition] 36

*If parallel straight lines are drawn, one in each of opposite hyperbolas, then the straight line joining their midpoints will be a diameter of the opposite hyperbolas* [40].

Let there be the opposite hyperbolas A and B, and let ΓΔ and EZ be drawn, one in each of them, and let them be parallel, and let them both be bisected at H and Θ, and let HΘ be joined.

I say that HΘ is a diameter of the opposite hyperbolas.

[Proof]. For if not, let HK be one [diameter]. Therefore the tangent to A [according to Proposition II.5] is parallel to ΓΔ, and so also to EZ. Therefore [according to Proposition I.48] EK is equal to KZ, and this is impossible since also EΘ is equal to ΘZ. Therefore HK is not a diameter of the opposite hyperbolas. Therefore HΘ is [the diameter].

## [Proposition] 37

*If a straight line not through the center cuts the opposite hyperbolas, then the straight line joined from its midpoint to the center is a so-called upright diameter of the opposite hyperbolas, and the straight line drawn from the center parallel to the bisected straight line is a transverse diameter conjugate to it* [41].

Let there be the opposite hyperbolas A and B let some straight line ΓΔ not through the center cut the hyperbola A and B and let it be bisected at E, and let X be the center of the hyperbolas, and let XE is joined, and through X let AB be drawn parallel to ΓΔ.

I say that AB and EX are conjugate diameters of the hyperbolas.

[Proof]. For let ΔX be joined and continued to Z, and let ΓZ be joined. Therefore [according to Proposition I.30] ΔX is equal to XZ. But also ΔE is equal to EΓ. Therefore EX is parallel ZΓ. Let BA be continued to H. And since ΔX is equal to XZ, therefore also EX is equal to ZH, and so also ΓH is equal to ZH. Therefore the tangent at A [according to Proposition II.5] is parallel to ΓZ, and so also to EX. Therefore EX and AB [according to Proposition I.16] are conjugate diameter.

[Proposition] 38

*If two straight lines meeting touch opposite hyperbolas, the straight line joined from the point of meeting to the midpoint of the straight line joining the points of contact will be a so-called upright diameter of the opposite hyperbolas and the straight line drawn through center parallel to the straight line joining of contact is a transverse diameter conjugate to it* [42]

Let there be the opposite hyperbolas A and B, and ΓX and XΔ touching the hyperbolas, and let ΓΔ be joined and bisected at E, and let EX be joined.

I say that the diameter EX is a so-called upright diameter, and the straight line drawn through the center parallel to ΓΔ is a transverse diameter conjugate to it.

[Proof]. For, if possible, let EZ be a diameter, and let Z be a point taken at random, therefore ΔX will meet EZ. Let it meet it at Z, and let ΓZ be joined, therefore [according to Proposition I.32] ΓZ will hit the hyperbola. Let it hit it as A, and through A let AB be drawn parallel to ΓΔ. Since then EZ is a diameter, and bisects ΓΔ, it also bisects [according to Definition 4] the parallels to it. Therefore AH is equal to HB. And since ΓE is equal to EΔ, and is on the triangle ΓEΔ, therefore also AH is equal to HK. And so also HK equal to HB, and this is impossible. Therefore EZ will be a diameter.

[Proposition] 39

*If two straight line meeting touch opposite hyperbolas, the straight line drawn through the center and the point of meeting of the tangents bisects straight line joining the points of contact* [43].

Let there be the opposite hyperbolas A and B, and let ΓE and EΔ be drawn touching A and B, and let ΓΔ be joined, and let EZ be drawn as a diameter.

I say that ΓZ is equal to ZΔ.

[Proof]. For if not, let ΓΔ be bisected as H, and let HE be joined, therefore HE [according to Proposition II.38] is [a diameter]. But EZ is also [a diameter], therefore [according the porism to Proposition I.31] E is the center. Therefore the point of meeting of the tangents is at the center of the hyperbolas, and this [according to Proposition II.32] is impossible.

Therefore, ΓZ is not unequal to ZΔ. Therefore [they are] equal.

[Proposition] 40

*If two straight lines touching opposite hyperbolas meet, and trough the point of meeting a straight line drawn parallel to straight line joining the points of contact, and meeting the hyperbolas, then the straight lines drawn from the points of meeting to the midpoint of the straight line joining the point of contact touch the hyperbolas* [44].

Let there be the opposite hyperbolas A and B, and let ΓE and EΔ be drawn touching A and B, and let ΓΔ be joined, and through E let ZEH be drawn parallel to ΓΔ, and let ΓΔ be bisected at Θ, and let ZΘ and ΘH be joined.

I say that ZΘ and ΘH touch the hyperbolas.

[Proof]. Let EΘ be joined, therefore EΘ is an upright diameter, and the straight line drawn through the center parallel to ΓΔ [according to Proposition II.38] is a transverse diameter conjugate to it. And let the center X be taken, and let AXB be drawn parallel to ΓΔ, Therefore ΘE and AB are conjugate diameter. And ΓΘ has been drawn as an ordinate to the second diameter, and ΓE has been drawn touching the section and meeting the second diameter. Therefore pl.EXΘ is equal to the square on the half of the second diameter [according to Proposition I.38], which is to the quarter of the *eidos* corresponding to AB. And since ZE has been drawn as an ordinate and ZΘ joined, therefore [according to Proposition I.38] ZΘ touches the hyperbola A. Likewise then also HΘ touches the hyperbola B. Therefore ZΘ and ΘH touch the hyperbolas A and B.

[Proposition] 41

*If in opposite hyperbolas two straight lines not through the center cut each to other, then they do not bisect each other*[45].

Let there be the opposite hyperbolas A and B, and in A and B let ΓB and AΔ not through the center cut each other at E.

I say that they do not bisect each other.

[Proof]. For if possible, let them bisect each other, and let X be the center of the hyperbolas, and let EX is be joined, therefore [according to Proposition II.37] EX is a diameter. Let XZ be drawn through X parallel to BΓ, therefore XZ is a diameter conjugate to EX and [according to Proposition II.37]to EX. Therefore the tangent at Z is parallel to EX [according to Definition 6].Then for the same reasons with ΘK drawn parallel to AΔ, the tangent at Θ is parallel to EX, and so also the tangent at Z is parallel to the tangent at Θ, and this is impossible for it has been shown [in Proposition II.31] that is it also meets it. Therefore ΓΒ and ΑΔ not through the center do not bisect each other.

[Proposition] 42

*If in conjugate opposite hyperbolas two straight lines not through the center cut each to other, then they do not bisect each other*[46] .

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and in Α, Β, Γ, and Δ let two straight lines not through the center, ΕΖ and ΗΘ, cut each other at Κ.

I say that they do not bisect each other.

[Proof]. For, if possible, let them bisect each other, and let the center of the hyperbola be Χ, and let ΑΒ be drawn parallel to ΕΖ and ΓΔ [parallel] to ΘΗ, and let ΚΧ be joined, therefore [according to Proposition II.37] ΚΧ and ΑΒ are conjugate diameters. Likewise ΧΚ and ΓΔ are also conjugate diameter. And so also the tangent at Α is parallel to the tangent at Γ, and this is impossible for it meets it, since the tangent at Γ [according to Proposition II.19] cuts the hyperbolas Α and Β, and the tangent at Α [cuts] the hyperbolas Γ and Δ, it is evident also that their point of meeting [according to Proposition II.21] is in the place under the angle ΑΧΓ. Therefore ΕΖ and ΗΘ not through the center do not bisect each other.

[Proposition] 43

*If a straight line cuts one of conjugate opposite hyperbolas at two points, and through the center one straight line is drawn to the meet point of the cutting straight line, and another straight line is drawn parallel to the cutting straight line, they will be conjugate diameter of the opposite hyperbolas*[47].

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let some straight line cut the hyperbola Α at two points Ε and Ζ, and let ΖΕ be bisected at Η, and let Χ be the center, and let ΧΗ be joined, and let ΓΧ be drawn parallel to ΕΖ.

I say that ΑΧ and ΧΓ are conjugate diameters.

[Proof]. For since ΑΧ is a diameter, and bisects ΕΖ, the tangent at Α [according to Proposition II.5] is parallel to ΕΖ, and so also to ΓΧ. Since then they are opposite hyperbolas, and a tangent has been drawn to one of them, Α at Α, and from the center Χ one straight line ΧΑ is joined to the point of contact, and another ΓΧ has been drawn parallel to the tangent, therefore ΧΑ and ΓΧ are conjugate diameter for this has been shown before [in Proposition II.20].

[Proposition] 44 [Problem]

*Given a section of a cone, to find a diameter* [48].

Let there be the given conic section on which are the point Α, Β, Γ, Δ, and Ε. Then it is required to find a diameter.

[Solution]. Let it have been done, and let it be ΓΘ than with ΔΖ and ΕΘ drawn as ordinates and continued ΔΖ is equal to ΖΒ, and ΕΘ is equal to ΘΑ.

If then we fix ΒΔ and ΕΑ in position to be parallel, the points Θ and Ζ will be given. And so ΘΖΓ will be given in position.

Then the synthesis[49] to this problem is as follows. Let there be the given conic section on which are the points Α, Β, Γ, Δ, and Ε, and let ΒΔ and ΑΕ be drawn parallel and bisected at Ζ and Θ. And ΖΘ joined will be [according to Proposition II.28] a diameter of the section. And in the same way we could also find an indefinite number of diameter.

[Proposition] 45 [Problem]

*Given an ellipse or a hyperbola, to find the center*[50].

And this is evident: for if two diameters of the section ΑΒ and ΓΔ, are drawn [according to Proposition II.44] through point at which they cut each other will be the center of the section, as indicated.

[Proposition] 46 [Problem]

*Given a section of a cone, to find the axis* [51].

Let the given section if a cone first be a parabola, on which are the point Ζ, Γ, and Ε. Then it is required to find its axis.

[Solution]. For let ΑΒ be drawn as a diameter of it [according to Proposition II.44]. If then ΑΒ is an axis, what was enjoined would have been done, but it not, let it have been done, and let ΓΔ be the axis: therefore the axis ΓΔ is parallel to ΑΒ [according to the porism to Proposition I.51] and bisects the straight lines drawn perpendicular to it[according to Definition 7] And the perpendiculars to ΓΔ are also perpendiculars to ΑΒ, and so ΓΔ bisects the perpendicular to ΑΒ. If then we fix ΕΖ, a perpendicular to ΑΒ, it will be given in position, and therefore ΕΔ is equal to ΔΖ, therefore Δ is given.

Therefore through the given point Δ, ΓΔ has been drawn parallel to ΑΒ, which is given in position, therefore ΓΔ is given in position.

Then  the synthesis of this problem is as follows. Let there be parabola on which are points Z, E, and A, and let AB, a diameter of it, be drawn [according to Proposition II.44] and let BE be drawn perpendicular to it, and let it be continued to Z. If then EB is equal to BZ, it is evident that AB is the axis [according to Definition 7], but if not, let EZ be bisected at Δ and let XΔ be drawn parallel to AB. Then it is evident that XΔ is the axis of the section for it is parallel to the diameter it is also a diameter it bisects EZ at right angles. Therefore ΓΔ has been found as the axis of the given parabola.

And it is evident that the parabola has one only axis for if there is another as AB, it will be parallel to ΓΔ [ according the porism to Proposition I.51]. And its cuts EZ and so it also bisects it [according to Definition 4].

Therefore BE is equal to BZ, and this is impossible.

[Proposition] 47 [Problem]

*Given a hyperbola or an ellipse, to find the axis* [52] .

Let there be the hyperbola or the ellipse ABΓ, then it is required to find its axis.

[Solution]. Let it have been found, and let it be KΔ, and K the center of the section, therefore KΔ bisects the ordinates to it and at right angles [according to Definition 7].

[Solution]. Let the perpendicular ΓΔA be drawn, and let KA and KΓ be joined. Since then ΓΔ is equal to ΔA, therefore ΔK is equal to KA.

If then we fix the given point Γ, ΓK will be given. And so the circle described, ΓK will be given. And so the circle  with the center K and the radius KΓ will also pass through A and will be given in position. And the section ABΓ is also given in position, therefore A is given. But Γ is also given, therefore ΓA is given in position. Also ΓΔ is equal to ΔA, therefore Δ is given. But also is given, therefore ΔK is given in position.

Then  the synthesis of thus: problem is as follows.  Let there be given the hyperbola or the ellipse ABΓ, and let K be taken as its center, and let a point be taken as random on the section, and let the circle ΓEA with the center K and the radius KΓ be described, and let ΓA be joined and bisected at Δ, and let KΓ, KD, and KA be joined, and let KΔ be drawn through B.

Since then  AΔ is equal ΔΓ, and ΔK is common, therefore ΓΔ and ΔK are equal to AΔ and ΔK, and the base KA is equal to the base KΓ. Therefore KBΔ bisects AΔΓ at right angles. Therefore KΔ is an axis [according to Definition 7], Let MKN be drawn through K parallel to ΓA, therefore MN air the axis of the hyperbola conjugate to BK [according to Definition 8].

[Proposition] 48 [Problem]

*Then with these reasons shown, let it be next in order to show that there are no other axes of the same section*[53].

[Solution]. For, if possible, let there also be another axis KH. Then in the same way as before with AΘ drawn perpendicular [according to Definition 4] AΘ is equal to ΘΛ and so also AK is equal to KΛ. But also AK is equal to KΓ, therefore KΛ is equal to KΓ, and this is impossible.

Now that the circle AEΓ does not hit the section also at another point between A, B, and Γ, is evident in the case of the hyperbola, and in the case of the ellipse the perpendiculars ΓP and ΛΣ be drawn. Since then KΓ is equal to KΛ for they are radii, also sq.KΓ is equal to sq.KΛ. But the sum of sq.ΓP and sq.PK is equal to sq.ΓK, therefore the sum sq.ΓP and sq.PK is equal to the sum sq.KΣ and sq.ΣΛ.

Therefore the difference between sq.ΓP and sq.ΣΛ is equal to the difference between sq.KΣ and sq.PK.

Again since the sum pl.MPN and sq.PK is equal to sq.KM, and also [according to Proposition II.5 of Euclid] the sum pl.MΣN and sq.ΣK is equal to sq.KM, therefore the sum pl.MPN and sq.PK is equal to the sum pl.MΣN and sq.ΣK. Therefore the difference between sq.ΣK and sq.KP is equal to the difference between pl.MPN and pl.MΣN.

And it was shown that the difference between sq.ΣK and sq.KP is equal to the difference between sq.ΓP and sq.ΣΛ, therefore the difference between sq.ΓP and sq.ΣΛ is equal to the difference between pl.MPN and pl.MΣN. And since ΓP and ΛΣ are ordinates [according to Proposition I.21] as sq.ΓP is to pl.MPN, so sq.ΣΛ is to pl.MΣN.

But the same difference was also shown for both, therefore sq.ΓP is equal to pl.MPN, and [according to Propositions V.9, V.16, and V.17 of Euclid] sq.ΣΛ is equal to pl.M

Therefore the line ΛΓM is a circle and this is impossible for it is supposed an ellipse.

[Proposition] 49 [Problem]

*Given a section of a cone and a point both with in the section, to draw from this point a straight line touching the section* [54] .

Let the given section of a cone first a parabola whose axis is ΒΔ. Then it is required to draw a straight line as prescribed from the given point that is not within the section.

Then the given point is either on the line or on the axis or somewhere else outside.

Now let it be on the line, and let it be Α, and let it have been done, and let it be ΑΕ, let ΑΔ be drawn perpendicular, then it will be given in position. And [according to Proposition I.35] ΒΕ is equal to ΒΔ, and ΒΔ is given, therefore ΒΕ is also given. And Β is given, therefore Ε is also given. But Α also [is given], therefore ΑΕ is given in position.

Then the synthesis of this problem is as follows. Let ΑΔ be drawn perpendicular from Α, and let ΒΕ be made equal to ΒΔ, and let ΑΕ be joined.
Then it is evident that it [according to Proposition I.33] touches the section.

Again let the given point Ε be on the axis, and let it have been done, and let ΑΕ be drawn tangent, and let ΑΔ be drawn perpendicular, therefore [according to Proposition I.35] ΒΕ is equal to ΒΔ. And ΒΕ is given, therefore also ΒΔ is given. And Β is given, therefore Δ is also given. And ΔΑ is perpendicular, therefore ΔΑ is given in position. Therefore Α is given. But also Ε [is given], therefore ΑΕ is given in position.

Then the synthesis of this problem is as follows. Let ΒΔ be made equal to ΒΕ, and from Δ let ΔΑ be drawn perpendicular to ΕΔ, and let ΑΕ be joined.

Then it is evident that ΑΕ touches [according to Proposition I.33].

And it is evident also that, even if the given point is the same as Β, the straight line drawn from Β perpendicular touches the section [according to Proposition I.17].

Then let Γ be let the given point, ad let it have been done, and let ΓΑ be it, and through Γ let ΓΖ be drawn parallel to the axis, that is to ΒΔ, therefore ΓΖ is given in position. And from Α let ΑΖ be drawn as an ordinate to ΓΖ, then [according to Proposition I.35] ΓΗ is equal to ΖΗ. And Η is given, therefore Ζ is also given. And ΖΑ has been erected as an ordinate, which is parallel to the tangent as Η [according to Proposition I.32], therefore ΖΑ is given in position. Therefore Α is also given, but also Γ [is given]. Therefore ΓΑ is given in position.

Then the synthesis of this problem is as follows. Let ΓΖ be drawn through Γ parallel to ΒΔ, and let ΖΗ be made equal to ΓΗ, and let ΖΑ be drawn parallel to the tangent at Η, and let ΑΓ be joined. It is evident then that this will do the problem [according to Proposition I.33].

Again let it be a hyperbola whose axis is ΔΒΓ and center Θ, and asymptotes ΘΕ an ΘΖ. Then the given point will be given either on the section or on the axis or within the angle EΘZ or in the adjacent place or on one of the asymptotes containing the section or in the place between the straight lines containing the angle vertical to the angle EΘZ.

Let A first be on the section, and let it have been done, and let AH be tangent, and let AΔ be drawn perpendicular, and let BΓ be the *latus transversum* of the *eidos*, then [according to Propositions I.36] as ΓΔ is to ΔB, so ΓH is to HB. And the ratio ΓΔ to ΔB is given for both these straight lines are given, therefore also the ratio ΓH to ΓB is given. And BΓ is given, therefore H is given. But also A [is given], therefore AH is given in position.

Then the synthesis of this problem is as follows. Let AΔ be drawn perpendicular from A, and let as ΓH is to HB, so ΓΔ is to ΔB, and let AH be joined then it is evident that AH touches the section [according to Proposition I.34].

Then again let the given point H be on the axis, and let it have been done, and let AH be drawn tangent, and let AΔ be drawn perpendicular. Then for the same reason [according to Proposition I.36] as ΓH is to HB, so ΓΔ is to ΔB. And BΓ is given, therefore Δ is given. And ΔA is perpendicular, therefore ΔA is given in Position. And also the section is given in position, therefore A is given. But also H [is given], therefore AH is given in position.

Then the synthesis of this problem is as follows. Let all other be supposed the same, and let it be contrived that as ΓH is to HB, so ΓΔ is to ΔB, and let ΔA be drawn perpendicular, and let AH be joined. Then it is evident AH does the Problem [according to Proposition I.34], and that from H another tangent to the section could be drawn on the other side.

With the same suppositions let the given point K be in the place inside the angle EΘZ, and let it be required to draw a tangent to the section from K. Let it have been done, and it be KA, and let KΘ be joined an continued, and let ΘN be made equal to ΛΘ, therefore they are all given. Then also ΛN will be given. Then let AM be drawn as an ordinate to MN, then also as NK is to KΛ, so MN is to MΛ.

And the ratio NK to KΛ is given, therefore the ratio NM to MΛ is given. And Λ is given, therefore also M is given. And MA has been erected parallel to the tangent at Λ, therefore MA is given in position.

And also the section AΛB is given in position, therefore A is given. But K is also given, therefore AK is given.

Then the synthesis of this problem is as follows. Let all other be supposed the same, and the given point K, and KΘ be joined and continued, and let ΘN be made equal to ΘΛ, and let it be contrived that as NK is to KΛ, so NM is to MΛ, and let MA be drawn parallel to the tangent at Λ, and let KA be joined, therefore [according to Proposition I.34] KA touches the section.

And it is evident that a tangent to the section could also be drawn to the other side.

With the same suppositions the given point Z be on one of the asymptotes containing the section, and let it be required to draw from Z a tangent to the section. And let it have been done, and let it be ZAE, and through A let AΔ be drawn parallel to EΘ, then ΔΘ is equal to ΔZ, since also [according to Proposition II.3] ZA is equal to AE. And ZΘ is given, therefore also Δ is given. And through the given point Δ ΔA has been drawn parallel in position to EΘ, therefore ΔA is given in position. And the section is also given in position, therefore A is given. But Z also given therefore ZAE is given in position.

Then the synthesis of this problem is as follows. Let there be the section AB, and asymptotes EΘ and ΘZ, and the given point Z on one of the asymptotes containing the section, and let ZΘ be bisected as Δ, and through Δ let ΔA be drawn parallel to ΘE and let ZA be joined. And since ZΔ is equal to ΔΘ therefore also ZA is equal to AE.

And so by the shown before [in Proposition II.9] ZAE touches the section.

With the same supposition let the given point be in the place under the angle adjacent to the straight lines containing the section, and let it be K, it is required then to draw a tangent to the section from K. And let it have been done, and let be KA, and let KΘ be joined and continued, then it will be given in position. If then a given point Γ is taken on the section, and through Γ ΓΔ is drawn parallel to KΘ it will be given in position. And if ΓΔ is bisected at E, and ΘE is joined and continued, it will be in position a diameter conjugate to KΘ [according to Definition 6]. Then let ΘH be made equal to BΘ, and through A let AΛ be drawn parallel to BΘ, then because KΛ and BH are conjugate diameters, and AK a tangent, and AΛ a straight line drawn parallel to BH, therefore pl.KΘΛ is equal to the quarter of the *eidos* corresponding to BH [according to Proposition I.38]. Therefore pl.KΘΛ is given. And KΘ is given, therefore ΘΛ is also given. But it is also given in position, and Θ is given, therefore Λ is also given. And through Λ ΛA has been drawn parallel in position to BH, therefore ΛA is given in position. And the section is also given in position, therefore A is given. But also K [is given], therefore AK is given in position.

Then the synthesis is as follows. Let the other supposition be the same, and let the given point K be in the mentioned place, and let KΘ be joined and continued, and let some point ΓZ be taken, and let ΓΔ be drawn parallel to KΘ, and let ΓΔ bisected at E and let EΘ be joined and continued, and ΘH be made equal to BΘ, therefore HB is a transverse diameter conjugate to KΘΛ [according to Definition 6] then let pl.KΘΛ be made equal to the quarter of the *eidos* corresponding to BH, and through Λ let ΛA be drawn parallel to BH, and let KA be joined, then it is clear that KA touches the section according to the converse of the theorem [Proposition I.38].

And if it is given in the place between EΘ and ΘΠ, the problem is impossible for the tangent will cut HΘ. And so it will meet both ZΘ and ΘΠ, and this is impossible according to shown in the theorem 31 of the book I [Proposition I.31] and in the theorem 3 of this book [Proposition II.3].

With the same suppositions let the section be an ellipse and the given point A on the section, and let it be required to draw from A tangent to the section. Let it have been done, and let it be AH, and let AΔ be drawn from A as an ordinate to the axis BΓ, then Δ will be given, and [according to Proposition I.36] as ΓΔ is to ΔB, so ΓH is to ΓB.

And the ratio ΓΔ to ΔB is given, therefore the ratio ΓH to ΓB is also given. Therefore H is given. But also A [is given],therefore AH is given in position.

Then the synthesis of this problem is as follows. Let AΔ be drawn perpendicular, and let as ΓH is to HB, so ΓΔ is to ΔB, and let AH be joined. Then it is evident that AH touches, as also in the case of the hyperbola [according to Proposition I,34].

Then again let the given point be K, and let it be required to draw a tangent. Let it have been done, and let it be KA, and let KΛΘ be joined to the center Θ and continued to N, then will be given in position. And if AM is drawn as an ordinate, then [according to Proposition I.36] as NK is to KΛ, so NM is to MΛ. and the ratio NK to KΛ is given, therefore the ratio MN to ΛM is also given. Therefore M is given. And MA has been erected as an ordinate for it is parallel to the tangent at Λ, therefore MA is given in position. Therefore A is given. But also K [is given], Therefore KA is given in position.

And the synthesis of this problem is the same as for the preceding.

[Proposition] 50 [Problem]

*Given the section of a cone, draw a tangent, which will make with the axis, on the same side as the section, an angle equal to a given acute angle*[55].

Let the section of a cone first be a parabola whose axis is ΑΒ, then it is required to draw a tangent to the section that will make with the axis ΑΒ on the same side as the section an angle equal to the given acute angle.

[Solution]. Let the have been done, and let it be ΓΔ, therefore the angle ΒΔΓ is given, let ΒΓ is drawn perpendicular, then the angle at Β is also given. Therefore the ratio ΔΒ to ΒΓ is given. But the ratio ΒΔ to ΒΑ is given, therefore also the ratio ΑΒ to ΒΓ is given. And the angle at Β is given, therefore the angle ΒΑΓ is also given. And it is [given] with respect to ΒΑ, which is given in position, and with respect to the given point Α, therefore ΓΑ is given in position. And the section is also given in position, therefore Γ is given. And ΓΔ touches, therefore ΓΔ is given in position.

Then the synthesis of this problem is as follows. Let the given section of a cone first be a parabola whose axis is ΑΒ, and the given acute angle ΕΖΗ, and let some point Ε be taken on ΕΖ, and let ΕΗ be drawn perpendicular, and let ΖΗ be bisected at Θ, and let ΘΕ be joined, and let the angle ΒΑΓ be made equal to the angle ΗΘΕ, and let ΒΓ be drawn perpendicular, and let ΑΔ be made equal to ΒΑ, and let ΓΔ be joined. Therefore ΓΔ [according to Proposition I.33] is tangent to the section.

I say then that the angle ΓΔΒ is equal to the angle ΕΖΗ. For since as ΖΗ is to ΗΘ, so ΔΒ is to ΒΑ, and as ΘΗ is to ΗΕ, so ΑΒ is to ΒΓ, therefore *ex*[56] as ΖΗ is to ΗΕ, so ΔΒ is to ΒΓ.

And the angles at Η and Β are right, therefore the angle at Ζ is equal to the angle at Δ.

Let the section be a hyperbola, and let it have been done, and let ΓΔ be tangent, and let the center of the section Χ be taken, and let ΓΧ be joined and let ΓΕ be perpendicular, therefore the ratio pl.ΧΕΔ to sq.ΓΕ is given for [according to Proposition I.37] it is the same as the ratio of the *latus transversum* to the *latus rectum*. And the ratio sq,ΓΕ to sq.ΕΔ is given for each of the angles ΓΔΕ and ΔΕΓ is given. Therefore the ratio pl.ΧΕΔ to sq.ΕΔ is given, and so also the ratio ΧΕ to ΕΔ is given. And the angle at Ε is given, therefore the angle at Χ is also given. Then some straight line ΓΧ has been drawn across in position with respect to ΧΕ and to the given point Χ at a given angle, therefore ΓΧ is given in position. And the section is also given in position, therefore Γ is given. And ΓΔ has been drawn across as tangent, therefore ΓΔ is given in position.

Let the asymptote to the hyperbola ΧΖ be drawn, therefore ΓΔ continued [according to Proposition II,3] meet the asymptote. Let it meet it at Ζ. Therefore the angle ΖΔΕ is greater than the angle ΖΧΔ.

Therefore for the construction the given acute angle will have to be greater than the half the angle between the asymptotes.

Then the synthesis of his problem is as follows. Let there be the given hyperbola whose axis is AB, the asymptote XZ, and the given acute angle KΘH greater than the angle AXZ and let the angle KΘΛ equal to the angle AXZ and let AZ be drawn from A perpendicular to AB and let some point H be taken on HΘ, and let HK be drawn from it perpendicular to ΘK. Since then the angle ZXA is equal to the angle ΛΘK, and also the angles at A and K are right, therefore as XA is to AZ, so ΘK is to KΛ, and [the ratio] ΘK to KΛ is greater than [the ratio] ΘK to KH, therefore also [the ratio] XA to AZ is greater [the ratio] ΘK to KH. And so also [the ratio] sq.XA to sq.AZ is greater than [the ratio] sq.ΘK to sq.KH .

But [according to Proposition II.1] as sq.XA is to sq.AZ, so the *latus transversum* is to the *latus rectum*, therefore also [the ratio] the *latus transversum* to the *latus rectum* is greater than [the ratio] sq.ΘK to sq.KH.

If then we shall contrive that as sq.XA is to sq.AZ, so some other is to sq.KH, it will be greater than sq.ΘK. Let it be pl.MKΘ, and let HM be joined. Since then sq.MK is greater than pl.MKΘ, therefore [the ratio] sq.MK to sq.KH is greater than [the ratio] pl.MKΘ to sq.KH, which is greater than [the ratio] sq.XA to sq.AZ.

And if we shall contrive that as sq.MK is to sq.KH, so sq.XA is to some other, it will be to a magnitude less than sq.AZ, and the straight line joined from X to the point taken will make similar triangles, and therefore the angle ZXA is greater than the angle HMK. Let the angle AZΓ be made equal to the angle HMK, therefore XΓ will cut the section [according to Proposition II.2]. Let is cut it at Γ, and from Γ let ΓΔ be drawn tangent to the section [according to Proposition II.49], and ΓE drawn perpendicular, therefore the triangle ΓXE is similar to the triangle HMK. Therefore as sq.XE is to sq.EΓ, so sq.MK is to sq.KH.

But also [according to Proposition I.37] as the *latus transversum* is to the *latus rectum*, so pl.XEΔ is to sq.EΓ, and as the *latus transversum* is to the *latus rectum*, so pl.MKΘ is to sq.KH. And inversely as sq.ΓE is to pl.XEΔ, so sq.HK is to pl.MKΘ, therefore ex as sq.XE is to pl.XEΔ, so sq.MK is to pl.MKΘ. And therefore as XE is to EΔ, so MK is to KΘ. But also we had as ΓE is to EX, so HK is to KM, therefore ex as ΓE is to EΔ, so HK is to KΘ.

And the angles at E and K are right, therefore the angle at Δ is equal to the angle HΘK.

Let the section be an ellipse whose axis is AB. Then it is required to draw a tangent to the section that with the axis will contain on the same side as the section an angle equal to the given acute angle.

Let it have been done, and let it be ΓΔ. Therefore the angle ΓΔA is given. Let ΓE be drawn perpendicular, therefore the ratio sq.ΔE to sq.EΓ is given. Let X be the center of the section, and let ΓX be joined. Then the ratio sq.ΓE to pl.ΔEX is given for [according to Proposition I.37] it is the same as the ratio of the *latus rectum* to the *latus transversum*, and therefore the ratio sq.ΔE to pl.ΔEX is given, and therefore the ratio ΔE to EX is given. And [the ratio] ΔE to EΓ [also is given], therefore the ratio ΓE to EX is given.

And the angle at E is right, therefore the angle at X is given. And it is given respect to a straight line given in position and to a given point, therefore Γ is given. And from the given point Γ let ΓΔ be drawn tangent, therefore ΓΔ is given in position.

Then the synthesis of this problem is as follows. Let there be the given acute angle ZHΘ, and let some point Z be taken on ZH, and let ZΘ be drawn perpendicular, and let it be contrived that as the *latus rectum* is to the *latus transversum*, so sq.ZΘ is to pl.HΘK, and let KZ be joined, and let X be the center of the section, and let the angle AXΓ be made equal to the angle AKZ, and let ΓΔ be drawn tangent to the section [according to Proposition II.49].

I say that ΓΔ does the problem, that is the angle ΓΔE is equal to the angle ZHΘ. For since as XE is to EΓ, so KΘ is to ZΘ, therefore also as sq.XE is to sq.EΓ, so sq.KΘ is to sq.ZΘ. But also as sq.EΓ is to pl.ΔEX, so sq.ZΘ is to pl.KΘH for each is the same ratio as that of the *latus rectum* to the *latus transversum* [according to Proposition I.37]. And therefore ex as sq.XE is to pl.ΔEX, so sq.KΘ is to pl.KΘH. And therefore as XE is to EΔ, so KΘ is to ΘH.

But also as XE is to EΓ, so KΘ is to ZΘ, therefore ex as ΔE is to EΓ, so ΘH is to ZΘ.

And the sides about the right angles are proportional, therefore the angle ΓΔE is equal to the angle ZHΘ. Therefore ΓΔ does the problem.

[Proposition] 51 [Problem]

*Given a section of a cone, to draw a tangent, which with the diameter drawn through the point of contact will contain an angle equal to a given acute angle* [57] .

Let the given section of a cone first be a parabola whose axis is AB, and the given angle is Θ, then it is required to draw a tangent to the parabola which with the diameter from the point of contact will contain an angle equal to the angle Θ.

[Solution]. Let it have been done, and let ΓΔ be drawn a tangent making with the diameter ΕΓ drawn through the point of contact the angle ΕΓΔ equal to the angle Θ, and let ΓΔ meet the axis at Δ [according to Proposition I.24]. Since then ΑΔ is parallel to ΕΓ [according the porism to Proposition I.51] the angle ΑΔΓ is equal to the angle ΕΓΔ.

But the angle ΕΓΔ is given for it is equal to the angle Θ, therefore the angle ΑΔΓ is also given.

Then the synthesis of this problem is as follows. Let there be a parabola whose axis is ΑΒ, and the given angle is Θ. Let ΓΔ be drawn a tangent to the section making with the axis the angle ΑΔΓ equal to the angle Θ [according to Proposition II.50], and through Γ let ΕΓ be drawn parallel to ΑΒ. Since then the angle Θ is equal to the angle ΑΔΓ, and the angle ΑΔΓ is equal to the angle ΕΓΔ, therefore also the angle Θ is equal to the angle ΕΓΔ.

Let the section a hyperbola whose axis is ΑΒ, and center Ε and asymptote ΕΤ, and the given acute angle Ω, and let ΘΔ be tangent and let ΓΕ be joined doing the problem. And let ΓΗ be drawn perpendicular. Therefore the ratio of the *latus transversum* to the *latus rectum* is given, and so also the ratio pl.ΕΗΔ to sq.ΓΗ [according to Proposition I.37]. Then let some given straight line ΖΘ be laid out, and on it let there be described an arc of a circle admitting an angle equal to the angle Ω [according to Proposition III.33 of Euclid], therefore it will greater than a semicircle [according to Proposition III.31 of Euclid]. And from some point Κ of those on the circumference let ΚΛ be drawn perpendicular making as pl.ΖΛΘ is to sq.ΛΚ, so the *latus transversum* is to the *latus rectum*, and let ΖΚ and ΚΘ be joined. Since then the angle ΖΚΘ is equal to the angle ΕΓΔ, but also as pl.ΕΗΔ is to sq.ΗΓ, so the *latus transversum* is to the *latus rectum*, and as pl.ΖΛΘ is to sq.ΛΚ, so the *latus transversum* is to the *latus rectum*, therefore the triangle ΚΖΛ is similar to the triangle ΕΓΗ, and the triangle ΖΘΚ [is similar]to the triangle ΕΓΔ. And so the angle ΘΖΚ is equal to the angle ΓΕΔ.

Then the synthesis of this problem is as follows. Let there be the given hyperbola ΑΓ, and axis ΑΒ, and center Ε, and given acute angle Ω, and let the given ratio of the *latus transversum* to the *latus rectum* be the same as ΧΨ to ΧΦ, and let ΦΨ be bisected at Υ, and let a given straight line ΖΘ be laid out, and on it let there be described an arc of a circle greater than semicircle and admitting an angle equal to the angle Ω [according to Proposition III.31 and III,33], and let it be ΖΚΘ, and let the center of the circle Ν be taken, and from Ν let ΝΟ be drawn perpendicular to ΖΘ, and let ΝΟ be cut at Π in the ratio ΥΦ to ΦΧ, and through Π let ΠΚ be drawn parallel to ΖΘ and from Κ let ΚΛ be drawn perpendicular to ΖΘ continued, and let ΖΚ and ΚΘ be joined, and let ΛΚ be continued to M, and from N let NΞ be drawn perpendicular to it, therefore it is parallel to ZΘ.

And therefore as NΠ is to ΠO or YΦ is to ΦX, so ΞK is to KΛ.

And doubling the antecedents as ΨΦ is to ΦX, so MK is to KΛ, and *componendo* as ΨX is to XΦ, so MΛ is to ΛK. But as MΛ is to ΛK, so pl.MΛK is to sq.ΛK, therefore as ΨX is to XΦ, so pl.MΛK is to sq.ΛK, and [according to Proposition III.36 of Euclid] pl.ZΛΘ is to sq.ΛK.

But as ΨX is to XΦ, so the *latus transversum* is to the *latus rectum*, therefore also as pl.ZΛΘ is to sq.ΛK, so the *latus transversum* is to *latus rectum*.

Then let AT be drawn from A perpendicular to AB. Since then [according to Proposition II.1] as sq.EA is to sq.AT so the *latus transversum* is to the *latus rectum*, and also as the *latus transversum* is to the *latus rectum*, so pl.ZΛΘ is to sq.ΛK, and [the ratio] sq.ZΛ to sq.ΛK is greater than [the ratio] pl.ZΛΘ to sq.ΛK, therefore also [the ratio] sq.ZΛ to sq.ΛK is greater than [the ratio] sq.EA to sq.AT.

And the angles at A and Λ are right, therefore the angle Z is less than the angle E.

Then let the angle AEΓ be made equal to the angle ΛZK, therefore EΓ will [according to Proposition II.2] meet the section. Let it meet it at Γ. Then let ΓΔ be drawn tangent from Γ [according to Proposition II.49], and let ΓH be drawn perpendicular, then [according to Proposition I.37] as the *latus transversum* is to *latus rectum*, so pl.EHΔ is to sq.ΓH. Therefore also as pl.ZΛΘ is to sq.ΛK, so pl.EHΔ is to sq.ΓH, therefore the triangle KZΛ is similar to the triangle EΓH, and the triangle KΘΛ [is similar] to the triangle ΓHΔ, and the triangle KZΘ to the triangle ΓEΔ. And so the angle EΓΔ is equal to the angle ZKΘ and is equal to the angle Ω.

And if the ratio of the *latus transversum* to the *latus rectum* is equal to the equal, KΛ is touches the circle ZKΘ [according to Proposition III.37 of Euclid], and the straight line joined from the center to K will be parallel to ZΘ and it will do the problem.

[Proposition] 52

*If a straight line touches an ellipse making an angle with the diameter drawn through the point of contact, it is not less than the angle adjacent to the one contained by the straight lines deflected at the middle of the section* [58] .

Let there be an ellipse whose axes are AB and ΓΔ, and center E, and let

ΑΒ be the major axis, and let ΗΖΛ touch the section, and let ΑΓ, ΓΒ, and ΖΕ be joined, and let ΒΓ be continued to Λ.

I say that the angle ΛΖΕ is not less than the angle ΛΓΑ.

[Proof]. For ΖΕ is either parallel to ΛΒ or not.

Let it first be parallel, and ΑΕ is equal to ΕΒ, therefore also ΑΘ is equal to ΘΓ. And ΖΕ is a diameter, therefore [according to Proposition II.6] the tangent at Ζ is parallel to ΑΓ. But also ΖΕ is parallel to ΛΒ, therefore ΖΘΓΛ is a parallelogram, and therefore the angle ΛΖΘ is equal to the angle ΛΓΘ. And since ΑΕ and ΕΒ are each greater than ΕΓ, the angle ΑΓΒ is obtuse, therefore the angle ΛΓΑ is acute. And so also the angle ΛΖΕ [is acute]. And therefore the angle ΗΖΕ is obtuse.

Then let ΕΖ not be parallel to ΛΒ, and let ΖΚ be drawn perpendicular, therefore the angle ΛΒΕ is not equal to the angle ΖΕΑ. But the right angle at Ε is equal to the right angle at Κ, therefore it is not true that as sq.ΒΕ is to sq.ΕΓ, so sq.ΕΚ is to sq.ΚΖ. But [according to Proposition I.21] as sq.ΒΕ is to sq.ΕΓ, so pl.ΑΕΒ is to sq.ΕΓ, that is the *latus transversum* is to the *latus rectum*, and [according to Proposition I.37] as the *latus transversum* is to *latus rectum*, so pl.ΗΚΕ is to sq.ΚΖ. Therefore it is not true that as pl.ΗΚΕ is to sq.ΚΖ, so sq.ΚΕ is to sq.ΚΖ. Therefore ΗΚ is not equal to ΚΕ.

Let there be laid out an arc of a circle ΜΥΝ admitting an angle equal to the angle ΑΓΒ [according to Proposition III.33 of Euclid], and the angle ΑΓΒ is obtuse, therefore ΜΥΝ is an arc less than a semicircle [according to Proposition III.31 of Euclid]. Then let it be contrived that as ΗΚ is to ΚΕ, so ΝΞ is to ΞΜ, and from Ξ let ΥΞΧ be drawn at right angles, and let ΝΥ and ΥΜ be joined, and let ΜΝ be bisected at Τ, and let ΟΤΠ be drawn at right angle; therefore it is a diameter. Let the center be Ρ, and from it let ΡΣ be drawn perpendicular, and ΟΝ and ΟΜ be joined. Since then the angle ΜΟΝ is equal to the angle ΑΓΒ, and ΑΒ and ΜΝ have been bisected, one at Ε and other at Τ, and the angles at Ε and Τ are right, therefore the triangles ΟΤΝ and ΒΕΓ are similar. Therefore as sq.ΤΝ is to sq.ΤΟ, so sq.ΒΕ is to sq.ΕΓ. And since ΤΡ is equal to ΣΞ, and ΡΟ is greater than ΣΥ, therefore [the ratio] ΡΟ to ΤΡ is greater than [the ratio] ΣΥ to ΣΞ, and convertendo [the ratio] ΡΟ to ΟΤ is less than [the ratio] ΣΥ to ΥΞ.

And doubling the antecedents, therefore [the ratio] ΠΟ to ΤΟ is less [the ratio] ΧΥ to ΥΞ.

And separando [the ratio] ΠΤ to ΤΟ is less [the ratio] ΧΞ to ΥΞ.
But [according to Proposition I.21] as ΠΤ is to ΤΟ, so sq.ΤΝ is to sq.ΤΟ, that is sq.ΒΕ is to sq.ΕΓ, that is the *latus transversum* is to the *latus rectum*, and [according to Proposition I.37] as the *latus transversum* is to the *latus rectum*, so

pl.HKE is to sq.KZ.  Therefore [the ratio] pl.HKE to sq.KZ is less than [the ratio] XΞ to ΞY, that is less [the ratio] pl.XΞY to sq.ΞY, what is less [the ratio] pl.NΞM to sq.ΞY.

If then we contrive it that as pl.HKE is to sq.KZ, so pl.MΞN is to some other,  it will be greater than sq.ΞY. Let it be to sq.ΞΦ. Since then as HK is to KE, so NΞ is to ΞM, and KZ and XΦ are perpendicular, and as pl.HKE is to sq.KZ, so pl.MΞN is to sq.ΞΦ, therefore the angle HZE is equal to the angle MΦN. Therefore the angle MYN or the angle AΓB is greater than the angle HZE, and the adjacent angle ΛZΘ is greater than the angle ΛΓΘ.

Therefore the angle ΛZΘ is not less than the angle ΛΓΘ.

[Proposition] 53 [Problem]

*Given an ellipse, to draw a tangent which will make with the diameter drawn through the point of contact an angle equal to a given acute angle, then it is required that the given acute angle be not less than the angle adjacent to the angle contained by the straight lines deflected at the middle of the section*[59].

Let there be the given ellipse whose major axis is AB and minor axis ΓΔ, and center E, and let AΓ and ΓB be joined,  and let the angle Y be the given angle not less than the angle AΓH,  and so also the angle AΓB is not less than the angle X.

Therefore the angle Y is either greater for equal to the angle AΓH.

[Solution]. Let it first be equal, and through E let EK be drawn parallel to BΓ, and through K let KΘ be drawn tangent to the section [according to Proposition II.49]. Since then  AE is equal to EB, and as AE is to EB, so AZ is to ZΓ, therefore AZ is equal to ZΓ. And KE is a diameter therefore the tangent to the section at K, that is ΘKH, is parallel to ΓA [according to Proposition II.6]. And also EK is parallel to HB, therefore KZΓH is a parallelogram, and therefore the angle HKZ is equal to the angle HΓZ. And the angle HΓZ is equal to the given angle, which is Y, therefore also the angle HKE is equal to the angle Y.

Then let the angle Y is greater than the angle AΓH, then inversely the angle X is less than the angle AΓB.

Let a circle be laid out, and let an arc be taken from it, and let it be MNΠ admitting  an angle equal to the angle X, and let MΠ be bisected at O, and from O let NOP be drawn at right angles to MΠ, and let NM and NΠ be joined, therefore the angle MNΠ is less than the angle ATB.

But the angle ΜΝΟ is equal to the half of the angle ΜΝΠ, and the angle ΑΓΕ is equal to the half of the angle ΑΓΒ, therefore the angle ΜΝΟ is less than the angle ΑΓΕ, And the angle at Ε and Ο are right, therefore [the ratio] ΑΕ to ΕΓ is greater than [the ratio] ΟΜ to ΟΝ. And so also [the ratio] sq.ΑΕ to sq.ΕΓ is greater than [the ratio] sq.ΜΟ to sq.ΝΟ.

But sq.ΑΕ is equal to pl.ΑΕΒ, and [according to Proposition III.35 of Euclid} sq.ΜΟ is equal to pl.ΜΟΠ, and is equal to pl.ΝΟΡ, therefore [the ratio] pl.ΑΕΒ to sq.ΕΓ for the *latus transversum* to the *latus rectum* [according to Proposition I.21] is greater than [the ratio] ΡΟ to ΟΝ.

Then let it be that as the *latus transversum* is to the *latus rectum*, so Ω □ is to □ς[60] , and let Ως be bisected at Ϙ. Since then [the ratio] the *latus transversum* to the *latus rectum* is greater than [the ratio] ΡΟ to ΟΝ, also [the ratio] Ω□ to □ς is greater than [the ratio] ΡΟ to ΟΝ.

And *componendo* [the ratio] Ως to ς□ is greater than [the ratio] ΡΝ to ΝΟ.

Let the center of the circle be Φ, and so also [the ratio] Ϙς to ς□ is greater than [the ratio] ΦΝ to ΝΟ.

And separando [the ratio]□Ϙ to □ς is grater than [the ratio] ΦΟ to ΟΝ.

Then let it be contrived that as □Ϙ is to □ς, so ΦΟ is to less than ΟΝ such as ΙΟ, and let ΙΞ and ΞΤ and ΦΨ be drawn parallel. Therefore as □Ϙ is to □ς, so ΦΟ is to ΟΙ, and is to ΨΣ is to ΣΞ, and *componendo* as Ϙς is to ς□, so ΨΞ is to ΞΣ.

Doubling the antecedents, as Ως is to ς□, so ΤΞ is to ΞΣ.

Separando as Ω□ is to □ς or the *latus transversum* to the *latus rectum*, so ΤΣ is to ΣΞ.

Then let ΜΞ and ΞΠ be joined, and let the angle ΑΕΚ be made on ΑΕ at Ε equal to the angle ΜΠΞ, and through Κ let ΚΘ be drawn touching the section [according to Proposition II.49], and let ΚΛ be dropped as an ordinate. Since then the angle ΜΠΞ is equal to the angle ΑΕΚ, and the right angle at Σ is equal to the right angle at Λ, therefore the triangle ΞΣΠ is equiangular with the triangle ΚΕΛ. And as the *latus transversum* is to the *latus rectum*, so ΤΣ is to ΣΞ, that is pl.ΤΣΞ is to sq.ΕΞ, that is pl.ΜΞ,ΣΠ is to sq.ΣΞ. Therefore the triangle ΚΛΕ is similar to the triangle ΣΞΠ, and the triangle ΜΞΠ [is similar] to the triangle ΚΘΕ and therefore the angle ΜΞΠ is equal to the angle ΘΚΕ.

But the angle ΜΞΠ is equal to the angle ΜΝΠ is equal to the angle Χ, therefore also the angle ΘΚΕ is equal to the angle Χ. And therefore the adjacent angle ΗΚΕ is equal to the adjacent angle Υ. Therefore ΗΘ has been drawn across tangent to the section and making with the diameter ΚΕ drawn through the point of contact, the angle HKE equal to the given angle Y, and this it was required to do [61].

## BOOK THREE

### [Proposition] 1

*If straight lines touching a section of a cone or the circumference of a circle meet, and diameters are drawn through the points of contact meeting the tangents, the resulting vertically related triangles will be equal*[1] .

Let there be the section of a cone or the circumference of a circle ΑΒ, and let ΑΓ and ΒΔ meeting at Ε touch ΑΒ, and let the diameters of the section ΓΒ and ΔΑ be drawn through Α and Β meeting the tangents at Γ and Δ.

I say that the triangle ΑΔΕ is equal to the triangle ΕΒΓ.

[Proof]. For let ΑΖ be drawn from Α parallel to ΒΔ, therefore it has been dropped as an ordinate [according to Proposition I.32]. Then in the case of the parabola [according to Proposition I.42] the parallelogram ΑΔΒΖ is equal to the triangle ΑΓΖ, and with the common area ΑΕΒΖ subtracted, the triangle ΑΔΕ is equal to the triangle ΓΒΕ.

And in the case of the other sections let the diameters meet at the center Η. Since then ΑΖ has been dropped as an ordinate, and ΑΓ touches [according to Proposition I.37] pl.ΖΗΓ is equal to sq.ΒΗ. Therefore as ΖΗ is to ΗΒ, so ΒΗ is to ΗΓ, therefore also [according to the porism to Proposition VI.19 of Euclid] as ΖΗ is to ΗΓ, so sq.ΖΗ is to sq.ΗΒ.

But [according to Proposition VI.19 of Euclid] as sq.ΖΗ is to sq.ΗΒ, so the triangle ΑΗΖ is to the triangle ΔΗΒ, and as ΖΗ is to ΗΓ, so the triangle ΑΗΖ is to the triangle ΑΗΓ, therefore also as the triangle ΑΗΖ is to the triangle ΑΗΓ, so the triangle ΑΗΖ is to the triangle ΔΗΒ. Therefore the triangle ΑΗΓ is equal to the triangle ΔΗΒ.

Let the common area ΑΗΒΖ be subtracted, therefore as remainders, the triangle ΑΕΔ is equal to the triangle ΓΕΒ.

### [Proposition] 2

*With the same suppositions if some point is taken on the section of a cone or the circumference of a circle, and through it parallels to the tangents are drawn as far as the diameters, then the quadrangle under one of the tangents, and one of the diameters will be equal to the triangle constructed on the same tangent and the other diameter*[2] .

Let there be the section of a cone or the circumference of a circle ΑΒ and let ΑΕΓ and ΒΕΔ be tangents, and ΑΔ and ΒΓ diameters, and let some point Η be taken on the section, and ΗΚΛ and ΗΜΖ be drawn parallel to the tangent.

I say that the triangle ΑΙΜ is equal to the quadrangle ΓΛΗΙ.

[Proof]. For the triangle ΗΚΜ [in Propositions I.42 and I.43] has been shown that it is equal to the quadrangle ΑΛ, let the common quadrangle ΙΚ be added or subtracted, and the triangle ΑΙΜ is equal to the quadrangle ΓΗ.

## [Proposition] 3

With the same suppositions if two points are taken on the section or the circumference of a circle, and through them parallels to the tangents are drawn as far as the diameters, the quadrangles under the straight lines drawn, and standing on the diameters as bases, are equal to each other [3].

Let there be the section and tangents and diameters as said before, and let two points at random Ζ and Η be taken on the section, and through Ζ let the straight lines ΖΘΚΛ and ΝΖΙΜ be drawn parallel to the tangents, and through Η the straight lines ΗΞΟ and ΘΠΡ.

I say that the quadrangle ΛΗ is equal to the quadrangle ΜΘ, and the quadrangle ΛΝ is equal to the quadrangle ΡΝ.

[Proof]. For since it has already been shown [in Proposition III.2] the triangle ΡΠΑ is equal to the quadrangle ΓΗ, and the triangle ΑΜΙ is equal to the quadrangle ΓΖ, and the triangle ΡΠΑ is equal to the sum of the triangle ΑΜΙ and the quadrangle PM therefore also the quadrangle ΓΗ is equal to the sum of the quadrangles ΓΖ and ΠΜ, and so the quadrangle ΓΗ is equal to the sum of the quadrangles ΓΘ and ΡΖ.

Let the common quadrangle ΓΘ be subtracted, therefore as remainders the quadrangle ΛΗ is equal to the quadrangle ΘΜ.

And therefore as wholes the quadrangle ΛΝ is equal to the quadrangle ΡΝ.

## [Proposition] 4

*If two straight lines touching opposite hyperbolas meet each other, and diameters are drawn through the points of contact meeting the tangents, then the triangles at the tangents will be equal* [4] .

Let there be the opposite hyperbolas Α and Β and let the tangents to them, ΑΓ and ΒΓ, meet at Γ, and let Δ be the center of the hyperbolas, and let

ΑΒ and ΓΔ be joined, and ΓΔ continued to Ε, and let ΔΑ and ΒΔ also be joined and continued to Ζ and Η.

I say that the triangle ΑΗΔ is equal to the triangle ΒΔΖ, and the triangle ΑΓΖ is equal to the triangle ΒΓΗ.

[Proof]. For let ΘΛ be drawn through Θ tangent to the section, therefore [according to Proposition I.44] it is parallel to ΑΗ. And since [according to Proposition I.30] ΑΔ is equal to ΔΘ, and [according to Proposition VI.19 of Euclid] the triangle ΑΗΔ is equal to the triangle ΔΘΛ.

But [according to Proposition III.1] the triangle ΔΘΛ is equal to the triangle ΒΔΖ, therefore also the triangle ΑΗΔ is equal to the triangle ΒΔΖ.

And so also the triangle ΑΓΖ is equal to the triangle ΒΓΗ.

[Proposition] 5

*If two straight lines touching opposite hyperbolas meet, and some point is taken on either of the hyperbolas, and from it two straight lines are drawn, one parallel to the tangent, other parallel to the line joining the points of contact, then the triangle constructed by them on the diameter drawn through the point of meeting differs from the triangle cut off at the point of meeting of the tangents by the triangle cut off on the tangent and the diameter drawn through the point of contact* [5].

Let there be opposite hyperbolas whose center is Γ, and let tangents ΕΔ and ΔΖ meet at Δ, and let ΕΖ and ΓΔ be joined, and let ΓΔ be continued, and let ΖΓ and ΕΓ be joined and continued, and let some point Η be taken on the section, and through it let ΘΗΚΛ be drawn parallel to ΕΖ, and ΗΜ parallel to ΔΖ.
I say that the triangle ΗΘΜ differ from the triangle ΚΘΔ by the triangle ΚΛΖ.

[Proof].For since ΓΔ has been shown [in Propositions II.38 and II.39] to be a diameter of the opposite hyperbolas and [according to Definition 5 and Proposition II.38] ΕΖ to be an ordinate to it, and ΗΘ has been drawn parallel to ΕΖ, and ΜΗ parallel to ΔΖ, therefore the triangle ΗΘΜ differs from the triangle ΓΛΘ by the triangle ΓΔΖ [according to Propositions I.44 or I.45]. And so the triangle ΑΘΜ differs from the triangle ΚΘΔ by the triangle ΚΛΖ. And it is evident that the triangle ΚΛΖ is equal to the quadrangle ΜΗΚΔ.

[Proposition] 6

*With the same suppositions if some point is taken on one of the opposite hyperbolas, and from it parallels to the tangents are drawn meeting the tan-*

*gents and the diameters, then the quadrangle under one of the tangents and one of the diameters will be equal to the triangle constructed on the same tangent and the other diameter* [6].

Let there be opposite hyperbolas of which ΑΕΓ and ΒΕΔ are diameters, and let ΑΖ and ΒΗ touch the hyperbola ΑΒ meeting each other at Θ, and let some point Κ be taken on the section, and from it let ΚΜΛ and ΚΝΞ be drawn parallel to the tangents.

I say that the quadrangle ΚΖ is equal to the triangle ΛΙΝ.

[Proof]. Now since ΑΒ and ΓΔ are opposite hyperbolas, and ΑΖ, meeting ΒΔ, touches the hyperbola ΑΒ, and ΚΛ has been drawn parallel to ΑΖ, therefore [according to Proposition III.2] the triangle ΛΙΝ is equal to the quadrangle ΚΖ.

[Proposition] 7

*With the same suppositions if points are taken on each of the hyperbolas, and from them parallels to the tangents are drawn meeting the tangents and the diameter, then the quadrangles under the straight lines drawn and standing on the diameters as bases, will be equal to each other* [7].

With the mentioned suppositions let Κ and Λ be taken on both hyperbola, and through them let ΜΚΠΡΧ and ΝΣΤΛΩ be drawn parallel to ΑΖ, and ΝΙΟΚΞ and ΧΦΥΛΨ parallel to ΒΗ.

I say that what was said in the enunciation will be so.

[Proof]. For since [according to Proposition III.2] the triangle ΑΟΙ is equal to the quadrangle ΡΟ, let the quadrangle ΕΟ be added to both, therefore the whole triangle ΑΕΖ is equal to the quadrangle ΚΕ.

But also [according to Proposition III.5] the triangle ΒΗΕ is equal to the quadrangle ΛΕ, and [according to Proposition III.1] the triangle ΑΕΖ is equal to the triangle ΒΗΕ, therefore the quadrangle ΛΕ is equal to the quadrangle ΙΚΡΕ.

Let the common quadrangle ΝΕ be added, therefore as the whole quadrangle ΤΚ is equal to the quadrangle ΙΛ, and also the quadrangle ΚΥ is equal to the quadrangle ΡΛ.

[Proposition] 8

With the same suppositions instead of Κ and Λ let there be taken Γ and Δ of which the diameters hit the hyperbolas, and through them the parallels to the tangents be drawn [8].

I say that the quadrangle ΔH is equal to the quadrangle ΖΓ, and the quadrangle ΞΙ is equal to the quadrangle ΟΤ.

[Proof]. For since it was shown [in Proposition III.1] the triangle ΑΗΘ is equal to the triangle ΘΒΖ, and the straight line from Α to Β is parallel to the straight line from Η to Ζ, therefore as ΑΕ is to ΕΗ, so ΒΕ is to ΕΖ, and convertendo as ΕΑ is to ΑΗ, so ΕΒ is to ΒΖ. And also as ΓΑ is to ΑΕ, so ΔΒ is to ΒΕ for each is double the other, therefore ex as ΓΑ is to ΑΗ, so ΔΒ is to ΒΖ. And the triangles are similar because of the parallels, therefore [according to Proposition VI.19 of Euclid] as the triangle ΓΤΑ is to the triangle ΑΘΗ, so the triangle ΞΒΔ is to the triangle ΘΒΖ. And alternately [as the triangle ΓΤΑ is to the triangle ΞΒΔ, so the triangle ΑΘΗ is to the triangle ΘΒΖ]. But [according to Proposition III.1] the triangle ΑΘΗ is equal to the triangle ΘΒΖ, therefore the triangle ΓΤΑ is equal to the triangle ΞΒΔ.

As parts of these it was shown that the triangle ΑΘΗ is equal to the triangle ΘΒΖ, therefore also as remainders of the quadrangle ΔΘ is equal to the quadrangle ΓΘ. And so also the quadrangle ΔΗ is equal to the quadrangle ΓΖ. And since ΓΟ is parallel to ΑΖ, the triangle ΓΟΕ is equal to the triangle ΑΕΖ.

And likewise also the triangle ΔΕΙ is equal to the triangle ΒΕΗ. But [according to Proposition III.1] the triangle ΒΕΗ is equal to the triangle ΑΕΖ, therefore also the triangle ΓΟΕ is equal to the triangle ΔΕΙ. And also the quadrangle ΔΗ is equal to the quadrangle ΓΖ.

Therefore as wholes the quadrangle ΞΙ is equal to the quadrangle ΟΤ.

## [Proposition] 9

With the same suppositions if one of the points is between the diameters as Κ and other is the same with one of Γ and Δ, for instance Γ, and the parallels are drawn. I say that the triangle ΓΕΟ is equal to the quadrangle ΚΕ, and the quadrangle ΛΟ is equal to the quadrangle ΛΜ [9].

And this is evident for since it was shown that the triangle ΓΕΟ is equal to the triangle ΑΕΖ, and [according to Proposition III.5] the triangle ΑΕΖ is equal to the quadrangle ΚΕ,therefore also the triangle ΓΕΟ is equal to the quadrangle ΚΕ And so also the triangle ΓΡΜ is equal to the quadrangle ΚΟ, and the quadrangle ΚΓ is equal to the quadrangle ΛΟ.

## [Proposition] 10

With the same suppositions let Κ and Λ be taken not as points at which the diameters hit the hyperbolas. Then it is to be shown that the quadrangle ΛΤΡΧ is equal to the quadrangle ΩΧΚΙ [10].

[Proof]. For since AZ and BH touches and AE and BE are diameters through the points of contact, and ΛT and KI are parallel to the tangents, [according to Proposition I.44] the triangle TYE is equal to the sum of the triangles YΩΛ and EZA. And likewise also the triangle ΞEI is equal to the sum of the triangle ΞPK and BEH.

But [according to Proposition III.1] the triangle EZA is equal to the triangle BEH, therefore the triangle TYE without the triangle YΩΛ is equal to the triangle ΞEI without the triangle ΞPK.

Therefore the sum of the triangles TYE and ΞPK is equal to the sum of the triangles ΞEI and YΩΛ.

Let the common area KΞEYΛX be added, therefore the quadrangle ΛTPX is equal to the quadrangle ΩXKI.

[Proposition] 11

*With the same suppositions if some point is taken on either of the hyperbolas, and from it parallels are drawn, one parallel to the tangent and other parallel to the straight line joining the points of contact, then the triangle constricted by them on the diameter drawn through the point of meeting of the tangents differs from the triangle cut off on the tangent and the diameter drawn through the point of contact by the triangle cut off at the point of meeting of the tangents*[11].

Let there be the opposites hyperbola AB and ΓΔ, and let the tangents AE and ΔE meet at E, and let the center be Θ, and let AΔ and EΘH be joined, and let some point B be taken at random on the hyperbola AB, and through it let BZΛ has been dropped to EZ parallel to AH, and BM parallel to AE.

I say that the triangle BZM differs from the triangle AKΛ by the triangle KEZ

[Proof]. For it is evident that AΔ is bisected by EΘ [according to Propositions II.29 and II.39], and that EΘ is a diameter conjugate to the diameter drawn through Θ parallel to AΔ [according to Proposition II.38], and so AH is an ordinate to EH [according to Definition 6].

Since then HE is a diameter, and AE touches, and AH is an ordinate, and with the point B taken on the hyperbola AB, let BZ be dropped to EH parallel to AH and BM parallel to AE, therefore it is clear that [according to Propositions II.43 and II. 45] the triangle BMZ differs from the triangle ΛΘZ by the triangle ΛΘZ by the triangle ΘAE.

And so also the triangle BMZ differs from the triangle AKΛ by the triangle KZE.

And it has been proved at the same time that the quadrangle BKEM is equal to the triangle ΛKA.

[Proposition] 12

*With the same suppositions if of one hyperbola two points are taken and parallels are drawn from each of them, likewise the quadrangles under them will be equal* [12].

Let there be the same suppositions as before, and let B and K be taken at random on the hyperbola AB, and through them let ΛBMN and KΞOΥΠ be drawn parallel to AΔ, and BΞP and ΛKΣ parallel to AE.

I say that the quadrangle BΠ is equal to the quadrangle KP.

[Proof]. For since it has been shown [according to Proposition III.11] that the triangle AOΠ is equal to the quadrangle KOEΣ, and the triangle AMN is equal to the quadrangle BEMP, therefore, as remainder, either the quadrangle KP without the quadrangle BO is equal to MΠ or the sum of the quadrangles KP and BO is equal to the quadrangle MΠ.
And with the common quadrangle BO added or subtracted the quadrangle BP is equal to the quadrangle ΞΣ.

[Proposition] 13

*If in conjugate opposite hyperbolas straight line tangent to the adjacent hyperbola meet, and diameters are drawn through the points of contact, then the triangles whose common vertex is the center of the opposite hyperbolas will be equal* [13] .

Let there be conjugate opposite hyperbolas on which there are the points A, B, Γ, and Δ, and let BE and EK meeting at E touch the hyperbolas A and B, and let Θ be the center, and let AΘ and BΘ be joined and continued to Δ and Γ.

I say that the triangle BZΘ is equal to the triangle AHΘ.

[Proof]. For let AK and ΛΘM be drawn through A and Θ parallel to BE. Since then BZE touches the hyperbola B, and ΔΘB is a diameter through the point of contact, and ΛM is parallel to BE, ΛM a diameter conjugate to the diameter BΔ, the so-called second diameter [according to Proposition II.20], and therefore AK has been drawn as an ordinate to BΔ. And AH touches, therefore [according to Proposition I.38] pl.KΘH is equal to sq.BΘ .

Therefore as ΚΘ is to ΘΒ, so ΒΘ is to ΗΘ. But as ΚΘ is to ΘΒ, so ΚΑ is to ΒΖ, and ΑΘ is to ΘΖ, therefore also as ΑΘ is to ΘΖ, so ΒΘ is to ΗΘ..

And the angles ΒΘΖ and ΗΘΖ are equal to two right angles, therefore the triangle ΑΗΘ is equal to the triangle ΒΘΖ

[Proposition] 14

*With the same suppositions if some point is taken on any one of the hyperbola, and from it parallels to the tangents are drawn as far as the diameters, then the triangle constructed at the center will differ from the triangle constructed about the same angle by the triangle having the tangent as base, and center as vertex* [14].

Let the other be the same, and let some point Ξ be taken on the hyperbola Β, and through it let ΞΡΣ be drawn parallel to ΑΗ and ΞΤΟ parallel to ΒΕ. I say that the triangle ΟΘΤ differs from the triangle ΞΣΤ by the triangle ΘΒΖ

[Poof]. For let ΑΥ be drawn from Α parallel to ΒΖ. Since then, because of the same reasons as before, ΛΘΜ is a diameter of the hyperbola ΑΛ, and ΔΘΒ is a second diameter conjugate to it [according to Proposition II.20] and ΑΗ is a tangent at Α, and ΑΥ has been dropped parallel to ΛΜ, therefore [according to Proposition I.40 the ratio] ΑΥ to ΥΗ is compounded of [the ratios] ΘΥ to ΥΑ and the *latus transversum* of the *eidos* corresponding to ΛΜ to the *latus rectum*.

But as ΑΥ is to ΥΗ, so ΞΤ is to ΤΣ, and as ΘΥ is to ΥΑ, so ΘΤ is to ΤΟ, and ΘΒ is to ΒΖ, and [according to Proposition I.60] as the *latus transversum* of the *eidos* corresponding to ΛΜ is to the *latus rectum*, so the *latus rectum* of the *eidos* corresponding to ΒΔ is to the *latus transversum*.

Therefore [the ratio] ΞΤ to ΤΣ is compounded of [the ratios] ΘΒ to ΒΖ and the *latus rectum* of the *eidos* corresponding to ΒΔ to the *latus transversum* or [the ratio] ΞΤ to ΤΣ is compounded of [the ratios] ΘΤ to ΤΟ and the latus rectum of the *eidos* corresponding to ΒΔ to the *latus transversum*.

And by the shown in the theorem 41 of Book I [that is Proposition I.41] the triangle ΤΘΟ differs from the triangle ΞΤΣ by the triangle ΒΖΘ.

And so also [according to Proposition III.13] by the triangle ΑΗΘ.

[Proposition] 15

*If straight lines touching one of the conjugate opposites hyperbolas meet, and diameters are drawn through the points of contact, and some point is taken* *on one of the conjugate hyperbolas, and from it parallels to the tangents are drawn as far as the diameters, then the triangle constructed by them at the hyperbola is greater than the triangle constructed at the center by the triangle having the tangent as base and the center of the opposite hyperbolas as vertex*[15].

Let there be conjugate opposite hyperbolas ΑΒ, ΗΣ, Τ, and Ξ whose center is Θ and let ΑΔΕ and ΒΔΓ touch the hyperbola ΑΒ, and let the diameters ΑΘΖΦ and ΒΘΤ be drawn through the points of contact Α and Β, and let some point Σ be taken on the hyperbola ΗΣ, and through it let ΣΖΛ be drawn parallel to ΒΓ and ΣΥ parallel to ΑΕ.

I say that the triangle ΣΛΥ is equal to the sum of the triangles ΘΛΖ and ΘΓΒ.

[Proof]. For let ΞΘΗ be drawn through Θ parallel to ΒΓ, and ΚΙΗ through Η parallel to ΑΕ, and ΣΟ parallel to ΒΤ, then it is evident that ΞΗ is a diameter conjugate to ΒΤ [according to Proposition II.20], and that ΣΟ is parallel to ΒΤ dropped as an ordinate to ΘΗΟ, and that ΣΛΘΟ is a parallelogram.

Since then ΒΓ touches, and ΒΘ is through the point of contact, and ΑΕ is another tangent, let it be contrived that as ΔΒ is to ΒΕ, so ΜΝ is to double ΒΓ, therefore ΜΝ is the so-called the *latus rectum* of the *eidos* corresponding to ΒΤ [according to Proposition I.50]. Let ΜΝ be bisected at Π, therefore as ΔΒ is to ΒΕ, so ΜΠ is to ΒΓ.

Then let it be contrived that as ΞΗ is to ΤΒ, so ΤΒ is to Ρ, then Ρ also will be so-called the *latus rectum* of the *eidos* applied to ΞΗ [according to Propositions I.16 and I.60].

Since then as ΔΒ is to ΒΕ, so ΜΠ is to ΒΓ, but as ΔΒ is to ΒΕ, so sq.ΔΒ is to pl.ΔΒΕ, and as ΜΠ is to ΒΓ, so pl.ΜΠ,ΒΘ is to pl.ΓΒΘ, therefore as sq.ΔΒ is to pl.ΔΒΕ, so pl.ΜΠ,ΒΘ is to pl.ΓΒΘ. And pl.ΜΠ,ΒΘ is equal to sq.ΘΗ because as sq.ΔΒ is to pl.ΔΒΕ, so pl.ΜΠ,ΒΘ is to pl.ΓΒΘ. And pl.ΜΠ,ΒΘ is equal to sq.ΘΗ, because [according to Proposition I.16] sq.ΞΗ is equal to pl.ΤΒ,ΜΝ, and pl.ΜΠ,ΒΘ is equal to the quarter of pl.ΤΒ,ΜΝ ,and sq.ΘΗ is equal to the quarter of sq.ΞΗ, therefore as sq.ΔΒ is to pl.ΔΒΕ, so sq.ΘΗ is to pl.ΓΒΘ. And correspondingly sq.ΔΒ is to sq.ΘΗ, so pl.ΔΒΕ is to pl.ΓΒΘ . But as sq.ΔΒ is to sq.ΘΗ, so the triangle ΔΒΕ is to the triangle ΗΘΙ for they are similar, and as pl.ΔΒΕ is to pl.ΓΒΘ, so the triangle ΔΒΕ is to the triangle ΓΒΘ, therefore as the triangle ΔΒΕ is to the triangle ΗΘΙ, so the triangle ΔΒΕ is to the triangle ΓΒΘ.

Therefore the triangle ΗΘΙ is equal to the triangle ΓΒΘ.

Again since [the ratio] ΘΒ to ΒΓ is compounded of [the ratios] ΘΒ to ΜΠ and ΜΠ to ΒΓ, but as ΘΒ is to ΜΠ, so ΤΒ is to ΜΝ, and Ρ to ΞΗ, and as ΜΠ is to ΒΓ, so ΔΒ is to ΒΕ, therefore [the ratio] ΘΒ to ΒΓ is compounded of [the ratios] ΔB to BE and P to ΞH. And since BΓ is parallel to ΣΛ, and the triangle ΘΓB is similar to the triangle ΘΛZ, and as ΘB is to BΓ, so ΘΛ is to ΛZ, therefore [the ratio] ΘΛ to ΛZ is compounded of [the ratios] P to ΞH and ΔB to BE or [the ratio] ΘΛ to ΛZ is compounded of [the ratios] P to ΞH and ΘH to HI.

Since then HΣ is a hyperbola having ΞH as a diameter, and P as the *latus rectum*, and from some point Σ let ΣO be dropped as an ordinate, and the figure ΘIH let be described on the radius ΘH, and the figure ΘΛZ let be described on the ordinate ΣO or its equal ΘΛ, and on ΘO the straight line between the center and the ordinate, or on ΣΛ, its equal, the figure ΣΛY let be described similar to the figure ΘIH described on the radius, and there are compounded ratios as already given, therefore the triangle ΣΛY is equal to the sum of the triangles ΘΛZ and ΘΓB [according to Proposition I.41].

## [Proposition] 16

*If two straight lines touching a section of a cone or the circumference of a circle meet, and from some point on the section a straight line is drawn parallel to one tangent and cutting the section and the other tangent, then as the squares on the tangents are to each other, so the plane under the straight lines between the section and the tangent will be to the square cut off at the point of contact* [16].

Let there be the section of a cone or the circumference of a circle AB, and let AΓ and ΓB meeting at Γ touch it, and let some point Δ be taken on the section AB, and through it let EΔZ be drawn parallel to ΓB.

I say that as sq.BΓ is to sq.AΓ, so pl.ZEΔ is to sq.EA.

[Proof]. For let the diameters AHΘ and KBΛ be drawn through A and B, and ΔMN through Δ parallel to AΛ, it is at once evident that [according to Propositions I.46 and I.47] ΔK is equal to KZ, and [according to Proposition III.2] the triangle AEH is equal to the quadrangle ΛΔ, and [according to Proposition III.1] the triangle BΛΓ is equal to the triangle AΓΘ. Since then ΔK is equal to KZ and ΔE added, as the sum of pl.ZEΔ and sq.ΔK is equal to sq.KE. And since the triangle EΛK is similar to the triangle ΔNK, as sq.EK is to sq.KΔ, so the triangle EKΛ is to the triangle ΔNK. And alternately as the whole sq.EK is to the whole triangle EΛK, so the sum of the subtracted part of sq.ΔK is to the subtracted part of the triangle ΔNK.Therefore also as the remainder of pl.ZEΔ is to the remainder of the quadrangle ΔΛ, so sq.EK is to the triangle EΛK. But as sq.EK is to the triangle EΛK, so sq.ΓB is to the triangle BΛΓ, therefore also as pl.ZEΔ is to the quadrangle ΛΔ, so sq.ΓB is to the triangle ΛΓB. But the quadrangle ΛΔ is equal to the triangle ΑΕΗ, and the triangle ΒΛΓ is equal to the triangle ΑΓΘ, therefore also as pl.ΖΕΔ is to sq.ΓΒ, so the triangle ΑΕΗ is to the triangle ΑΓΘ.Alternately [as pl.ΖΕΔ is to sq.ΕΑ, so sq.ΓΒ is to sq.ΑΓ].

[Proposition] 17

*If two straight lines touching a section of a cone or the circumference of a circle meet, and two points are taken at random on the section, and from them in the section are drawn parallel to the tangents straight lines cutting each other and the line of the section, then as the squares on the tangents are to each other, so will the rectangular planes under the straight lines taken similarly* [17].

Let there be the section of a cone or the circumference of a circle ΑΒ, and tangents to ΑΒ,ΑΓ and ΓΒ meeting at Γ, and let Δ and Ε be taken at random on the section, and through them at ΕΖΙΚ and ΔΖΗΘ be drawn parallel to ΑΓ and ΓΒ.

I say that as sq.ΓΑ is to sq.ΓΒ, so pl.ΚΖΕ is to pl.ΘΖΔ.

[Proof]. For let the diameters ΑΛΜΝ and ΒΟΞΠ be drawn through Α and Β, and let the tangents and parallels be continued to the diameters, and let ΔΞ and ΕΜ be drawn from Δ and Ε parallel to the tangents, then it is evident that [according to Propositions i.46 and i.47] ΚΙ is equal to ΙΕ, ΘΗ is equal to ΗΔ. Since then ΚΕ has been cut equally at I and unequally at Ζ [according to Proposition II.5 of Euclid] the sum of pl.ΚΖΕ and sq.ΖΙ is equal to sq.ΕΙ . And since the triangles are similar because of the parallels, as the whole sq.ΕΙ is to the whole triangle ΙΜΕ, so the subtracted part of sq.IZ is to the subtracted part of the triangle ΖΙΛ. Therefore also as the remainder of pl.ΚΖΕ is to the remainder of the quadrangle ΖΜ, so the whole sq.ΕΙ is to the whole triangle ΙΜΕ. But as sq.ΕΙ is to the triangle ΙΜΕ, so sq.ΓΑ is to the triangle ΓΑΝ. Therefore as pl.ΚΖΕ is to the quadrangle ZM, so sq.ΓΑ is to the triangle ΓΑΝ. But the triangle ΓΑΝ is equal to the triangle ΓΠΒ [according to Proposition III.1] and the quadrangle ΖΜ is equal to the quadrangle ΖΞ [according to Proposition III.3], therefore as pl.ΚΖΕ is to the quadrangle ΖΞ, so sq.ΓΑ is to the triangle ΓΠΒ. Then likewise it could be shown that as pl.ΘΖΔ is to the quadrangle ΖΞ, so sq.ΓΒ is to the triangle ΓΠΒ. Since then as pl.ΚΖΕ is to the quadrangle ΖΞ, so sq.ΓΑ is to the triangle ΓΠΒ, and inversely as the quadrangle ΖΞ is to pl.ΘΖΔ, so the triangle ΓΠΒ is to sq.ΓΒ, therefore ex as sq.ΓΑ is to sq.ΓΒ, so pl.ΚΖΕ is to pl.ΘΖΔ.

[Proposition] 18

*If two  straight lines touching opposite hyperbolas meet, and some point is taken on either one of the hyperbolas, and from it some straight line is drawn parallel to one of the tangents cutting the section and the other tangent, then as the squares on the tangents are to each other, so will the rectangular plane under the straight lines between the section and the tangent be to the square on the straight line cut off at the point of contact* [18].

Let there be the opposite hyperbolas ΑΒ and ΜΝ, the tangents ΑΓΛ and ΒΓΘ, and through the points of contact the diameters ΑΜ and ΒΝ, and let some point Δ be taken at random on the hyperbola ΜΝ, and through it let ΕΔΖ be drawn parallel to ΒΘ.

I say that as sq.ΒΓ is to sq.ΓΑ, so pl.ΖΕΔ is to sq.ΑΕ .

[Proof]. For let ΔΞ be drawn through Δ parallel to ΑΕ. Since then ΑΒ is a hyperbola and BN its diameter and ΒΘ a tangent and ΔΖ parallel to ΒΘ, therefore [according to Proposition I.48] ΖΟ is equal to ΟΔ. And ΕΔ is added, therefore [according to Proposition II.6 of Euclid] the sum of pl.ΖΕΔ and sq.ΔΟ is equal to sq.ΕΟ. And since ΕΛ is parallel to ΔΞ, the triangle ΕΟΛ is similar to the triangle ΔΞΟ. Therefore as the whole sq.ΕΟ is to the whole triangle ΕΟΛ, so the subtracted part of sq.ΔΟ is to the subtracted part of the triangle ΔΞΟ, therefore also as the remainder of pl.ΔΕΖ is to the remainder of the quadrangle ΔΛ, so sq.ΕΟ is to the triangle ΕΟΛ. But as sq.ΟΕ is to the triangle ΕΟΛ, so sq.ΒΓ is to the triangle ΒΓΛ, therefore also as pl.ΖΕΔ is to the quadrangle ΔΛ, so sq.ΒΓ is to the triangle ΒΓΛ. And [according to Proposition III.6] the quadrangle ΔΛ is equal to the triangle ΑΕΗ, and [according to Proposition III.1] the triangle ΒΓΛ is equal to the triangle ΑΓΘ, therefore as pl.ΖΕΔ is to the triangle ΑΕΗ, so sq.ΒΓ is to the triangle ΑΓΘ. But also as the triangle ΑΕΗ is to sq.ΕΑ, so the triangle ΑΓΘ is to sq.ΑΓ, therefore ex as sq.ΒΓ is to sq.ΑΓ, so pl.ΖΕΔ is to  sq.ΕΑ.

[Proposition] 19

*If two straight lines touching opposite hyperbolas meet parallels to the tangents are drawn cutting each  other and the section, then as the squares on the tangents are each other, so will the rectangular plane under the straight lines between the section and the point of meeting of the straight lines be to the rectangular plane under the straight lines taken similarly* [19].

Let there be the opposite hyperbolas whose diameters are ΑΓ and ΒΔ and the center at Ε, and let the tangents ΑΖ and ΖΔ meet at Ζ, and let ΗΘΙΚΛ and ΜΝΞΟΛ be drawn  from any points parallel to ΑΖ and ΖΔ.

I say that as sq.ΑΖ is to sq.ΖΔ, so pl.ΗΛΙ is to pl.ΜΛΞ.

[Proof]. Let ΙΠ and ΞΡ be drawn through I and Ξ parallel to ΑΖ and ΖΔ. And since as sq.ΑΖ is to the triangle ΑΖΣ, so sq.ΘΛ is to the triangle ΘΛΟ, and sq.ΘΙ is to the triangle ΘΙΠ, therefore as the remainder of pl.ΗΛΙ is to the remainder of the quadrangle ΙΠΟΛ, so sq.ΑΖ is to the triangle ΑΖΣ. But [according to Proposition III.4] the triangle ΑΖΣ is equal to the triangle ΔΤΖ, and [according to Proposition III.7] the quadrangle ΙΠΟΛ is equal to the quadrangle ΚΡΞΛ, therefore also as sq.ΑΖ is to the triangle ΔΤΖ, so pl.ΗΛΙ is to the quadrangle ΑΡΞΛ . But [likewise] as the triangle ΔΤΖ is to sq.ΖΔ, so the quadrangle ΚΡΞΛ is to pl.ΜΛΞ, and therefore ex as sq.ΑΖ is to sq.ΖΔ, so pl.ΗΛΙ is to pl.ΜΛΞ.

[Proposition] 20

*If two straight lines touching the opposite hyperbolas meet, and through the point of meeting some straight line is drawn parallel to the straight line joining the points of contact and meeting each of the hyperbolas, and some other straight line is drawn parallel to the same straight line and cutting the hyperbolas and the tangents, then as the rectangular plane under the straight lines drawn from the point of meeting to cut the hyperbolas is to the square on the tangent, so is the rectangular plane under the straight lines between the hyperbolas and the tangent to the square on the straight line cut off at the point of contact* [20].

Let there be the opposite hyperbolas ΑΒ and ΓΔ whose center is Ε and tangents ΑΖ and ΓΖ, and let ΑΓ be joined, and let ΕΖ and ΑΕ be joined and continued, and let ΒΖΘ be drawn through Ζ parallel to ΑΓ, and let the point Κ be taken at random, and through it let ΚΛΣΜΝΞ be drawn parallel to ΑΓ.

I say that as pl.ΒΖΔ is to sq.ΖΑ, so pl.ΚΛΞ is to sq.ΑΛ.

[Proof]. For let ΚΠ and ΒΡ be drawn from Κ and B parallel to ΑΖ. Since then as sq.ΒΖ is to the triangle ΒΖΡ, so sq.ΚΣ is to the triangle ΚΣΠ, so sq.ΛΣ is to the triangle ΛΣΖ, and as sq.ΚΣ is to the triangle ΚΣΠ, so the remainder of pl.ΚΛΞ [according to Proposition II.5 of Euclid] is to the remainder of the quadrangle ΚΛΖΠ [according to Proposition V.19 of Euclid] and ΒΖ is equal to pl.ΒΖΔ [according to Propositions II.38 and II 39] and the triangle ΒΡΖ is equal to the triangle ΑΖΘ [according to Proposition III.11], therefore as pl.ΒΖΔ is to the triangle ΑΖΘ, so pl.ΚΛΞ is to the triangle ΑΛΝ.

And as pl.ΒΖΔ is to sq.ΖΑ, so pl.ΚΛΞ is to sq.ΑΛ.

[Proposition] 21

*With the same suppositions if two points are taken on the section, and through them straight lines are drawn, one parallel to the tangent, other parallel to the straight line joining the points of contact and cutting each other and the hyperbolas, then as the rectangular plane under the straight lines drawn from the point of meeting to cut hyperbola is to the square on the tangent, so will the rectangular plane under the straight lines between the section and the point of meeting* [21].

Let there be the same suppositions as before, and let Η and Κ be taken, and through them let ΝΞΗΟΠΡ and ΚΣΤ be drawn parallel to ΑΖ, and ΗΛΜ and ΚΟΦΙΧΨΩ parallel to ΑΓ.

I say that as pl.ΒΖΔ is to sq.ΖΑ, so pl.ΚΟΩ is to pl.ΝΟΗ.

[Proof]. For since as sq.ΑΖ is to the triangle ΑΖΘ, so sq.ΑΛ is to the triangle ΑΛΜ, and sq.ΞΟ is to the triangle ΞΟΨ, and as sq.ΞΟ is to the triangle ΞΟΨ, so sq.ΞΗ is to the triangle ΞΗΜ, therefore the whole sq.ΞΟ is to the whole triangle ΞΟΨ, so the subtracted part of sq.ΞΗ is to the subtracted part of the triangle ΞΗΜ, therefore also as the remainder of pl.ΝΟΗ is to the remainder of the quadrangle ΗΟΨΜ, so sq.ΑΖ is to the triangle ΑΖΘ.

But [according to Proposition III.11] the triangle ΑΖΘ is equal to the triangle ΒΥΖ and [according to Proposition III.12] the quadrangle ΗΟΨΜ is equal to the quadrangle ΚΟΡΤ, therefore as sq.ΑΖ is to the triangle ΒΖΥ, so pl.ΝΟΗ is to the quadrangle ΚΟΡΤ. But it was shown [in Proposition III.20] as the triangle ΒΥΖ is to sq.ΒΖ or pl.ΒΖΔ [according to Propositions II,38 and II.39], so the quadrangle ΚΟΡΤ is to pl.ΚΟΩ, therefore ex as sq.ΑΖ is to pl.ΒΖΔ, so pl.ΝΟΗ is to pl.ΚΟΩ. And inversely as pl.ΒΖΔ is to sq.ΖΑ, so pl.ΚΟΩ is to pl.ΝΟΗ.

[Proposition] 22

*If two parallel straight lines touch opposite hyperbolas, and two straight lines are drawn cutting each other and the hyperbolas, one parallel to the tangent, other parallel to the straight line joining the points of contact, then as the latus transversum of the eidos corresponding to the straight line joining the points of contact is to the* latus rectum*, so the rectangular plane under the straight lines between the section and the point of meeting will be to the rectangular plane under the straight lines between the section and the point of meeting* [22].

Let there be the opposite hyperbolas Α and Β, and let ΑΓ and ΒΔ be parallel and tangent to them, and let ΑΒ be joined. Then let ΕΞΗ be drawn across parallel to ΑΒ and ΚΕΛΜ parallel to ΑΓ.

I say that as ΑΒ is to the *latus rectum* of the *eidos*, so pl.ΗΕΞ is to pl.ΚΕΜ.

[Proof]. Let ΞΝ and ΗΖ be drawn through Η and Ξ parallel to ΑΓ for since ΑΓ and ΒΔ are parallels tangent to the hyperbolas, ΑΒ is a diameter [according to Proposition II.31], and ΚΛ, ΞΝ, and ΗΖ are ordinates to it [according to Proposition I.32]. Then [according to Proposition I.21] as ΑΒ is to the *latus rectum*, so pl.ΒΛΑ is to sq.ΛΚ, and so pl.ΒΝΑ is to sq.ΝΞ or sq.ΛΚ. Therefore the whole pl.ΒΛΑ is to the whole sq.ΛΚ, so the subtracted part of pl.ΒΝΑ is to the subtracted part of sq.ΛΕ, or as pl.ΒΛΑ is to sq.ΛΚ, so pl.ΖΑΝ is to sq.ΛΕ for [according to Proposition I.21] ΝΑ is equal to ΒΖ, therefore also as the remainder of pl.ΖΛΝ is to the remainder of pl.ΚΕΜ, so ΑΒ is to the *latus rectum*. But pl.ΖΛΝ is equal to pl.ΗΕΞ, therefore as ΑΒ, that is the *latus transversum* of the *eidos*, is to the *latus rectum*, so pl.ΗΕΞ is to pl.ΚΕΜ.

[Proposition] 23

*If in conjugate opposite hyperbolas two straight lines touching contrary hyperbolas meet in a hyperbola at random, and two straight lines are drawn parallel to the tangents and cutting each other and the other of opposite hyperbolas, then as the squares on the tangents are to each other, so the rectangular plane under the straight lines between the section and the point of meeting will be to the rectangular plane under the straight lines similarly taken* [23].

Let there be the conjugate opposite hyperbolas ΑΒ, ΓΔ, ΕΖ, and ΗΘ and their center Κ, and let ΑΦΓΛ and ΕΧΔΛ, tangents to the hyperbolas ΑΒ and meet at Λ, and let ΑΚ and ΕΚ be joined and continued to Β and Ζ, and let ΗΜΝΞΟ be drawn from H parallel to ΑΛ, and ΘΠΡΞΣ from Θ parallel to ΕΛ.

I say that at sq.ΕΛ is to sq.ΛΑ, so pl.ΘΞΣ is to pl.ΗΞΟ.

[Proof]. For let ΣΤ be drawn through Σ parallel to ΑΛ, and ΟΥ from Ο parallel to ΕΛ. Since then ΒΕ is a diameter of the conjugate opposite hyperbolas ΑΒ, ΓΔ, ΕΖ, and ΗΘ, and ΕΛ touches the section, and ΘΣ has been drawn parallel to it, [according to Proposition II.20 and Definition 5] ΘΠ is equal to ΠΣ , and for the same reasons ΗΜ is equal to ΜΟ. And since as sq.ΕΛ is to the triangle ΕΦΛ, so sq.ΠΣ is to the triangle ΠΤΣ, and so sq.ΠΞ is to the triangle ΠΝΞ, also as the remainder of pl.ΘΞΣ is to the remainder of the quadrangle ΤΝΞΣ , so sq.ΕΛ is to the triangle ΦΛΕ. But [according to Proposition III.4] the triangle

ΕΦΛ is equal to the triangle ΑΛΧ, and [according to Proposition III.15] the quadrangle ΤΝΞΣ is equal to the quadrangle ΞΡΥΟ, therefore as sq.ΕΛ is to the triangle ΑΛΧ, so pl.ΘΞΣ is to the quadrangle ΞΡΥΟ. But as the triangle ΑΧΛ is to sq.ΑΛ, so the quadrangle ΞΡΥΟ is to pl.ΗΞΟ, therefore ex as sq.ΕΛ is to sq.ΑΛ, so pl.ΘΞΣ is to pl.ΗΞΟ.

[Proposition] 24

*If in conjugate opposite hyperbolas two straight lines are drawn from the center to the hyperbolas, one of them is taken as the transverse diameter and other as the upright diameter, and two straight lines are drawn parallel to two diameters and meeting each other and the hyperbolas, and the point of meeting of the straight lines is the place between four hyperbolas, then the rectangular plane under the segments of the parallel to the transverse diameter together with the plane under the segments of the parallels to the upright diameter has the ratio which the square on the upright diameter has to the square on the transverse diameter, will be equal to the double square on the half of the transverse diameter* [24] .

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ whose center is Ε, and from Ε let the transverse diameter ΑΕΓ and the upright diameter ΔΕΒ be drawn through, and let ΖΗΘΙΚΛ and ΜΝΞΟΠΡ be drawn parallel to ΑΓ and ΔΒ and meeting each other at Ξ, and first let Ξ be within the angle ΣΕΦ or the angle ΥΕΤ.

I say that pl.ΖΞΛ together with pl.ΡΞΜ has the ratio sq.ΔΒ to sq.ΑΓ is equal to the double sq.ΑΕ.

[Proof]. For let the asymptotes of the hyperbolas ΣΕΤ and ΥΕΦ be drawn, and through Α let ΣΗΑΦ tangent to the hyperbola be drawn. Since then [according to Propositions I.60 and II.1] pl.ΣΑΦ is equal to sq.ΔΕ, therefore as pl.ΣΑΦ is to sq.ΕΑ, so sq.ΔΕ is to sq.ΕΑ.

And [the ratio] pl.ΣΑΦ to sq.ΑΕ is compounded of [the ratios] ΣΑ to ΑΕ and ΦΑ to ΑΕ.

But as ΣΑ is to ΑΕ, so ΝΞ is to ΞΘ, and as ΦΑ is to ΑΕ, so ΠΞ is to ΞΚ; therefore [the ratio] sq.ΔΕ to sq.ΑΕ is compounded of [the ratio] ΝΞ to ΞΘ and ΠΞ to ΞΚ.

But [the ratio] ΠΞΝ to pl.ΚΞΘ is compounded of [the ratios] ΝΞ to ΞΘ and ΡΞ to ΞΚ, therefore as sq.ΔΕ is to sq.ΑΕ, so pl.ΡΞΝ is to pl.ΚΞΘ.

Therefore also as [sq.ΔΕ is to sq.ΑΕ][25], so the sum of sq.ΔΕ and pl.ΡΞΝ is to the sum of sq.AE and pl.ΚΞΘ. And sq.ΔE is equal to pl.ΠMN [according to Proposition II.11] and is equal to pl.PNM [according to Proposition II.16], and sq.AE is equal to pl.KZΘ [according to Proposition II.11] and is equal to pl.ΛΘZ [according to Proposition II.16], therefore as sq.ΔE is to sq.AE, so the sum of pl.PΞN and pl.PNM is to the sum of pl.KΞΘ and pl.ΛΘZ. And the sum of pl.PΞN and pl.PNM is equal to pl.PΞM, therefore as sq.ΔE is to sq.AE, so pl.PΞM is to the sum of pl.KΞΘ and pl.KZΘ.

Then it must be shown that the sum pl.ZΞΛ and pl.KΞΘ and pl.KZΘ is equal to the double sq.AE.

Let the common sq.AE, that is pl.KZΘ be subtracted, therefore is remains to be shown that the sum of pl.ZΞΛ and pl.KΞΘ is equal to sq.AE.

And this is so four the sum pl.ZΞΛ and pl.KΞΘ is equal to pl.ΛΘZ, and the sum pl.ZΞΛ and pl.KΞΘ is equal to KZΘ [according to Proposition II.16] and is equal to sq.AE [according to Proposition II.11].

Then let ZΛ and MP meet on one of the asymptotes at Θ. Then pl.ZΘΛ is equal to sq.AE, and pl.MΘP is equal to sq.ΔE [according to Propositions II.11 and II.16], therefore as sq.ΔE is to sq.AE, so pl.MΘ,EP is to pl.ZΘΛ.

And so we want the double pl.ZΘΛ to be equal the double sq.AE, and it does.

And let Ξ be within the angle ΣEK or the angle ΦET. Then likewise by the composition of ratios as sq.ΔE is to sq.AE, so pl.ΠΞN is to pl.KΞΘ. And sq.ΔE is equal to pl.ΠM,PN, so is equal to pl.PNM, and sq.AΛ is equal to pl.ZΘΛ, therefore as pl.PNM is to pl.ZΘΛ, so the subtracted part of pl.ΠΞN is to the subtracted part of pl.KΞΘ. Therefore also as pl.PNM is to pl.ZΘΛ, so the remainder of pl.PΞM is to the remainder of sq.AE without pl.KΞΘ.

Therefore it must shown that pl.ZΞΛ together sq.AE without pl.KΞΘ are equal to the double sq.AE.

Let common sq.AE, that is pl.ZΘΛ, be subtracted, therefore it remains to be shown that pl.KΞΘ together with sq.AE without pl.KΞΘ are equal to sq.AE.

And this is so for pl.KΞΘ together with sq.AE without pl.KΞΘ is equal to sq.AE.

## [Proposition] 25

With the same suppositions let the point of meeting of the parallels to AΓ and BΔ be within one of the hyperbolas Δ and B, as set out for instance at Ξ [26].

I say that the rectangular plane under the segment of the parallels to the transverse diameter, that is pl.ΟΞΝ, will be greater than the plane to which the plane under the segments of the parallels to the upright diameter, that is pl.ΡΞΜ, has the ratio that the square on the upright diameter has to the square on the transverse diameter by the double square on the half of the transverse diameter.

[Proof]. For the same reason as sq.ΔΕ is to sq.ΑΕ, so pl.ΠΞΘ is to pl.ΣΞΛ, and sq.ΔΕ is equal to pl.ΠΜΘ, and [according to Proposition II.11] sq.ΑΕ is equal to pl.ΛΟΣ, therefore also as sq.ΔΕ is to sq.ΑΕ, so pl.ΠΜΘ is to pl.ΛΟΣ.

And since [according to Proposition II.22] the whole pl.ΠΞΘ is to the whole pl.ΛΞΣ, so the subtracted part of pl.ΠΜΘ is to the subtracted part of pl.ΛΟΣ or pl.ΣΤΛ, therefore also the remainder of pl.ΡΞΜ is to the remainder of pl.ΤΞΚ, so sq.ΔΕ is to sq.ΑΕ.

Therefore it must be shown that pl.ΟΞΝ is equal to the sum of pl.ΤΞΚ and the double sq.ΑΕ.

Let the common pl.ΤΞΚ be subtracted, therefore it must be shown that pl.ΟΤΝ [according to Proposition III.24] is equal to the double sq.ΑΕ.

And it is [according to Proposition II.23 ] the mentioned equality.

[ Proposition ] 26

*And if the point of meeting of the parallels at Ξ is within one of the hyperbolas Α and Γ, as set out before then the rectangular plane under the segments of the parallels to the transverse diameter, that is pl.ΛΞΖ , will be less than the plane to which the plane under the segments of the other parallel, that is pl.ΡΞΗ has the ratio which the square on the upright diameter has to the square on the transverse diameter by the double square on the half of the transverse diameter.*

[Proof]. For, since for the same reasons as before as sq.ΔΕ is to sq.ΑΕ, so pl.ΦΞΣ is to pl.ΚΞΘ, therefore also as the whole pl.ΡΞΗ is to the whole pl.ΚΞΘ together with sq.ΑΕ, so square on the upright diameter is to square on the transverse diameter. Therefore it must be shown that as the sum of pl.ΛΞΖ and the double sq.ΑΕ is equal to the sum of pl.ΚΞΘ and sq.ΑΕ.

Let the common sq.ΑΕ be subtracted, therefore it remains to be shown that the sum of pl.ΛΞΖ and sq.ΑΕ is equal to pl.ΚΞΘ or the sum of pl.ΛΞΖ and pl.ΛΘΖ is equal to pl.ΚΞΘ [according to Propositions II.11 and II.16].

And it is for the sum of pl.ΛΘΖ and pl.ΛΞΖ is equal to pl.ΚΞΘ.

[Proposition] 27

*If the conjugate diameters of an ellipse or the circumference of a circle are drawn, and one of them is called the upright diameter, and other the transverse diameter, and two straight lines meeting each other and the line of the section are drawn parallel to them, then the squares on the straight lines cut off on the straight line drawn parallel to the transverse diameter between the point of meeting of the straight lines and the line of the section increased by the figures described on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the line of the section, figures similar and similarly situated to the* eidos *corresponding to the upright diameter will be equal to the square on the transverse diameter* [27].

Let there be the ellipse or the circumference of a circle ΑΒΓΔ, whose center is Ε, let two of its conjugate diameters be drawn, the upright diameter ΑΕΓ and the transverse diameter ΒΕΔ, and let ΝΗΖΘ and ΚΖΛΜ be drawn parallel to ΑΓ and ΒΔ.

I say that sq.ΝΖ and sq.ΖΘ increased by the figures described on ΚΖ and ΖΜ similar and similarly situated to the *eidos* corresponding to ΑΓ will be equal to the sq.ΒΔ.

[Proof]. For let ΝΞ be drawn from Ν parallel to ΑΕ, therefore it has been dropped as an ordinate to ΒΔ. And let ΒΠ be the *latus rectum*. Now since [according to Proposition I.15] as ΒΠ is to ΑΓ, so ΑΓ is to ΒΔ, therefore as ΒΠ is to ΒΔ, so sq.ΑΓ is to sq.ΒΔ. And sq.ΒΔ is equal to the *eidos* corresponding to ΑΓ, therefore as ΒΠ is to ΒΔ, so sq.ΑΓ is to the *eidos* corresponding to ΑΓ. And as sq.ΑΓ is to the *eidos* corresponding to ΑΓ, so sq.ΝΞ is to the figure on ΝΞ similar to the *eidos* corresponding to ΑΓ [according to Proposition VI.22 of Euclid], therefore also as ΒΠ is to ΒΔ, so sq.ΝΞ is to the figure on ΝΞ similar to the *eidos* corresponding to ΑΓ. And also as ΒΠ is to ΒΔ, so sq.ΝΞ is to pl.ΒΞΔ [according to Proposition I.21], therefore the figure on ΝΞ or ΖΛ similar to the *eidos* corresponding to ΑΓ is equal to pl.ΒΞΔ.

Then likewise we could show that the figure on ΚΛ similar to the *eidos* corresponding to ΑΓ is equal to pl.ΒΛΔ.

And since ΝΘ has been cut equally at Η and unequally at Ζ the sum of sq.ΘΖ and sq.ΖΝ is equal to the sum of the double sq.ΘΗ and the double sq.ΗΖ is equal to the sum of the double sq.ΝΗ and the double sq.ΗΖ [according to Proposition VI.9 of Euclid].

Then for the same reasons also the sum of sq.MZ and sq.ZK is equal to the double sq.KΛ and the double sq.ΛZ, and the figures on MZ and ZK similar to the *eidos* corresponding to ΑΓ are equal to the double similar figures on KΛ and ΛZ. And the sum of the figures on KΛ and ZΛ is equal to the sum of pl.BΞΔ and pl.ΠΛΔ. And the sum of the figures on KΛ and ZΛ is equal to pl.BΞΔ and pl.BΛΔ, and the sum of sq.NH and sq.HZ is equal to the sum of sq.ΞE and sq.ZΛ, therefore the sum of sq.NZ and sq.ZΘ and the figures on KZ and ZN similar to the *eidos* corresponding to ΑΓ is equal to the sum of the double pl.BΞΔ and the double pl.BΛΔ , and the double sq.ΞE and the double sq.EΛ. And since BΔ has been cut equally at E and unequally at Ξ , the sum of pl.BΞΔ and sq.ΞE is equal to sq.ΞE [according to Proposition II.5 of Euclid].

Likewise also the sum of pl.BΛΔ and sq.ΛE is equal to sq.BE.

And so the sum of pl.BΞΔ and pl.BΛΔ and sq.ΞE and sq.ΛE is equal to the double sq.BE.

Therefore sq.NZ and sq.ZΘ together with the figures on KZ and on ZM similar to the *eidos* corresponding to ΓΑ are equal to the double of sq.BE. But also sq.BΔ is equal to the double of sq.BE, therefore sq.NZ and sq.ZΘ together the figures on KZ and ZM similar to the *eidos* corresponding to ΑΓ are equal to the sq.BΔ.

## [Proposition] 28

*If in conjugate opposite hyperbolas conjugate diameters are drawn, one of them is so-called the upright diameter, and other the transverse diameter, and two straight lines are drawn parallel to them and meeting each other and the hyperbolas, then the squares on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the hyperbolas have to the squares on the straight lines cut off on the straight line drawn parallel to the transverse diameter between the point of meeting of the straight lines and the hyperbolas the ratio which the square on the upright diameter has to the square on the transverse diameter* [28].

Let there be the conjugate opposite hyperbolas Α, Β, Γ, and Δ, and let ΑΕΓ be the upright diameter and ΒΕΔ the transverse diameter, and let ZHΘK and ΛHMN be drawn parallel to them and cutting each other and the hyperbolas.

I say that as the sum of sq.ΛH and sq.HN is to the sum of sq.ZH and sq.HK, so sq.ΑΓ is to sq.ΒΔ.

[Proof]. For let ΛΞ and ZO be drawn as ordinates from Z and Λ, therefore they are parallel to ΑΓ and ΒΔ. And from Β let the *latera recta* corresponding to ΒΔ and ΒΠ be drawn, then it is evident that as ΠΒ is to ΒΔ., so sq.ΑΓ is to sq.ΒΔ [according to Proposition I.15] , so sq.ΑΕ is to sq.ΕΒ, and as sq.ZO is to pl.ΒΟΔ [according to Proposition I.21], so pl.ΓΞΑ is to sq.ΛΞ [according to Propositions I.21 and I.60].

Therefore as one of the antecedents is to one of consequents, so are all of the antecedents to all of the consequents [according to Proposition V.12 of Euclid], therefore as sq.ΑΓ is to sq.ΒΔ, so the sum of pl.ΓΞΑ and sq.ΑΕ and sq.ΟZ is to the sum of pl.ΔΟΒ and sq.ΒΕ and sq.ΛΞ or as sq.ΑΓ is to sq.ΒΔ, so the sum of pl.ΓΞΑ and sq.ΑΕ, and sq.ZΘ is to the sum of pl.ΔΟΒ and sq.ΒΕ and sq.ΜΕ.

But the sum of pl.ΓΞΑ and sq.ΑΕ is equal to sq.ΞΕ, and the sum of pl.ΔΟΒ and sq.ΒΕ is equal to sq.ΟΕ [according to Proposition II.6 of Euclid], therefore as sq.ΑΓ is to sq.ΒΔ, the sum of sq.ΞΕ and sq.ΕΘ is to the sum of sq.ΟΕ and sq.ΕΜ so the sum of sq.ΛΜ and sq.ΜΗ is to the sum sq.ZΘ and sq.ΘΗ.

And as has been shown, the sum of sq.ΝΗ and sq.ΗΛ is equal to the sum of the double of sq.ΛΜ and the double of sq.ΜΗ, and [according to Proposition II.7 of Euclid]the sum of sq.ZΗ and sq.ΗΚ is equal to the sum of the double sq.ZΘ and the double sq.ΘΗ, therefore also as sq.ΑΓ is to sq.ΒΔ, so the sum of sq.ΝΗ and sq.ΗΛ is to the sum of sq.ZΗ and sq.ΗΚ.

## [Proposition] 29

*With the same suppositions if the parallel to the upright diameter cuts the asymptotes, then the squares on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the asymptotes together with the half of the square on the upright diameter has to the squares on the straight lines cut off on the straight line drawn parallel to the transverse diameter between the point of meeting of the straight lines and the hyperbolas the ratio which the square on the upright diameter has the square on the transverse diameter*[29].

Let there be the same construction as before, and let ΝΛ cut the asymptotes at Ξ and Ο. It is to be shown that as the sum of sq.ΞΗ and sq.ΗΟ and the half of sq.ΑΓ is to the sum of sq.ZΗ and sq.HK, so sq.ΑΓ is to sq.ΒΔ or as the sum of sq.ΞΗ and sq.ΗΟ, and the double sq.ΑΕ is to the sum of sq.ZΗ and sq.ΗΚ, so sq.ΑΓ is to sq.ΒΔ.

[Proof] . For since [according to Proposition II.16] ΛΞ is equal to ON,the sum of sq.ΛH and sq.HN and the double pl.NΞΛ is equal to the sum of sq.ΞH and sq.HO, therefore the sum of sq.ΞH and sq.HO and the double sq.AE is equal to the sum of sq.ΛH and sq.HN. And as the sum of sq.ΛH and sq.HN is to the sum of sq.ZH and sq.HK, so sq.AΓ is to sq.BΔ [according to Proposition III.28], therefore also as the sum of sq.ΞH and sq.HO and the double sq.AE is to the sum of sq.ZH and sq.HK, so sq.AΓ is to sq.BΔ.

[Proposition] 30

*If two straight lines touching a hyperbola meet, and through the points of contact a straight line is continued, and through the point of meeting a straight line is drawn parallel to one of the asymptotes and cutting both the hyperbola and the straight line joining the points of contact, then the straight line between the point of meeting and the strait line joining the points of contact will be bisected by the hyperbola* [30].

Let there be the hyperbola ABΓ, and let AΔ and ΔΓ be tangents and EZ and EH asymptotes, and let AΓ be joined, and through Δ parallel to ZE let ΔKΛ be drawn.

I say that ΔK is equal to KΛ.

[Proof].For let ZΔBM be joined and continued both ways, and let ZΘ be made equal to BZ, and through B and K let BE and KN be drawn parallel to AΓ. Therefore they have been dropped as ordinates. And since the triangle BEZ is similar to the triangle ΔNK, therefore as sq.ΔN is to sq.NK, so sq.BZ is to sq.BE. And as sq.BZ is to sq.BE, so ΘB is to the *latus rectum* [according to Proposition II.1], therefore also as sq.ΔN is to sq.NK, so ΘB is to the *latus rectum*.

But as ΘB is to the *latus rectum*, so pl.ΘNB is to sq.NK [according to Proposition I.21], therefore also as sq.ΔN is to sq.NK, so pl.ΘNB is to sq.NK. Therefore pl.ΘNB is equal to sq.ΔN. And also [according to Proposition i.37] pl.MZΔ is equal to sq.ZB because AΔ touches and AM has been dropped as an ordinate, and so also the sum of pl.ΘNB and sq.ZB is equal to the sum of pl.MZΔ and sq.ΔN.

But the sum of pl.ΘNB and sq.ZB is equal to sq.ZN [according to Proposition II.6 of Euclid], and therefore the sum of pl.MZΔ and sq.ΔN is equal to sq.ZN. Therefore ΔN has been bisected at N with added ΔZ [according to Proposition II.6 of Euclid]. And KN and ΛM are parallel, therefore ΔK is equal to KΛ.

[Proposition] 31

*If two straight lines touching opposite hyperbolas meet, and a straight line is continued through the points of contact, then and through the point of meeting a straight line is drawn parallel to the asymptote and cutting both the section and the straight line joining the points of contact, then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section* [31].

Let there be the opposite hyperbolas Α and Β, and tangents ΑΓ and ΓΒ, and let ΑΒ be joined and continued, and let ΖΕ be an asymptote and through Γ let ΓΗΘ be drawn parallel to ΖΕ.

I say that ΓΗ is equal to ΗΘ.

[Proof]. For let ΓΕ be joined and continued to Δ, and through Ε and Η let ΝΕΚΜ and ΗΞ be drawn parallel to ΑΒ, and through Η and Κ let ΚΖ and ΗΛ be drawn parallel to ΓΔ. Since the triangle ΚΖΕ is similar to the triangle ΜΛΗ, as sq.ΚΕ is to sq.ΚΖ, so sq.ΜΛ is to sq.ΛΗ. And it has been shown that as sq.ΚΕ is to sq.ΚΖ, so pl.ΝΛΚ is to sq.ΛΗ [according to Proposition III.30] .

Therefore pl.ΝΛΚ is equal to sq.ΜΛ. Let sq.ΚΕ be added to each [side of this equality], therefore the sum of pl.ΝΛΚ and sq.ΚΕ is equal to sq.ΛΕ, that is sq.ΗΞ, is equal to the sum of sq.ΜΛ and sq.ΚΕ. And [according to Propositions V.12 and VI.4 of Euclid] as sq.ΗΞ is to the sum of sq.ΜΛ and sq.ΚΕ, so sq.ΞΓ is to the sum of sq.ΛΗ and sq.ΚΖ, therefore sq.ΞΓ is equal to the sum of sq.ΛΗ and sq,ΚΖ. And sq.ΛΗ is equal to sq.ΞΕ, and sq.ΚΖ is equal to the square on the half of the second diameter [according to Proposition II.1], and is equal to pl.ΓΕΔ [according to Proposition I.38], therefore sq.ΞΓ is equal to the sum of sq.ΞΕ and pl.ΓΕΔ.

Therefore ΓΔ has been cut equally at Ξ and unequally at Ε , and we use the Proposition II.5 of Euclid.

And ΔΘ is parallel to ΗΞ, therefore ΓΗ is equal to ΗΘ. [32–33] .

[Proposition] 32

If two straight lines touching a hyperbola meet, and a straight line is continued through the points of contact, and a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and a straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of asymptotes, then the straight line cut off between this midpoint and the parallel will be bisected by the hyperbola [34].

Let there be the hyperbola ΑΒΓ whose center is Δ,and asymptote ΔΕ, and let ΑΕ and ΖΓ touch, and let ΓΑ and ΖΔ be joined and continued to Η and Θ, then it is evident that ΑΘ is equal to ΘΓ. Then let ΖΚ be drawn through Ζ parallel to ΑΓ, and ΘΛΚ through it parallel to ΔΕ.

I say that ΚΛ is equal to ΘΛ.

[Proof] . For let ΛΜ and ΒΕ be drawn through Β and Λ parallel to ΑΓ, then, as has been already shown [in Proposition III.30], as sq.ΔΒ is to sq.ΒΕ, so sq.ΘΜ is to sq.ΜΛ, and pl.ΒΜΗ is to sq.ΜΛ, therefore pl.ΗΜΒ is equal to sq.ΜΘ. And also pl.ΘΔΖ is equal to sq.ΔΒ because ΑΖ touches, and ΑΘ has been dropped as an ordinate [according to Proposition I.37], therefore the sum of pl.ΗΜΒ and sq.ΔΒ is equal to the sum of pl.ΘΔΖ and sq.ΜΘ equal to sq.ΔΜ [according to Proposition II.6 of Euclid].

Therefore ΖΘ has been bisected at Μ with added ΔΖ.

And ΚΖ and ΛΜ are parallel, therefore ΚΛ is equal to ΛΘ.

[Proposition] 33

*If two straight lines touching opposite hyperbolas meet, and one straight line is drawn through the points of contact, and another straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and still another straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of asymptotes and meeting the section, and the parallel drawn through the point of meeting, then the straight line between the midpoint and the parallel will be bisected by the section* [35].

Let there be the opposite hyperbolas ΑΒΓ and ΔΕΖ , and tangents ΑΗ and ΔΗ and center Θ, and asymptote ΚΘ, and let ΘΗ be joined and continued, and also let ΑΛΔ be joined, then it is evident that it is bisected at Λ [according to Proposition II.30]. Then let ΒΘΕ and ΓΗΖ be drawn through Η and Θ parallel to ΑΔ, and ΛΜΝ through Λ parallel to ΘΚ.

I say that ΛΜ is equal to ΜΝ.

[Proof]. For let ΕΚ and ΜΞ be dropped from E and Μ parallel to ΗΘ, and ΜΠ through Μ parallel to ΑΔ.

Since then through already shown [in Proposition III.30] that as sq.ΘΕ is to ΕΚ, so pl.ΒΞΕ is to sq.ΞΜ, therefore as sq.ΘΕ is to sq.ΕΚ, so the sum of pl.BΞE and sq.ΘE is to the sum of sq.KE and sq.ΞM [according to Proposition V.12 of Euclid] or as sq.ΘE is to sq.EK, so sq.ΘΞ is to the sum of sq.KE and sq.ΞM [according to Proposition II.6 of Euclid].

But it has been shown [in Propositions I.38 and II.1] that sq.EK is equal to pl.HΘΛ, and sq.ΞM is equal to sq.ΘΠ, therefore as sq.ΘE is to sq.EK, so sq.ΘΞ for sq.MΠ is to the sum of pl.HΘΛ and sq.ΘΠ. And [according to Proposition VI.4 of Euclid} as sq.ΘE is to sq.EK, so sq.MΠ is to sq.ΠΛ, therefore as sq.MΠ is to sq.ΠΛ, so sq.MΠ is to the sum of pl.HΘΛ and sq.ΘΠ. Therefore sq.ΠΛ is equal to the sum of pl.HΘΛ and sq.ΘΠ.

Therefore, ΛH has been cut equally at Π and unequally at Θ [and we use Proposition II.5 of Euclid]. MΠ and HN are parallel, therefore ΛM is equal to MN.

[Proposition] 34

*If some point is taken on one of asymptotes of a hyperbola, and a straight line from it touches the hyperbola, and through the point of contact a parallel to the asymptote is drawn, then the straight line drawn from the taken point parallel to other asymptote will be bisected by the section* [36].

Let there be the hyperbola AB, and asymptotes ΓΔ and ΔE, and let a point Γ be taken at random on ΓΔ, and through it let ΓBE be drawn touching the section, and through B let ZBH be drawn parallel to ΓΔ, and through Γ let ΓAH be drawn parallel to ΔE.

I say that ΓA is equal to AH.

[Proof]. For let AΘ be drawn through A parallel to ΓΔ, and BK through B parallel to ΔE. Since then [according to Proposition II.3] ΓB is equal to BE, therefore also ΓK is equal to KΔ, and ΔZ is equal to ZE.

And since [according to Proposition II,12] pl.KBZ is equal to pl.ΓAΘ, and BZ is equal to ΔK and is equal to ΓK, and AΘ is equal to ΔΓ, therefore pl.ΔΓA is equal to pl.HΓK. Therefore as ΔΓ is to ΓK, so HΓ is to ΓA, and ΓΔ is equal to the double ΓK, therefore also HΓ is equal to the double ΓA.

Therefore ΓA is equal to AH.

[Proposition] 35

*With the same suppositions, if from the taken point some straight line is drawn cutting the section at two points, then as the whole straight line is* *to the straight line cut off outside, so will the segments of the straight line cut off inside be to each other*[37].

Let there be the hyperbola ΑΒ and the asymptotes ΓΔ and ΔΕ, and ΓΒΕ touching and ΘΒ parallel to ΓΔ, through Γ let some straight line ΓΑΛΖΗ be drawn across cutting the section at Α and Ζ.

I say that as ΖΓ is to ΓΑ, so ΖΛ is to ΑΛ.

[Proof]. For let ΓΝΞ, ΚΑΜ, ΟΠΒΡ and ΖΥ be drawn through Γ,Α, Β, and Ζ parallel to ΔΕ, and ΑΠΣ and ΤΕΡΜΞ through Α and Ζ parallel to ΓΔ.

Since then [according to Proposition II.8] ΑΓ is equal to ΖΗ, therefore also [according to Proposition VI.4 of Euclid] ΚΑ is equal to ΤΗ.

But ΚΑ is equal to ΔΣ, therefore also ΤΗ is equal to ΔΣ. And so also ΓΚ is equal to ΔΥ. And since ΓΚ is equal to ΔΥ, also ΔΚ is equal to ΓΥ, therefore as ΔΚ is to ΓΚ, so ΓΥ is to ΓΚ, and as ΓΥ is to ΓΚ, so ΖΓ is to ΑΓ, and as ΖΓ is to ΑΓ, so ΜΚ is to ΚΑ, and [according to Proposition VI.1 of Euclid] as ΜΚ is to ΚΑ, so the parallelogram ΜΔ is to the parallelogram ΔΑ, and as ΔΚ is to ΓΚ, so the parallelogram ΘΚ is to the parallelogram ΚΝ, therefore also as the parallelogram ΜΔ is to the parallelogram ΔΑ, so the parallelogram ΘΚ is to the parallelogram KN.

But the parallelogram ΔΑ is equal to the parallelogram ΔΒ [according to Proposition II.12] and is equal to the parallelogram ΟΝ for [according to Proposition II.3] ΓΒ is equal to ΒΕ and ΔΟ is equal to ΟΓ, therefore as the parallelogram ΜΔ is to the parallelogram ΟΝ, so the parallelogram ΘΚ is to the parallelogram ΚΝ. And as the remainder of the parallelogram ΜΘ is to the remainder of the parallelogram ΒΚ, so the whole parallelogram ΜΔ is to the whole parallelogram ΟΝ. And since the parallelogram ΔΑ is equal to the parallelogram ΔΒ, let the common parallelogram ΔΠ be subtracted, therefore the parallelogram ΚΠ is equal to the parallelogram ΠΘ.

Let the common parallelogram ΑΒ be added, therefore the whole parallelogram ΒΚ is equal to the whole parallelogram ΑΘ. Therefore as the parallelogram ΜΔ is to the parallelogram ΔΑ, so the parallelogram ΜΘ is to the parallelogram ΑΘ.

But as the parallelogram ΜΔ is to the parallelogram ΔΑ , so ΜΚ is to ΚΑ, and so ΖΓ is to ΑΓ, and as the parallelogram ΜΘ is to the parallelogram ΑΘ, and so ΜΦ is to ΦΑ, and so ΖΛ is to ΛΑ, therefore as ΖΓ is to ΑΓ, so ΖΛ is to ΛΑ, therefore also as ΖΓ is to ΑΓ, so ΖΛ is to ΛΑ.

[Proposition] 36

*With the same suppositions if the straight line drawn across from the point neither cuts the section at two points nor is parallel to the asymptote, it will meet the opposite hyperbola, and as the whole straight line is to the straight line between the section and the parallel through the point of contact, so will the straight line between the opposite hyperbola and the asymptote be to the straight line between the asymptote and the other hyperbola* [38].

Let there be the opposite hyperbolas Α and Β whose center is Γ and asymptotes ΔΕ and ΖΗ, and let some point H be taken on ΓΗ, and from it let ΗΒΕ be drawn tangent, and ΗΘ neither parallel to ΓΕ nor cutting the section at two points [according to Proposition I.26].

It has been shown that ΗΘ continued meets ΓΔ and therefore also the hyperbola Α. Let it meet at Α, and let ΚΒΛ be drawn through Β parallel to ΓΗ.

I say that as AK is to ΚΘ, so ΑΗ is to ΗΘ.

[Proof}. For let ΘΜ and ΑΝ be drawn from Α and Β parallel to ΓΗ, and ΒΞ, ΗΠ, and ΡΘΣΝ from Β, Η, and Θ parallel to ΔΕ. Since then [according to Proposition II.16] ΑΔ is equal to ΗΘ, as ΑΗ is to ΗΘ, so ΔΘ is to ΘΗ.

But as ΑΗ is to ΗΘ, so ΝΣ is to ΣΘ, and as ΔΘ is to ΗΘ, so ΓΣ is to ΣΗ.

And therefore as ΝΣ is to ΣΘ, so ΓΣ is to ΣΗ. But as ΝΣ is to ΣΘ, so the parallelogram ΝΓ is to the parallelogram ΓΘ, and as ΓΣ is to ΣΗ, so the parallelogram ΡΓ is to the parallelogram ΡΗ, therefore also as the parallelogram ΝΓ is to the parallelogram ΓΘ, so the parallelogram ΡΓ is to the parallelogram ΡΗ. And as one is to one, so are all to all, therefore the parallelogram ΝΓ is to the parallelogram ΓΘ, so the whole parallelogram ΝΛ is to the sum of the whole parallelogram ΓΘ and the parallelogram ΡΗ. And since ΖΒ is equal to ΒΗ, also ΛΒ is equal to ΒΠ, and the parallelogram ΛΞ is equal to the parallelogram ΒΗ.

And [according to Proposition II.12] the parallelogram ΛΞ is equal to the parallelogram ΓΘ, therefore also the parallelogram ΒΗ is equal to the parallelogram ΓΘ.

Therefore as the parallelogram ΝΓ is to the parallelogram ΓΘ, so the whole parallelogram ΝΛ is to the sum of the whole parallelogram ΒΗ and the parallelogram ΡΗ or as the parallelogram ΝΓ is to the parallelogram ΓΘ, so the parallelogram ΝΛ is to the parallelogram ΡΞ.

But the parallelogram ΡΞ is equal to the parallelogram ΛΘ, since also [according to Proposition II.12] the parallelogram ΓΘ is equal to the parallelogram ΒΓ, and the parallelogram ΜΒ is equal to the parallelogram ΞΘ. Therefore as the parallelogram ΝΓ is to the parallelogram ΓΘ, so the parallelogram ΝΛ is to the parallelogram ΛΘ.

But as the parallelogram ΝΓ is to the parallelogram ΓΘ, so ΝΣ is to ΣΘ, and so ΑΗ is to ΗΘ, and as the parallelogram ΝΛ is to the parallelogram ΛΘ, so ΝΡ is to ΡΘ,and so ΑΚ is to ΚΘ, therefore also as ΑΚ is to ΚΘ,so ΑΗ is to ΗΘ

## [Proposition] 37

*If two straight lines touching a section of a cone or the circumference of a circle or opposite hyperbolas meet, and a straight line is joined to the points of contact, and from the point of meeting of the tangents some straight line is drawn across cutting the line [of the section] at two points, then as the whole straight line is to the straight line cut off outside, so will the segments continued by the straight line joining the points of contact be to each other*[39].

Let there be the section of a cone ΑΒ and tangents ΑΓ and ΓΒ and let ΑΒ be joined and let ΓΔΕΖ be drawn across.

I say that as ΓΖ is to ΓΔ, so ΖΕ is to ΕΔ.

[Proof]. For let the diameters ΓΘ and ΑΚ be drawn through Γ and Α,and through Ζ and Δ let ΔΠ, ΖΡ, ΛΕΜ, and ΝΔΟ parallel to ΑΘ and ΑΓ be drawn. Since then ΛΕΜ is parallel to ΞΔΟ as ΖΓ is to ΓΔ, so ΛΖ is to ΞΔ, and so ΖΜ is to ΔΟ, and so ΛΜ is to ΞΟ, and therefore as sq.ΛΜ is to sq.ΞΟ, so sq.ΖΜ is to sq.ΔΟ.

But as sq.ΛΜ is to sq.ΞΟ, so the triangle ΛΜΓ is to the triangle ΞΓΟ [according to Proposition VI.19 of Euclid], and as sq.ΖΜ is to sq.ΔΟ, so the triangle ΖΡΜ is to the triangle ΔΠΟ, therefore also as the triangle ΛΜΓ is to the triangle ΞΓΟ, so the triangle ΖΡΜ is to the triangle ΔΠΟ, and so the remainder of the quadrangle ΛΓΡΖ is to the remainder of the quadrangle ΞΓΠΔ.

But [according to Propositions III.2 and III.11] the quadrangle ΛΓΡΖ is equal to the triangle ΑΛΚ, and the quadrangle ΞΓΠΔ is equal to the triangle ΑΝΞ, therefore as sq.ΛΜ is to sq.ΞΟ, so the triangle ΑΛΚ is to the triangle ΑΝΞ.

But as sq.ΛΜ is to sq.ΞΟ, so sq.ΖΓ is to sq.ΓΔ, and as the triangle ΑΛΚ is to the triangle ΑΝΞ, so sq.ΛΑ is to sq.ΑΞ, and so sq.ΖΕ is to ΕΔ, therefore also as sq.ΖΓ is to sq.ΓΔ, so sq.ΖΕ is to sq.ΕΔ.

And therefore as ΖΓ is to ΓΔ, so ΖΕ is to ΕΔ.

## [Proposition] 38

*With the same suppositions if some straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and a straight line drawn through the midpoint of the straight* *line joining the points of contact cuts the section at two points and the straight line through the point of meeting parallel to the straight line joining the points of contact, then as the whole straight line drawn across is to the straight line cut off outside between the section and the parallel, so will the segments continued by the straight line joined to the points of contact be to each other* [40].

Let there be the section AB and tangents AΓ and BΓ and AB is the straight line joining the points of contact, and AN and ΓM are diameters, then it is evident that AB has been bisected at E [according to Propositions II.30 and II.39]. Let ΓO be drawn from Γ parallel to AB, and let ZEΔO be drawn across through E

I say that as ZO is to OΔ so ZE is to EΔ.

[Proof]. For let ΛZKM and ΔΘHΞN be drawn through Z and Δ parallel to AB, and through Z and H let ZP and HΠ be drawn parallel to ΛΓ. Then likewise as before [in Proposition III.37] it will be shown that as sq.ΛM is to sq.ΞΘ, so sq.ΛA is to sq.AΞ. And as sq.ΛM is to sq.ΞΘ, so sq.ΛΓ is to sq.ΓΞ, and so sq.ZO is to sq.OΔ, and as sq.ΛA is to sq.AΞ, so sq.ZE is to sq.EΔ, therefore as sq.ZO is to sq.OΔ, so sq.ZE is to sq.EΔ, and as ZO is to OΔ, so ZE is to EΔ.

## [Proposition] 39

*If two straight lines touching opposite hyperbolas meet, and a straight line is drawn through the points of contact, and a straight line drawn from the point of meeting of the tangents cuts both hyperbolas and the straight line joining the points of contact, then as the whole straight line drawn across is to the straight line cut off outside between the section and the straight line joining the points of contact, so will the segments of the straight line drawn by the segments and the point of meeting of the tangents be to each other* [41].

Let there be the opposite hyperbolas A and B whose center is Γ ,and tangents AΔ and ΔB, and let AB and ΓΔ be joined and continued, and through Δ let some straight line EΔZH be drawn across.

I say that as EH is to HZ, so EΔ is to ΔZ.

[Proof]. For let AΓ be joined and continued, and through E and Z let EΘΣ and ZΛMNΞO be drawn parallel to AB, and parallel to AΔ, EΠ, and ZP.

Since then ZΞ and EΣ are parallel, and EZ, ΞΣ, and ΘM have been drawn through them, as EΘ is to ΘΣ, so ZM is to MΞ. And alternately as EΘ is to ZM, so ΘΣ is to MΞ, therefore also as sq.EΘ is to sq.ZM, so sq.ΘΣ is to sq.MΞ.

But as sq.EΘ is to sq.ZM, so the triangle EΘΠ is to the triangle ZPM, and as sq.ΘΣ is to sq. MΞ, so the triangle ΔΘΣ is to the triangle ΞMΔ, therefore also as the triangle EΘΠ is to the triangle ZPM, so the triangle ΔΘΣ is to the triangle ΞMΔ. And [according to Proposition III.11] the triangle EΘΠ is equal to the sum of the triangles AΣK and ΔΘΣ, and the triangle ZPM is equal to the sum of the triangles AΞN and ΞMΔ, therefore as the triangle ΔΘΣ is to the triangle ΞMΔ, so the sum of the triangles AΣK and ΔΘΣ is to the sum of the triangles AΞN and ΞMΔ, and the remainder of the triangle AΣK is to the remainder of the triangle ANΞ, so the triangle ΔΘΣ is to the triangle ΞMΔ.

But as the triangle AΣK is to the triangle ANΞ, so sq.KA is to sq.AN, and so sq.EH is to sq.ZH, and as the triangle ΔΘΣ is to the triangle ΞMΔ, so sq.ΘΔ is to sq.ΔM, and so sq.EΔ is to sq.ΔZ. Therefore also as EH is to ZH, so EΔ is to ΔZ.

## [Proposition] 40

*With the same suppositions, if a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and if a straight line drawn from the midpoint of the straight line joining the points of contact cuts both hyperbolas and the straight line parallel to the straight line joining the points of contact, then as the whole straight line drawn across is to the straight line cut off outside between the parallel and the hyperbola, so will the straight line's segments drawn by the hyperbolas and the straight line joining the points of contact be to each other*[42].

Let there be the opposite hyperbolas A and B whose center is Γ, and tangents AΔ and ΔB, and let AB and ΓΔE be joined, therefore [according to Proposition II.39] AE is equal to EB. And from Δ let ZΔH be drawn parallel to AB, and from E let ΛE be drawn at random.

I say that as ΘΛ is to ΛK, so ΘE is to EK.

[Proof]. From Θ and K let NMΘΞ and KOP be drawn parallel to AB, and ΘK and KΣ parallel to AΔ, and let ΞAΓT be drawn through.

Since then ΞAY and MAΠ have been drawn across the parallels ΞM and KΠ, as ΞA is to AY, so MA is to AΠ.

But as ΞA is to AY, so ΘE is to EK, and as ΘE is to EK, so ΘN is to KO because of the similarity of the triangles ΘEN and KEO, therefore as ΘN is to ΛO, so MA is to AΠ, therefore also as sq.ΘN is to sq.KO, so sq.MA is to sq.AΠ.

But as sq.ΘN is to sq.KO, so the triangle ΘBN is to the triangle KΣO, and as sq.MA is to sq.AΠ, so the triangle ΞMA is to the triangle AYΠ, therefore also as the triangle ΘBN is to the triangle ΚΣΟ, so the triangle ΞΜΑ is to the triangle ΑΥΠ.

And [according to Proposition III.11] the triangle ΘΝΡ is equal to the sum of the triangles ΞΜΑ and ΜΝΔ, and the triangle ΚΣΟ is equal to the sum of the triangles ΑΥΠ and ΔΟΠ, therefore also as the sum of the triangles is ΞΜΑ and ΜΝΔ is to the sum of the triangles ΑΥΠ and ΔΟΠ, so the triangle ΞΜΑ is to the triangle ΑΥΠ, therefore also as the remainder of the triangle ΝΜΔ is to the remainder of the triangle ΔΟΡ, so the whole is to the whole.

But as the triangle ΞΜΑ is to the triangle ΑΥΠ, so sq.ΞΑ is to sq.ΑΥ, and as the triangle ΝΜΔ is to the triangle ΔΟΠ, so sq.ΜΝ is to sq.ΠΟ, therefore also as sq.ΜΝ is to sq.ΠΟ, so sq.ΞΑ is to sq.ΑΥ.

But as sq.ΜΝ is to sq.ΠΟ, so sq.ΝΔ is to sq.ΟΔ, and as sq.ΞΑ is to sq.ΑΥ, so sq.ΘΕ is to sq.ΕΚ, and as sq.ΝΔ is to sq.ΔΟ, so sq.ΘΛ is to sq.ΛΚ, therefore also as sq.ΘΕ is to sq.ΕΚ, so sq.ΘΛ is to sq.ΛΚ.

Therefore as ΘΕ is to ΕΚ, so ΘΛ is to ΛΚ.

## [Proposition] 41

*If three straight lines touching a parabola meet each other, they will be cut in the same ratio* [43].

Let there be the parabola ΑΒΓ, and tangents ΑΔΕ, ΕΖΓ and ΔΒΖ.

I say that as ΓΖ is to ΖΕ, so ΕΔ is to ΔΑ, and so ΖΒ is to ΒΔ.

[Proof]. For let ΑΓ be joined and bisected at Η. Then it is evident [according to Proposition II.29] that the straight line from Ε to Η is a diameter of the parabola. If then is goes through Β ΔΖ is parallel to ΑΓ [according to Proposition II.5] and will be bisected by ΕΗ, and therefore [according to Proposition I.35] ΑΔ is equal to ΔΕ, and ΓΖ is equal to ΖΕ, and what was sought is apparent.

Let it not go through Β, but through Θ, and let ΚΘΛ be drawn through Θ parallel to ΑΓ, therefore it will touch the parabola at Θ [according to Proposition I.32], and because of already said [in Proposition I.35] ΑΚ is equal to ΚΕ, and ΛΓ is equal to ΛΕ.

Let ΜΝΒΞ be drawn through Β parallel to ΕΗ, and ΑΟ and ΓΠ through Α and Γ parallel to ΔΕ. Since then ΜΒ is parallel to ΕΘ, ΜΒ is a diameter [according to Propositions I.40 and I.51], and ΔΖ touches at Β, therefore ΑΟ and ΓΠ have been dropped as ordinates [according to Proposition II.5 and Definition 4]. And since ΜΒ is a diameter, and ΓΜ a tangent, and ΓΠ an coordinate [according to Proposition I.35] ΜΒ is equal to ΒΠ, and so also ΜΖ is equal to ΖΓ.

And since MZ is equal to ZΓ, and EΛ is equal to ΛΓ, as MΓ is to ΓZ, so EΓ is to ΓΛ, and corresponding as MΓ is to EΓ, so ΓZ is to ΓΛ.

But as MΓ is to EΓ, so ΞΓ is to ΓH, therefore also as ΓZ is to ΓΛ, so ΞΓ is to ΓH. And as ΓΛ is to EΓ, so ΓH is to ΓA, therefore ex as ΓA is to ΞΓ, so EΓ is to ΓZ, and convertendo as EΓ is to ZE, so ΓA is to AΞ, and *separando* as ΓZ is to ZE, so ΞΓ is to AΞ.

Again since MB is a diameter and AN a tangent and AO an ordinate [according to Proposition I,35] NB is equal to BO, and NΔ is equal to ΔA. And also EK is equal to KA, therefore as AE is to KA, so NA is to ΔA, and correspondingly as AE is to NA, so KA is to ΔA.

But as AE is to NA , so HA is to AΞ, therefore also as KA is to ΔA, so HA is to AΞ. And also an AE is to KA, so ΓA is to HA, therefore *ex aequa* as AE is to ΔA, so ΓA is to AΞ, and separando as EΔ is to ΔA, so ΞΓ is to AΞ.

And it was also shown that as ΞΓ is to AΞ, so ΓZ is to ZE, therefore as ΓZ is to EZ, so EΔ is to ΔA.

Again since as ΞΓ is to AΞ, so ΓΠ is to AO, and ΓΠ is equal to the double BZ, and ΓM is equal to the double MZ, and AO is equal to the double BΔ, and AN is equal to the double NΔ, therefore as ΞΓ is to AΞ, so ZB is to BΔ, and so ΓZ is to ZE, and so EΔ is to ΔA.

## [ Proposition ] 42

*If in a hyperbola or an ellipse or the circumference of a circle or opposite hyperbolas straight lines are drawn from the vertices of the diameter parallel to an ordinate, and some other straight line at random is drawn tangent, it will cut off from them straight lines under which the rectangular plane equal to the quarter of the eidos corresponding to the same diameter* [44] .

Let there be some of the mentioned sections, whose diameter is AB, and from A and B let AΓ and ΔB be drawn parallel to an ordinate, and let some other straight line ΓEΔ be tangent at E.

I say that pl.AΓ,BΔ is equal to the mentioned part of the *eidos* corresponding to AB.

[Proof]. For let its center be Z, and through it let ZH be drawn parallel to AΓ and BΔ. Since then AΓ and BΔ are parallel, and ZH is also parallel, [to them], therefore [according to Definition 6] it is the diameter conjugate to AB, and so sq.ZH is equal to the quarter of the *eidos* corresponding to AB [according to Definition 11].

If then ZH goes through E in the case of the ellipse and circle [according to Propositions I.32 and I.33 of Euclid] ΑΓ is equal to ΖΗ and is equal to ΒΔ and it is immediately evident that pl.ΑΓ,ΒΔ is equal to sq.ΖΗ or the quarter of the *eidos* corresponding to ΑΒ.

Then let it not go through it, and let ΔΓ and ΒΑ continued meet at Κ, and let ΕΛ be drawn through Ε parallel to ΑΓ, and ΕΜ parallel to ΑΒ.

Since then pl.ΚΖΛ is equal to sq.ΑΖ [according to Proposition I.37], as ΚΖ is to ΑΖ, so ΑΖ is to ΖΛ, and [according to Proposition V.18 of Euclid] as ΚΑ is to ΑΛ, so ΚΖ is to ΑΖ or ΖΒ, inversely as ΖΒ is to ΚΖ, so ΑΛ is to KA, *componendo* or separando as ΒΚ is to ΚΖ ,so ΛΚ is to ΚΑ.

Therefore also as ΔΒ is to ΖΘ, so ΕΛ is to ΓΑ. Therefore pl.ΔΒ,ΓΑ is equal to pl.ΖΘ,ΕΛ , which is equal to pl.ΘΖΜ.

But [according to Proposition I.38] pl.ΘΖΜ is equal to sq.ΖΗ, which is equal [according to Definition 11] to the quarter of the *eidos* corresponding to ΑΒ, therefore also pl.ΔΒ,ΓΑ is equal to the quarter of the *eidos* corresponding to ΑΒ.

## [Proposition] 43

*If a straight line touches a hyperbola, it will cut off from the asymptote beginning with the center of the section straight lines containing a rectangular plane equal to the plane under the straight lines cut off by the tangent at the vertex of the hyperbola at its axis* [45].

Let there be the hyperbola ΑΒ, and asymptotes ΓΔ and ΔΕ, and the axis ΒΔ, and let ΖΒΗ be drawn through Β tangent, and some other tangent ΓΑΘ be drawn at random.

I say that pl.ΖΔΗ is equal to pl.ΓΔΘ.

[Proof]. For let ΑΚ and ΒΛ be drawn from Α and Β parallel to ΔΗ, and ΑΜ and ΒΝ parallel to ΓΔ. Since then ΓΑΘ touches[according to PropositionII.3] ΓΑ is equal to ΑΘ, and so ΓΘ is equal to the double ΑΘ, and ΓΔ is equal to the double ΑΜ, and ΔΘ is equal to the double ΑΚ.

Therefore pl.ΓΔΘ is equal to the quadruple pl.ΚΑΜ.
Then likewise it could be shown that pl.ΖΔΗ is equal to the quadruple pl.ΛΒΝ.

But [according to Proposition II.12] pl.ΚΑΜ is equal to pl.ΛΒΝ.

Therefore also pl.ΓΔΘ is equal to pl.ΖΔΗ, then likewise it could be shown, even if ΔΒ were some other diameter and not the axis.

## [Proposition] 44

*If two straight lines touching a hyperbola or opposite hyperbolas meet the asymptotes, then the straight lines drawn to the section will be parallel to the straight line joining the points of contact* [46].

Let there be either the hyperbola or the opposite hyperbolas ΑΒ, and asymptotes ΓΔ and ΔΕ, and tangents ΓΑΘΖ and ΕΒΘΗ, and let ΑΒ, ΖΗ, and ΓΕ be joined.

I say that they are parallel.

[Proof]. For since [according to Proposition III.43] pl.ΓΔΖ is equal to pl.ΗΔΕ, therefore as ΓΔ is to ΔΕ, so ΗΔ is to ΔΖ, therefore ΓΕ is parallel to ΖΗ. And therefore as ΘΖ is to ΖΓ, so ΘΗ is to ΗΕ. And as ΖΓ is to ΑΓ, so ΗΕ is to HB. For each is the double [according to Proposition II.3], therefore ex as ΘΗ is to ΗΒ, so ΘΖ is to ΖΑ. Therefore ΖΗ is parallel to ΑΒ.

[Proposition] 45

*If in a hyperbola or an ellipse or the circumference of a circle or opposite hyperbolas straight lines are drawn from the vertex of the axis at right angles, and a rectangular plane equal to the quarter of the* eidos *is applied to the axis on each side and increased in the case of the hyperbola and the opposite hyperbolas, but decreased in the case of the ellipse, and some straight line is drawn tangent to the section, and meeting the perpendicular straight lines, then the straight lines drawn from the points of meeting to the points of the beginnings of application make right angles at the mentioned points* [47].

Let there be one of the mentioned sections whose axis is ΑΒ, and ΑΓ and ΒΔ are drawn at right angles, and ΓΕΔ is tangent, and let pl.ΑΖΒ and pl.ΑΗΒ equal to the quarter of the *eidos* be applied on each side [of ΑΒ] as it has been said, and let ΓΖ, ΓΗ, ΔΖ, and ΔΗ be joined.

I say that the angles ΓΖΔ and ΓΗΔ are right .

[Proof]. For since it has been shown that pl.ΑΓ,ΒΔ is equal to the quarter of the *eidos* corresponding to ΑΒ, and since also pl.ΑΖΒ is equal to the quarter of the *eidos* corresponding to ΑΒ, therefore pl.ΑΓ,ΒΔ is equal to pl.ΑΖΒ.

Therefore as ΑΓ is to ΑΖ, so ΖΒ is to ΒΔ. And the angles at Α and Β are right, therefore [according to Proposition VI.6 of Euclid] the angle ΑΓΖ is equal to the angle ΒΖΔ, and the angle ΑΖΓ is equal to the angle ΖΔΒ. And since the angle ΓΑΖ is right, therefore the sum of the angles ΑΓΖ and ΑΖΓ is equal to one right angle.

And it has also been shown that the angle ΑΓΖ is equal to the angle ΔΖΒ, therefore the sum of the angles ΑΖΓ and ΔΖΒ is equal to one right angle.

Therefore the angle ΔΖΓ is equal to one right angle.

Then likewise it could also be shown that the angle ΓΗΔ is equal to one right angle [48] .

[Proposition] 46

*With the same suppositions, the joined straight lines make equal angles with the tangents* [49].

For with the same suppositions I say that the angle ΑΓΖ is equal to the angle ΒΓΗ and the angle ΓΔΖ is equal to the angle ΒΔΗ.

[Proof]. For since it has been shown [in Proposition III.45] that both angles ΓΖΔ and ΓΗΔ are right, the circle described about ΓΔ as a diameter will pass through Ζ and Η, therefore the angle ΔΓΗ is equal to the angle ΔΖΗ for they are on the same arc of the circle. And it was shown that the angle ΔΖΗ is equal to the angle ΑΓΖ [according to Proposition III.45], and so the angle ΔΓΗ is equal to the angle ΑΓΖ.

And likewise also the angle ΓΔΖ is equal to the angle ΒΔΗ [50].

[Proposition] 47

*With the same suppositions the straight line drawn from the point of meeting of the joined straight lines to the point of contact will be perpendicular to the tangent* [51].

For let the same as before be supposed and let ΓΗ and ΖΔ meet each other at Θ, and let continued ΓΔ and ΒΑ meet at Κ, and let ΕΘ be joined.

I say that ΕΘ is perpendicular to ΓΔ.

[Proof]. For if not, let ΘΛ be drawn from Θ perpendicular to ΓΔ. Since then [according to Proposition III.46] the angle ΓΔΖ is equal to the angle ΒΔΗ, and also the right angle ΔΒΗ is equal to the right angle ΔΛΘ, therefore the triangle ΔΗΒ is similar to the triangle ΛΘΔ..Therefore as ΗΔ is to ΒΘ, so ΒΔ is to ΔΛ.

But as ΗΔ is to ΔΘ, so ΖΓ is to ΓΘ because the angles at Ζ and Η are right [according to Proposition III.45] and the angles at Θ are equal, but as ΖΓ is to ΓΘ, so ΑΓ is to ΓΛ because of the similarity of the triangles ΑΖΓ and ΛΓΘ [according to Proposition III.46], therefore as ΒΔ is to ΔΛ, so ΑΓ is to ΓΛ, and alternately as ΒΔ is to ΑΓ, so ΔΛ is to ΓΛ.

But as ΒΔ is to ΑΓ, so ΒΚ is to ΚΑ, therefore also as ΔΛ is to ΓΛ, so ΒΚ is to ΚΑ . Let ΕΜ be drawn from Ε parallel to ΑΓ, therefore it will have been dropped as an ordinate to AB [according to Proposition II.7], and as BK is to KA, so BM is to MA [according to Proposition I.36]. And as BM is to MA, so ΔE is to EΓ, therefore also as ΔΛ is to ΓΛ, so ΔE is to EΓ, and this is impossible. Therefore ΘΛ is not perpendicular, nor is any over straight line except ΘE [52].

[Proposition] 48

*With the same suppositions it must be shown that the straight lines drawn from the point of contact to the points produced by the application make equal angles with the tangent* [53].

For let to same suppositions, and let EZ and EH be joined.

I say that the angle ΓEZ is equal to the angle HEΔ.

[Proof]. For since [according to Propositions III.45 and III.47] the angles ΔHΘ and ΔEΘ are right the circle described about ΔΘ as a diameter will pass through E and H [according to Proposition III.31 of Euclid ] , and so the angle ΔΘH is equal to ΔEH [according to Proposition III.21 of Euclid] for they are in the same arc. Likewise then also the angle ΓEZ is equal to the angle ΓΘZ.

But the angle ΓΘZ is equal to the angle ΔΘH for they are vertical angles, therefore also the angle ΓEZ is equal to the angle ΔEH [54] .

[Proposition] 49

*With the same suppositions if from one of the points [of the beginnings of application] a perpendicular is drawn to the tangent, then the straight lines from that point to the ends of the axis make a right angle* [55].

For let the same be supposed, and let the perpendicular HΘ be drawn from H to ΓΔ, and let AΘ and BΘ be joined.

I say that the angle AΘB is right.

[Proof]. For since the angle ΔBH is right, and the angle ΔΘH also [is right], the circle described about ΔH as a diameter will pass through Θ and B, and the angle BΘH is equal to angle BΔH.

But it was shown [in Proposition III.45] that the angle AHΓ is equal to the angle BΔH, therefore also the angle BΘH is equal to the angle AHΓ ,which is equal to the angle AΘΓ [according to Proposition III.21 of Euclid]. And so also the angle ΓΘH is equal to the angle AΘB.

But the angle ΓΘH is right, therefore the angle AΘB also is right [56] .

[Proposition] 50

*With the same suppositions if from the center of the section there falls to the tangent a straight line parallel to the straight line drawn through the point of contact, and one of the points [of the beginning of application] , then it will be equal to the half of the axis* [57] .

Let there be the same as before, and let Θ be the center, and let EZ be joined, and let ΔΓ and BA meet at K, and through Θ let ΘΛ be drawn parallel to EZ.

I say that ΘΛ is equal to ΘB.

[Proof]. For let EH, AΛ, ΛB be joined, and through H let HM be drawn parallel to EZ. Since then [according to Proposition III.45] pl.AZB is equal to pl.AHB, therefore AZ is equal to HB.

But also AΘ is equal to ΘB, therefore also ZΘ is equal to ΘH. And so also EΛ is equal to ΛM.

And since it was shown [in Proposition III.48] that the angle ΓEZ is equal to the angle ΔEH, and the angle ΓEZ is equal to the angle EMH, therefore also the angle EMH is equal to the angle ΔEH. And therefore EH is equal to HM.

But it was also shown that EΛ is equal to ΛM, therefore HΛ is perpendicular to EM. And so through what was shown before [in Proposition III.49] that the angle AΛB is right, and the circle described about AB as a diameter will pass through Λ. And ΘA is equal to ΘB, therefore also, since ΘΛ is a radius of the semicircle, ΘΛ is equal to ΘB [58-59] .

[Proposition] 51

*If a rectangular plane equal to the quarter of the* eidos *is applied from both sides to the axis of a hyperbola or opposite hyperbolas and in creased and straight lines are deflected from the points of beginning of application to either one of the hyperbolas, then the greater of two straight lines increases the less by exactly as much as the axis* [60].

Let there be a hyperbola or opposite hyperbolas whose axis is AB and the center Γ, and let each of pl.AΔB and pl.AEB be equal to the quarter of the *eidos*, and from E and Δ let EZ and ZΔ be deflected to the line of the section.

I say that EZ is equal to the sum of ZΔ and AB.

[Proof]. For let ZKΘ be drawn tangent through Z, and HΓΘ through Γ parallel to ZΔ, therefore the angle KΘH is equal to the angle KZΔ for they are alternate. And [according to Proposition III.48] the angle KZΔ is equal to the angle HZΘ, therefore HZ is equal to HΘ. But HZ is equal to HE, since also AE is equal to ΒΔ, and ΑΓ is equal to ΓΒ, and therefore ΗΘ is equal to ΕΗ. And so ΖΕ is equal to the double ΗΘ.

And since it as been shown [in Proposition III.50] that ΓΘ is equal to ΓΒ, therefore ΖΕ is equal to the sum of the double ΗΓ and double ΓΒ.

But ΖΔ is equal to the double ΗΓ, and ΑΒ is equal to the double ΓΒ, therefore ΖΕ is equal to the sum of ΖΔ and ΑΒ. And so ΕΖ is greater than ΖΔ by ΑΒ.

[Proposition] 52

*If in an ellipse the rectangular plane equal to the quarter of the* eidos *is applied from both sides to the major axis and decreased , and from the points of beginnings of application straight lines are deflected to the line of the section, then they will be equal to the major axis* [61].

Let there be an ellipse whose major axis is ΑΒ, and let each of pl.ΑΓΒ and pl.ΑΔΒ be equal to the quarter of the *eidos*, and from Γ and Δ let ΓΕ and ΕΔ have been deflected to the line of the section.

I say that the sum ΓΕ and ΕΔ is equal to ΑΒ.

[Proof]. For let ΖΕΘ be drawn tangent, and Η be the center and through it let ΗΚΘ be drawn parallel to ΓΕ. Since then [according to Proposition III.48] the angle ΓΕΖ is equal to the angle ΘΕΚ ,and the angle ΓΕΖ is equal to the angle ΕΘΚ, therefore also the angle ΕΘΚ is equal to the angle ΘΕΚ.

Therefore ΘΚ is equal to ΚΕ. And since ΑΗ is equal to ΗΒ, and ΑΓ is equal to ΔΒ, therefore also ΓΗ is equal to ΗΔ, and so also ΕΚ is equal to ΚΔ.

And for this reason ΕΔ is equal to the double ΘΚ, and ΕΓ is equal to the double ΚΗ.

But also [according to Proposition III.50], ΑΒ is equal to the sum of ΕΔ and ΕΓ.

[Proposition] 53

*If in a hyperbola or an ellipse or the circumference of a circle or opposite hyperbolas straight lines are drawn from the vertex of a diameter parallel to an ordinate, and straight lines drawn from the same ends to the same point on the line of the section cut the parallels, then the rectangular plane under the straight lines cut off is equal to the* eidos *corresponding to the same diameter* [62].

Let there be one of the mentioned sections ΑΒΓ whose diameter is ΑΓ, and let ΑΔ and ΓΕ be drawn parallel to an ordinate, and let ΑΒΕ and ΓΒΔ be drawn across.

I say that pl.ΑΔ,ΕΓ is equal to the *eidos* corresponding to ΑΓ.

[Proof]. For let ΒΖ be drawn from Β parallel to an ordinate. Therefore [according to Proposition I.21 the ratio] pl.ΑΖΓ to sq.ΖΒ is compounded of [the ratios] the *latus transversum* to the *latus rectum* and sq.ΑΓ to the *eidos*.

But [the ratio] pl.ΑΖΓ to sq.ΕΒ is compounded of [the ratios] ΑΖ to ΖΒ and ΖΓ to ΖΒ, therefore [the ratio] the *eidos* to sq.ΑΓ is compounded of [the ratios] ΖΒ to ΑΖ and ΖΒ to ΖΓ,

But as ΑΖ is to ΖΒ, so ΑΓ is to ΓΕ, and as ΕΓ is to ΖΒ, so ΑΓ is to ΑΔ, therefore [the ratio] the *eidos* to sq.ΑΓ is compounded of [the ratios] ΓΕ to ΑΓ and ΑΔ to ΑΓ.

And also as pl.ΑΔ,ΓΕ is compounded of [the ratios] ΓΕ to ΑΓ and ΑΔ to ΑΓ, therefore as the *eidos* is to sq.ΑΓ, so pl.ΑΔ,ΓΕ is to sq.ΑΓ.

Therefore pl.ΑΔ,ΓΕ is equal to the *eidos* corresponding to ΑΓ.

## [Proposition] 54

*If two tangents to a section of a cone or to the circumference of a circle meet and through the points of contact parallels to the tangents are drawn, and from the points of contact, to the some point of the line of the section straight lines are drawn across cutting the parallels, then rectangular plane under the straight lines cut off to the square on the straight line joining the points of contact has a ratio compounded of the ratio which the inside segment joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact is equal in square to the remainder, and of the ratio which the plane under the tangents has to the quarter of the square on the straight line joining the points of contact* [63] .

Let there be a section of a cone or the circumference of a circle ΑΒΓ and tangents ΑΔ and ΓΔ, and let ΑΓ be joined and bisected at Ε, and let ΔΒΕ be joined, and let ΑΖ be drawn from Α parallel to ΓΔ, and ΓΗ from Γ parallel to ΑΔ, and let some point Θ on the section be taken, and let ΑΘ and ΓΘ be joined and continued to Η and Ζ.

I say that [the ratio] pl.ΑΖ,ΓΗ to sq.ΑΓ is compounded of [the ratios] sq.ΕΒ to sq.ΒΔ and pl.ΑΔΓ to the quarter of sq.ΑΓ or pl.ΑΕΓ.

[Proof]. For let ΚΘΟΞΛ be drawn from Θ parallel to ΑΓ, and from Β let ΜΒΝ be drawn parallel to ΑΓ, then it is evident that ΜΝ is tangent [according to Propositions II.5 , II,6 , and II.29]. Since then ΑΕ is equal to ΕΓ, also ΜΒ is equal to ΒΝ, and ΚΟ is equal to ΟΛ, and [according to Proposition II.7] ΘΟ is equal to ΟΞ, and ΚΘ is equal to ΞΛ.

Since then ΜΒ and ΜΑ are tangents and ΚΘΛ has been drawn parallel to ΜΒ [according to Proposition III.16] as sq.ΑΜ is to sq.ΜΒ, so sq.ΑΚ is to pl.ΞΚΘ or as sq.ΑΜ is to pl.ΜΒΝ, so sq.ΑΚ is to pl.ΛΘΚ.

And [according to Propositions V.18 and VI.2 of Euclid] as pl.ΝΓ,ΑΜ is to sq.ΑΜ, so pl.ΛΓ,ΑΚ is to sq.ΑΚ, therefore ex as pl.ΝΓ,ΑΜ is to pl.ΜΒΝ, so pl.ΛΓ,ΑΚ is to pl.ΛΘΚ.

But [the ratio] pl.ΛΓ,ΑΚ to pl.ΛΘΚ is compounded of [the ratios] ΛΓ to ΛΘ and ΑΚ to ΘΚ or [the ratio] pl.ΛΓ,ΑΚ to pl.ΛΘΚ is compounded of [the ratios] ΖΑ to ΑΓ and ΗΓ to ΓΑ, which is the same as pl.ΗΓ,ΖΑ to sq.ΓΑ. Therefore as pl.ΝΓ,ΑΜ is to pl.ΜΒΝ, so pl.ΗΓ,ΖΑ is to sq.ΓΑ.

But with pl.ΝΔΜ taken as a mean,[the ratio] pl.ΝΓ,ΑΜ to pl.ΜΒΝ, is compounded of [the ratios] pl.ΝΓ,ΑΜ to pl.ΝΔΜ and pl.ΝΔΜ to pl.ΜΒΝ, therefore [the ratio] pl.ΗΓ,ΖΑ to sq.ΓΑ is compounded of [the ratios] pl.ΝΓ,ΑΜ to pl.ΝΔΜ and pl.ΝΔΜ and pl.ΜΒΝ.

But as pl.ΝΓ,ΑΜ is to pl.ΝΔΜ, so sq.ΕΒ is to sq.ΒΔ, and as pl.ΝΔΜ is to pl.ΝΒΜ, so pl.ΓΔΑ is to pl.ΓΕΑ, therefore [the ratio] pl.ΗΓ,ΖΑ to sq.ΓΑ, is compounded of [the ratios] sq.ΒΕ to sq.ΒΔ and pl.ΓΔΑ to pl.ΓΕΑ.

## [Proposition] 55

*If two straight lines touching opposite hyperbolas meet, and through the point of meeting a straight line is drawn parallel to the straight line joining the point of contact, and from the points of contact parallels to the tangents are drawn across, and straight lines are drawn from the points of contact to the some point of one of the hyperbolas cutting the parallels, then the rectangular plane under the straight lines cut off will have to the square on the straight line joining the points of contact the ratio which the plane under the tangents is equal to the square of the straight line drawn through the point of meeting parallel to the straight line joining the points of contact as far as the section* [64].

Let there be the opposite hyperbolas ΑΒΓ and ΔΕΖ, and tangents to them ΑΗ and ΗΔ, and let ΑΔ be joined, and from Η let ΓΗΕ be drawn parallel to ΑΔ, and from Α let ΑΜ be drawn parallel to ΔΗ, and from Δ let ΔΜ be drawn parallel to ΑΗ, and let some point Ζ be taken on the hyperbola ΔΖ, and let ΑΝΖ and ΖΔΘ be joined.

I say that as sq.ΓΗ is to pl.ΑΗΔ, so sq.ΑΔ is to pl.ΘΑ,ΔΝ.

[Proof]. For let ZΛKB be drawn through Z parallel to AΔ. Since then it has been shown that [according to Proposition III.20] as sq.EH is to sq.HΔ, so pl.BΛZ is to sq.ΔΛ, and [according to Proposition II.38] ΓH is equal to EH and BK is equal to ΛZ, therefore as sq.ΓH is to sq.HΔ, so pl.KZΛ is to sq.ΔΛ. And also [according to Propositions VI.1 and VI.2 of Euclid] as sq.HΔ is to pl.AHΔ, so sq.ΔΛ is to pl.ΔΛ,AK ,therefore ex as sq.HΓ is to pl.AHΔ , so pl.KZΛ is to pl.ΔΛ,AK.

But [the ratio] pl.KZΛ to pl.ΔΛ,AK is compounded of [the ratios] KZ to AK and ZΛ to ΔΛ. But as KZ is to AK, so AΔ is to ΔN, and as ZΛ is to ΔΛ, so AΔ is to ΘA, therefore [the ratio] sq.ΓH to pl.AHΔ is compounded of [the ratios] AΔ to ΔN and AΔ to ΘA. And also [the ratio] sq.AΔ to pl.ΘA,ΔN is compounded of [the ratios] AΔ to ΔN and AΔ to ΘA, therefore as sq.ΓH is to pl.AHΔ, so sq.AΔ is to pl.ΘA,ΔN.

[Proposition] 56

*If two straight lines touching one of the opposite hyperbolas meet, and parallels to the tangents are drawn through the points of contact, and straight lines cutting the parallels are drawn from the point of contact to the some point of the other hyperbola, then the rectangular plane under the straight lines cut off will have to the square on the straight line joining the points of contact the ratio compounded of the ratio of the part of the straight line joining the point of meeting and the midpoint between the midpoint and the other hyperbola equal in square to the part between the same hyperbola and the point of meeting, and of the ratio of the plane under the tangents to the quarter of the square on the straight line joining the points of contact* [65].

Let there be the opposite hyperbolas AB and ΓΔ whose center is O, and tangents AEZH and BEΘK, and let AB be joined and be bisected at Λ. And let ΛE be joined and drawn across to Δ, and let AM be drawn from A parallel to BE, and BN from B parallel to AE, and let some point Γ be taken on the hyperbola ΓΔ, and let ΓBM and ΓAN be joined.

I say that [the ratio] pl.MA,BN to sq.AB is compounded of [the ratios] sq.ΛΔ to sq.ΔE and pl.AEB to quarter of sq.AB or pl.AΛB.

[Proof]. For let HΓK and ΘΔZ be drawn from Γ and Δ parallel to AB, then it is evident that ΘΔ is equal to ΔZ, and KΞ is equal to ΞH, and also ΞΓ is equal to ΞΠ, and so also ΓK is equal to HΠ.

And since ΑΒ and ΔΓ are opposite hyperbolas, and ΒΕΘ and ΘΔ are tangents, and ΚΗ is parallel to ΔΘ, therefore as sq.ΒΘ is to sq.ΘΔ, so sq.ΒΚ is to pl.ΠΚΓ [according to Proposition III.18].

But sq.ΘΔ is equal to pl.ΘΔΖ, pl.ΠΚΓ is equal to pl.ΚΓΗ, therefore as sq.ΒΘ is to pl.ΘΔΖ, so sq.BK is to pl.ΚΓΗ. And also as pl.ΖΑ,ΒΘ is to sq.ΒΘ, so pl.ΗΑ,ΒΚ is to sq.ΒΚ, therefore ex  as pl.ΖΑ,ΒΘ is to pl.ΘΔΖ, so pl.ΗΑ,ΒΚ is to pl.ΚΓΗ.

And with pl.ΘΕΖ taken as a mean, [the ratio] pl.ΖΑ,ΒΘ to pl.ΘΔΖ is compounded of [the ratios] pl.ΖΑ,ΘΒ to pl.ΘΕΖ and pl.ΘΕΖ to pl.ΘΔΖ, and as pl.ΖΑ,ΘΒ  is to pl.ΘΕΖ, so sq.ΛΔ is to sq.ΔΕ, and as pl.ΘΕΖ is to pl.ΘΔΖ, so pl.ΑΕΒ is to pl.ΑΛΒ, therefore [the ratio] pl.ΗΑ,ΒΚ to pl.ΚΓΗ is compounded of [the ratios] sq.ΛΔ to sq.ΔΕ and pl.ΑΖΒ to pl.ΑΛΒ. And [the ratio] pl.ΗΑ,ΒΚ to pl.ΚΓΗ is compounded of [the ratios] ΒΚ to ΚΓ and ΗΑ to ΓΗ.

But as ΒΚ is to ΚΓ, so ΜΑ is to ΑΒ, and as ΗΑ is to ΓΗ, so ΒΝ is to ΑΒ, therefore [the ratio] pl.ΜΑ,ΒΝ to sq.ΑΒ is compounded of [the ratios] ΜΑ to ΑΒ and ΒΝ to ΑΒ, that is the same as [the ratios] sq.ΛΔ to sq.ΔΕ and pl.ΑΕΒ to pl.ΑΛΒ.

# BOOK FOUR

Apollonius greets Attalus [1].

Earlier, I presented the first three books of my eight books treatise on conics to Eudemus of Pergamum, but with his having passed away I decided to write out the remaining books for you, because of your earnest desire to have them. To start, then, I am sending you the fourth book. This book treats of the greatest number of points at which sections of a cone can meet one another or meet a circumference of a circle, assuming that these do not completely coincide, and, moreover, the greatest number of points at which a section of a cone or a circumference of a circle can meet the opposite hyperbolas. Besides these questions, there are more that a few others of a similar character Conon of Samos presented the first mentioned question to Thrasydaeus without giving a correct proof, for which he was rightly attacked by Nicoteles of Cyrene [2]. As for the second question, Nicoteies, in replying to Conon only mentions that it can be proved, but I have found no proof either by him or by anyone else. Regarding the third and similar questions, however, I have not found them even noticed by anyone. And all these things just spoken of, whose demonstrations I have not found any where, require many and various striking theorems, of which most happen to be presented in the first three books of my treatise on conics, and the rest in this book. The investigation of these theorems is also of considerable use in the synthesis of problems and limits of possibility . So, Nicoteles was not speaking truly when, for the sake of his argument with Conon, he said that none of the things discovered by Conon were of any use for limits of possibility, but even if the limits of possibility are able to be obtained completely without these things yet, surely, some matters are more readily perceived by means of them, for example, whether a problem might be done in many ways, and in how many ways, or again, whether it might not be done at all. Moreover, this preliminary knowledge brings with it a solid starting point for investigations, and the theorems are useful for the analysis of limits of possibility. But apart from such usefulness, these things are also worthy of acceptance for the demonstrations themselves: indeed, we accept many things in mathematics for this and no other reason.

[Proposition] 1

*If a point is taken outside a section of a cone or the circumference of a circle, and from this point two straight lines are drawn towards the section, of which one touches the section and other cuts the section at two points, and if the straight line cut off inside the section is divided in that ratio which the whole straight line cut off has to the part outside bounded between the point and the section, so that homologous straight lines are at the same point, then the straight line drawn from the point of contact to the point of division will meet the line of the section, and the straight line drawn from the point of meeting to outside point will touch the section* [3].

Let there be the section of a cone or the circumference of a circle ΑΒΓ and let Δ be taken outside the section, from Δ let ΔΒ touch the section at Β and let ΔΕΓ cut the section at Ε and Γ, and let as ΓΖ is to ΖΕ, so ΓΔ is to ΔΕ.

I say that the straight line from Β to Ζ will meet the section, and the straight line drawn from the point of meeting to Δ will touch the section.

[Proof]. For let ΔΑ be drawn from Δ touching the section, and let ΒΑ be joined cutting ΕΓ, if possible, not at Ζ, but at Η. Now since ΒΔ and ΔΑ, touch the section, ΒΑ is drawn from the point of contact, and ΓΔ goes through ΑΒ cutting the section at Γ and Ε and meeting ΑΒ at Η, [according to Proposition III.37] as ΓΔ is to ΔΕ, so ΓΗ is to ΗΕ. But this is impossible for it was assumed that as ΓΔ is to ΔΕ, so ΓΖ is to ΖΕ. Therefore ΒΑ does not cut ΓΕ at a different point from Ζ, therefore it cuts ΓΕ at Ζ.

### [Proposition] 2

This is proved for all sections together. However regarding the hyperbola only, if ΔΕ touches the hyperbola and ΔΓ cuts it at two points Ε and Γ,.and if the point of contact, Β, is between Ε and Γ, and Δ is inside the angle between the asymptotes, then the proof is carried out similarly for from Δ it is possible to draw another straight line ΔΑ touching the hyperbola and the rest of the proof is done similarly [4].

### [Proposition] 3

With the same suppositions if Ε and Γ do not contain the point of contact, Β, between them, and let Δ be inside the angle between the asymptotes. Therefore from Δ it is possible to draw another straight line ΔΑ touching the section, and rest is proved as before [5].

[Proposition] 4

With the same suppositions if the points of the meeting Ε and Γ contain the point of contact, Β, and Δ is in the angle adjacent to the angle between the asymptotes, then the straight line from the point of contact to the point of division meets the opposite hyperbola, and the straight line drawn from the point of meeting to Δ will touch the opposite hyperbola [6].

[Proof]. For let Β and Θ be opposite hyperbolas, let ΚΛ and ΜΞΝ be asymptotes, and let Δ be in the angle ΛΞΝ. Furthermore let ΔΒ be drawn from Δ touching, and ΔΓ cut one of the hyperbolas, let the points of meeting Ε and Γ contain the point of contact Β, and let as ΓΖ is to ΖΕ , so ΓΔ is to ΔΕ. It is to be shown that the straight line joined from Β to Ζ will meet the hyperbola Θ, and that the straight line from the point of meeting to Δ will touch the hyperbola B.

Let ΔΘ be drawn from Δ touching the hyperbola, and let the straight line ΘΒ all fall, if possible, not at Ζ, but at Η. Therefore [according to Proposition III.37] as ΓΔ is to ΔΕ, so ΓΗ is to ΗΕ. But it is impossible for it was assumed that as ΓΔ is to ΔΕ, so ΓΖ is to ΖΕ.

[Proposition] 5

*With the same supposition if Δ is on an asymptote, the straight line drawn from Β to Ζ will be parallel to the asymptote* [7] .

[Proof]. For let the same be supposed, let Δ be on one of the asymptotes, ΜΝ. It is to be shown that the straight line drawn from Β parallel to ΜΝ will fall on Ζ. For if not, let the straight line, if possible, be ΒΗ. But then [according to Proposition III.35] as ΓΔ is to ΔΕ, so ΓΗ is to ΗΕ, but it is impossible.

[Proposition] 6

*If a point is taken outside a hyperbola, and from this point two straight lines are drawn to the hyperbola, one of which touches the hyperbola, and the other is parallel to one of the asymptotes, and if the segment of the latter straight line inside the hyperbola is equal to the segment cut off between the hyperbola and the point, then the straight line joined from the point of contact of the former straight line to the taken point will meet the hyperbola, and* *the straight line drawn from the point of meeting to the point outside will touch the hyperbola* [8].

Let there be the hyperbola ΑΕΒ, let Δ be some point taken outside it, and, to start, let Δ be inside the angle between the asymptotes, and from Δ let ΒΔ be drawn touching the hyperbola, let ΔΕΖ be parallel to the other of the asymptotes, and let ΕΖ be equal to ΔΕ.

I say that the straight line joining from Β and Ζ will meet the hyperbola and the straight line from the point of meeting to Δ will touch the hyperbola.

[Proof]. For let ΔΑ be drawn touching the hyperbola, and let ΒΑ be joined and cutting ΔΕ, if possible, not at Ζ but at some other point Η. Then [according to Proposition III.30] ΔΕ will be equal to ΕΗ. But it is impossible for it was assumed that ΔΕ is equal to ΕΖ.

[Proposition] 7

*With the same suppositions* Δ *be in the angle adjacent to the angle between the asymptotes.*

I say that the same will come to pass [9].

[Proof]. For let ΔΘ be drawn touching the hyperbola and let ΘΒ be joined and let, if possible, fall not on Ζ but on Η. Therefore [according to Proposition III.31] ΔΕ is equal to ΕΗ. But it is impossible for it was assumed that ΔΕ is equal to ΕΖ.

[Proposition] 8

*With the same suppositions if* Δ *is on one of the asymptotes and let the remaining constructions be the same.*

I say that the straight line drawn from the point of contact to the end of the straight line cut off will be parallel to the asymptote on which Δ is situated [10].

[Proof]. Let there be the construction just mentioned, and let ΕΖ be equal to ΔΕ, and from Β let ΒΗ be drawn, if possible, parallel to ΜΝ. Therefore [according to Proposition III.34] ΔΕ is equal to ΕΗ. But it is impossible for it was assumed that ΔΕ is equal to ΕΖ.

[Proposition] 9

*If from the some point two straight lines are drawn each cutting a section of a cone or the circumference of a circle at two points ,and if the segments cut off inside are divided in the same ratio as the wholes are to the segments cut off outside, so that the homologous straight lines are at the same point, then the straight line drawn through the points of division will meet the section at two points, and straight lines drawn from the points of meeting to the point outside will touch the section* [11].

Let there be the section described by us ΑΒ, and from a point Δ [outside it] let ΔΕ and ΔΖ be drawn cutting the section at Θ and Ε and at Ζ and Η, respectively. Furthermore let as ΕΛ is to ΛΘ, so ΔΕ is to ΘΔ, and at ΖΚ is to ΚΗ, so ΔΖ is to ΔΗ.

I say that the straight line joining Λ to Κ will meet the section at both ends, and the straight lines joining the points of meeting will touch the section.

[Proof]. For since ΕΔ and ΖΔ both cut the section at two points, it is possible to draw a diameter of the section through Δ, and with that also straight lines touching the section on either side. Let straight lines ΔΒ and ΔΑ be drawn touching section, and let ΒΑ be joined not passing through ΛΚ, if possible, but through only one of these two, or through neither. First, let it pass through Λ only and let it cut ΖΗ at M. Therefore [according to Proposition III.37] as ΖΔ is to ΔΗ, so ΖΜ is to ΜΗ, but this is impossible for it has been assumed that as ΖΔ is to ΔΗ, so ΖΚ is to ΚΗ.

If ΒΑ passes through neither Λ nor Κ then, the absurdity occurs with regards to each straight line ΔΕ and ΔΖ.

[Proposition] 10

*The reasons above are common for all sections. However regarding the hyperbola only, if the other reasons are assumed, and if the points of meeting of the one straight line are between the points of meeting of the other straight line, and if Δ is inside the angle between the asymptotes, the same reasons said above will happen as we said above in Theorem 2* [Proposition IV.2] [12].

[Proposition] 11

With the same suppositions if the points of meeting of one of the straight lines do not contain the points of meeting of the other straight line, then Δ is in the angle between the asymptotes and the diagram and the proof will be the same as in Theorem 9 [Proposition IV.9] [13].

[Proposition] 12

*With the same suppositions if the points of meeting of one of the straight lines contain those other straight lines, and if the chosen point is in the angle adjacent to the angle between the asymptotes, then the straight line drawn through the points of division and continued will meet the opposite hyperbola, and the lines drawn from the points of meeting to Δ will touch the opposite hyperbolas* [14].

Let there be the hyperbola ZH, and its asymptotes NΞ and OΠ, and its center be Π. Furthermore let Δ be in the angle ΞPΠ, let ΔE and ΔZ be drawn cutting the hyperbola each at two points, let E and Θ be between Z and H, and let be that EΔ is to ΔΘ, so EK is to KΘ, and that as ZΔ is to ΔH, so ZΛ is to ΛH.

It is to be shown that the [straight line] through K and Λ will meet both [the hyperbola] EZ and also the opposite hyperbola, and the lines from the points of meeting to Δ will touch the hyperbolas.

[Proof]. For let M be the opposite hyperbola, and from Δ let ΔM and ΔΣ be drawn touching the hyperbola, let MΣ be joined, and, if possible, let it not pass through K and Λ, but rather through only one of these two points for through neither.

First let it pass through K and cut ZH at X. Therefore [according to Proposition III.37] as ZΔ is to ΔH, so XZ is to XH. But this is impossible for it has been assumed that as ZΔ is to ΔH, so ZΛ is to ΛH.

If MΣ passes through neither K nor Λ, then the impossibility occurs with regards to each straight line EΔ and ΔZ.

[Proposition] 13

*With the same suppositions if Δ is on one of the asymptotes, and the remaining constructions are assumed to be the same, then the straight line drawn through the points of division will be parallel to the asymptote on which the point is situated and continued will meet the hyperbola. Moreover the straight line drawn from the point of meeting to the point situated on the asymptote will touch the section* [15].

Let there be a hyperbola and its asymptotes, and let Δ be taken on one of the asymptotes. Let straight lines be drawn and divided as we have said above, and let a straight line ΔB be drawn from Δ touching the hyperbola.

I say that the straight line drawn from B parallel to ΠO passes through K and Λ.

[Proof]. For let if not so, then surely it will pass through one of these points for two neither.

Let it pass through K only, therefore [according to Proposition III.35] as ZΔ is to ΔH, so ZX is to XH. But it is impossible. Therefore the straight line drawn through B parallel to ΠO will not pass through K only. Therefore it will pass through both points [K and Λ].

## [Proposition] 14

In the same suppositions if Δ is on one of the asymptotes, and ΔE cuts the hyperbola at two points, and ΔH parallel to the other asymptote cuts the hyperbola at H only, and if as ΔE is to ΔΘ, so EK is to KΘ, and HΛ is equal to ΔH is situated in a straight line with ΔH, then the straight line drawn through K and Λ will be parallel to the asymptote, and will meet the hyperbola, and the straight line drawn from the point of meeting to Δ will touch the hyperbola for similarity to what was said above, ΔB will touch the hyperbola.

I say that the straight line drawn from B parallel to the asymptote ΠO will pass through K and Λ.

[Proof]. Indeed, if it passed through K only, ΔH will not be equal to HΛ [according to Proposition III.34], which is impossible. And if it passes through Λ only then it will not be that [according to Proposition III.35] as EΔ is to ΔΘ., so EK is to KΘ, and if it passed neither through K nor through Λ, the impossibility will occur in both ways .Therefore it will pass through both points.

## [Proposition] 15

*If in opposite hyperbolas a point is taken between two hyperbolas, and if a straight line from this point touches one of opposite hyperbolas, and another straight line cuts each of opposite hyperbolas, and if as the straight line between the point and the one hyperbola which the first straight line does not touch is to the straight line between the point and the other hyperbola, so the greater straight line between the hyperbolas is to its excess over the latter, set in a straight line with it and with the homologous lines being at the same ends,* *then the straight line drawn from the end of the greater straight line to the point of contact will meet the section, and the straight line drawn from the point of meeting  to the taken point will touch the section* [17].

Let there be the opposite hyperbolas Α  and Β and let some point Δ be taken between the hyperbolas and in the angle between the asymptotes, and from this point let ΔΖ be drawn touching the section and ΑΔΒ be drawn cutting the section. Furthermore as ΑΓ is to ΓΒ, so ΑΔ is to ΔΒ. It is to be shown that the straight line drawn from Ζ to Γ will meet the section, and the straight line drawn from the point of meeting to Δ  will touch the section.

[Proof]. For let since Δ is  situated in the angle containing the section, it is possible to draw from Δ another straight line touching the section [according to Proposition II.49]. Let ΔΕ be drawn, let ΖΕ be drawn and let it pass, if possible, not through Γ, but through Η. It will then [according to Proposition III.37] that as ΑΔ is to ΔΒ, so ΑΗ will be to ΗΒ, which is impossible for it was assumed that as ΑΔ is to ΔΒ, so ΑΓ is to ΓΒ.

[Proposition] 16

If Δ is  situated in the angle adjacent to the angle between the asymptotes, and let the remaining construction be the same [18].

I say that the straight line joining Ζ to Γ will then continued to meet the opposite hyperbola, and the straight line from the point of meeting to Δ will touch the opposite hyperbola.

[Proof]. For let the same reason be as before, and let Δ be in the angle adjacent to the angle between the asymptotes, and let ΔΕ be drawn from Δ touching the hyperbola Α, let ΕΖ be joined and when continued let it not pass through Γ, but through Η, if possible. Then it will be that [according to Proposition III.39] as ΑΗ is to ΗΒ, so ΑΔ will be to ΔΒ, which is impossible for it was assumed that as ΑΔ is to ΔΒ, so ΑΓ is to ΓΒ.

[Proposition] 17

With the same suppositions let Δ be on an asymptote [19].

I say that the straight line drawn from Ζ to Γ will be parallel to the asymptote on which Δ is situated.

Let there be the same as before, let Δ be on one of asymptotes let a straight line be drawn through Ζ parallel to the asymptote, and , if possible, let it not fall on Γ but on H. It will then be [according to Proposition III.36] as AΔ is to ΔB, so AH will be to HB, which possible. Therefore the straight line from Z parallel to the asymptote will fall on Γ.

[Proposition] 18

*If in opposite hyperbolas a point is taken between the hyperbolas and from this point two straight lines are drawn cutting each of hyperbolas, and if as the straight lines between one of hyperbolas and the point are two those between the other hyperbola and the same point, so are straight lines greater than those cut off between the opposite hyperbola to their excess over the latter, then the straight line drawn through the ends of the greater straight lines will meet the hyperbolas, and the straight lines drawn from the points of meeting to the original taken point will touch the hyperbolas* [20].

Let there be the opposite hyperbolas A and B, and let Δ be between the hyperbolas. Let it be assumed first that Δ be in the angle between the asymptotes, and through Δ let AΔB, ΓΔΘ be drawn. AΔ is greater than ΔB, and ΓΔ is greater than ΔΘ since [according to Proposition II.16] BN is equal to AM.

Furthermore let as AK is to KB, so AΔ is to ΔB, and let as ΓH is to HΘ, so ΓΔ is to ΔΘ.

I say that the straight line through K and H meets the hyperbolas, and the straight lines from Δ to the points of meeting will touch the section.

[Proof]. For since Δ is inside of the angle between the asymptotes, it is possible to draw two straight lines touching the section [according to Proposition II.49]. Let ΔE and ΔZ be drawn, and let EZ be joined. It will, thus, pass through K and H for if it passes through one of these points only the other straight line will be cut in the same ratio by another point, which is impossible. If it passes through neither point, the same impossibility will occur in both straight lines.

[Proposition] 19

Let Δ be taken then in the angle adjacent to the angle between the asymptotes and let straight lines be drawn cutting the section and divided as said above[21].

I say that the straight line drawn through K and H will meet each of opposite hyperbolas, and the straight lines from the point of meeting to Δ will touch the section [21].

[Proof]. For let ΔE and ΔZ be drawn from Δ touching each of the hyperbolas. Therefore the straight line through E and Z will pass through K and H for if not so, it will surely go through one of two, or through neither, and again one will similarly inter from this an absurdity.

## [Proposition] 20

*If the point is taken on an asymptote, and the remaining constructions*
*are the same, then the straight line drawn through the ends of the greater straight lines will be parallel to the asymptote on which the point is situated, and the straight line drawn from the point of meeting of the section and the straight line drawn through the ends of the greater straight lines will touch the section* [22].

Let there be the opposite hyperbolas A and B, and let Δ be on one of the asymptotes, and let the remaining construction be the same.

I say that the straight line through K and H meets the section, and the straight line from the point of meeting to Δ will touch the section.

[Proof]. For let ΔZ be drawn from Δ touching the section, and a straight line be drawn from Z parallel to the asymptote on which Δ is situated, it will then pass through K and H for if not so, it will either pass through one of two or neither, and the same impossibilities will occur as before [according to Proposition III.36]

## [Proposition] 21

Again let there be the opposite hyperbolas A and B, and let Δ be on one of the asymptotes, let ΔBK be parallel to one of two asymptotes, meet the section at one point B only, but let ΓΔΘ meet both of hyperbolas. Furthermore let as ΓH be to HΘ, so ΓΔ be to ΔΘ, and let ΔB be equal to BK.

I say that the straight line through K and H will meet the section and will be parallel to the asymptote on which Δ is situated, and that the straight line drawn from the point of meeting to Δ will touch the section [23].

[Proof]. For let ΔZ be drawn touching the section, and let a straight line be drawn parallel to the asymptote on which Δ is situated. If will thus pass through K and H for if not so, the absurdity said before will occur [according to Proposition III.36]

[Proposition] 22

Similarly, let there be the opposite hyperbolas and their asymptotes, and let Δ be similarly taken. Let ΓΔΘ be taken cutting the hyperbolas, and ΔB be taken parallel to one of two asymptotes.
Moreover as ΓΔ is to ΔΘ, let ΓH be to HΘ, and let BK be equal to ΔB.

I say that the straight line through K and H will meet each of the opposite hyperbolas, and the straight lines from the points of meeting to Δ will touch the section [24].

[Proof]. For let ΔE and ΔZ be drawn touching the section, let EZ be joined, and, if possible, let it not pass through K and H, but through one of these two points or neither. If, on the one hand, it passes through H only, ΔB will not be equal to BK, but to some other straight line which [according to Proposition III.31] is impossible. If, on the other hand, it passed through K only, it will not be that as ΓΔ is to ΔΘ, so ΓH is to HΘ, but, some straight line to some other straight line [according to Proposition III.36]. If yet it passes through neither of K and H, then both impossibilities will occur.

[Proposition] 23

Again let there be the opposite hyperbolas A and B, and let Δ be in the angle adjacent to the angle between the asymptotes. Let BΔ be drawn cutting the hyperbola B at one point only, and thus parallel to one of two asymptotes, and let ΔA be drawn similarly to the hyperbola A, and let ΔB be equal to BH and ΔA to AK.

I say that the straight line through K and H meets the hyperbolas and the straight lines drawn from the points of meeting to Δ will touch the hyperbolas.

[Proof]. For let ΔE and ΔZ be drawn touching the hyperbolas, let EZ be joined, and, if possible, let it not pass through KH. So, either it will pass through one of these two points or through neither of them, and either ΔA will not be equal to AK, but some other straight line, which is impossible, or ΔB will not be equal to BH, or neither will be equal to neither, and again the same impossibility will occur in both cases [according to Proposition III.31]. Therefore EZ will pass through K and H.

[Proposition] 24

*A section of a cone will not meet a section of a cone or the circumference of a circle in such way that a part of them will be the same and another part will not be common* [26].

` [Proof]. For let, if possible, let the section of a cone ΔΑΒΓ meet [other section of a cone or] the circumference of the circle ΕΑΒΓ, let the same part ΑΒΓ of these sections be common and let ΑΔ and ΑΕ not be common. Let Θ be taken on this part, let ΘΑ be joined, and through an arbitrary point Ε draw ΔΕΓ parallel to ΑΘ. Moreover bisect ΑΘ at Η, and through Η draw the diameter ΒΗΖ. Therefore the straight line through Β parallel to ΑΘ touches each of the sections, and also will be parallel to ΔΕΓ. Also in one section ΔΖ will be equal to ΖΓ, and in other section [according to Propositions I.46 and I.47] ΕΖ will be equal to ΖΓ, so that also ΔΖ and ΖΕ are equal, but this is impossible [27].

[Proposition] 25

*A section of a cone does not cut a section of a cone or the circumference of a circle at more than four points* [28].

[Proof]. For let, if possible, them cut at five points Α, Β, Γ, Δ, Ε, and let the points of meeting Α, Β, Γ, Δ, Ε be taken in succession so the no point of meeting between them is left out, and let ΑΒ and ΓΔ be joined and continued. So, these straight lines will meet out side the section in the cases of the parabola and the hyperbola [according to Propositions II.24 and II.25]. Let them meet at Λ, and let as ΑΛ be to ΛΒ, so ΑΟ be to ΟΒ, and as ΔΛ be to ΛΓ, so ΔΠ be to ΠΓ.

Therefore the straight line from Π to Ο joined and continued will meet the section on each side and the straight lines joining the points of meeting and Λ [according to Proposition IV.9] will touch the section. Let the points of contact are Θ and Ρ and let ΘΛ and ΛΡ be joined. Hence they touch the section.

Therefore since there is no point of meeting between Β and Γ the straight line ΕΛ cuts each of the sections. Let it cut them at Μ and Η. Therefore in one hyperbola as ΕΝ is to ΝΗ, so ΕΛ is to ΛΗ, and in the other hyperbola as ΕΝ is to ΝΜ, so ΕΛ is to ΛΜ. But it is impossible, so that also what was assumed at the start is impossible.

If ΑΒ and ΔΓ are parallel, the sections will, of course, be the ellipses or the circumference of a circle. Let ΑΒ and ΓΔ be bisected at Ο and Π, and let ΟΠ be joined and continued on each side. Then it will meet the sections. So let it meet them at Θ and Ρ. Then ΘΡ will be a diameter of the sections, and ΑΒ and ΓΔ are drawn as ordinates [according to Proposition II.28]. Let ENMH be drawn from E parallel to AB and ΓΔ. Therefore EMH cuts ΘP each of the sections because there is no other meeting besides A, B, Γ, Δ. Then in one of the sections NM will be equal to EN, and in other section NE will be equal to NH [according to Definition 4], so that NM is equal to NH, but this is impossible [29-30] .

[Proposition] 26

*If the lines [of the sections] mentioned above some touch at one point, then they will not meet each other at more than two other points* [31].

Let two of the above mentioned lines touch at the point A.

I say that they will not meet each other at more than two other points.

[Proof]. For let, if possible, them meet at B, Γ, Δ, and let the points of meeting be taken in succession with no point of meeting between them be left out. Let BΓ be joined and continued, and from A let AΛ be drawn touching the section. Thus AΛ will touch both sections and meet ΓB. Let it meet it at Λ., and let it be that as ΓΛ is to ΛB, so ΓΠ is to ΠB.

Let AΠ be joined and continued. Thus it will meet the section and the straight lines drawn from the points of meeting to Λ will touch the section [according to Proposition IV.1] . Let it meet it at Θ and P, and let ΘΛ and ΛP be joined. These straight lines will touch the section. Therefore the straight line joining Δ to Λ will cut each of sections, and the earlier mentioned absurdity will occur. The section will not cut one another at more than two points.

If in an ellipse or the circumference of a circle ΓB is parallel to AΛ, the proof will be similar to that given above once AΘ is shown to be a diameter.

[Proposition] 27

*If the lines [of the sections] mentioned above some touch one another at two points, they will not meet one another at another point* [32].

Let two of lines mentioned above touch one another at two points A and B. I say that they will not meet one another at another point.

[Proof]. For let, if possible, them meet also at Γ, and to start let Γ be outside of the points of contact A and B, and let straight lines be drawn from A and B touching the sections. Therefore they will touch both lines. Let them touch and be continued to Λ, as in the first diagram, and let ΓΛ be drawn. Then it cuts each of the sections . Let it cut them at H and M, and let ANB be joined.

Therefore in one of the sections as ΓN will be to NH, so ΓΛ will be to ΛH, and in the other section as ΓN will be to NM, so [according to Proposition III.37] ΓΛ will be to ΛM, but this is impossible.

[Proposition] 28

If ΓH is parallel to the straight lines touching the sections at A and B as in the ellipses in the second diagram [33] , then joining AB we conclude that it is a diameter [according to Proposition II.27], so that each of ΓH and ΓM are bisected at N [according to Definition 4], but it is impossible. Therefore the lines [of the sections] do not meet one another at another point, but only at A and B

[Proposition] 29

Let Γ be between the points of contact, as in the third diagram [34] . It is evident that the lines [of the sections] do not touch one another at Γ since it has been assumed that the lines [of the sections] touch at two points only. Indeed, let them cut one another [point] at Γ. Let AΛ and ΛB be drawn from A
and B touching the sections, let AB be joined and bisected at Z. Therefore the straight line drawn from Λ to Z [according to Proposition II.29] will be a diameter. The diameter will surely not pass through Γ for if it did pass through it ,then the straight line drawn through Γ parallel to AB will touch each of the sections [according to Propositions II.5 and II.6] , and this is impossible.

So from Γ let ΓKHM be drawn parallel to AB, then in the one section ΓK will be equal to KH, and in the other section KM will be equal to KΓ, so that KM is equal to KH, but this is impossible.

Similarly if the straight lines touching the sections are parallel, the absurdity will be proved in the same way as above.

[Proposition] 30

*A parabola cannot touch a parabola at more points than one* [35].

[Proof]. For let, if possible, the parabolas AHB and AMB touch at A and B, and let AΛ and ΛB be drawn touching the parabolas. They will, thus, touch both sections and will meet at Λ. Let AB be joined and bisected at Z, and let ΛZ be drawn.

Now since two lines AHB and AMB touch one another at A and B, [according to Propositions IV.27, IV.28, and IV.29] they will not meet each other at another point, so that ΛZ cuts each of sections. Let it cuts them at H and M. In one section [according to Proposition I.35] ΛH will be equal to HZ, and in the other section ΛM will be equal to MZ, but it is impossible. Therefore a parabola cannot touch a parabola at more points than one.

[Proposition] 31

A parabola falling outside of a hyperbola will not touch the hyperbola at two points [36] .

[Proof]. For let there be the parabola AHB and the hyperbola AMB, and, if possible, let them touch at A and B. Let the straight lines be drawn from A and B touching each of sections that touch at A and B, and let these straight lines meet at Λ. Let AB be joined and bisected at Z, and let ΛZ be joined.

Now since the sections AHB and AMB touch at A and B, they will not meet at another point, therefore ΛZ cuts the sections at one and then another point. Let it cut them at H and M and let ΛZ be continued. It will [according to Proposition II.29] fall on the center Δ of the hyperbola. According to the properties of the hyperbola as ZΔ is to ΔM, so MΔ is to ΔΛ and the remainders ZM to MΛ [according to Proposition I.37] .Therefore ZM is greater than MΛ

But according to the properties of the parabola [proved in Proposition I.35] ZH is equal to HΛ, but this is impossible.

[Proposition] 32

*A parabola falling inside of an ellipse or the circumference of a circle will not touch the ellipse or the circumference of the circle at two points* [37].

[Proof]. For let there be the ellipse or the circumference of a circle AHB and the parabola AMB, and, if possible, let them touch at two points A and B, and let straight lines be drawn from A and B touching the sections and meeting at Λ, let AB be joined and bisected at Z, and let ΛZ be joined. ΛZ will cut each section at one point and then at another [point],as we said above. Let it cut them at H and M, and let ΛZ be continued to Δ, which is the center of the ellipse or of the circle. Therefore according to the properties of the ellipse and of the circle as ΛΔ is to ΔH, so ΔH is to ΔZ, and [according to Proposition I.37] that ratio is equal to the ratio of the remainders ΛH to HZ, and ΛΔ is greater than ΔH. Therefore ΛH is greater than HZ. But according to the properties of the parabola [proved in Proposition I.35] ΛM is equal to MZ, but this is impossible.

[Proposition] 33

*A hyperbola will not touch a hyperbola with the same center at two points* [38].

[Proof]. For let, if possible, the hyperbolas AHB and AMB with the same center Δ touch at A and B. Let AΛ and ΛB be drawn from A and B touching the hyperbolas and meeting one another, and let ΔΛ be joined and continued. Moreover let AB be joined. Therefore ΔZ bisects AB at Z. Then ΔZ [according to Proposition IV.29] cuts the hyperbolas at H and M . According to the properties of the hyperbola AHB pl.ZΔΛ will be equal to sq.ΔH, and according to the properties of the hyperbola AMB pl.ZΔΛ will be equal to sq.ΔM [according to Proposition I.37]. Therefore sq.MΔ is equal to sq.ΔH, but this is impossible.

[Proposition] 34

*If an ellipse touches an ellipse or the circumference of a circle with the same center at two points, then the straight line joining the points of contact passes through falls on the center* [39].

[Proof] . For let the above mentioned lines touch one another at A and B. Let AB be joined, and let straight lines touching the sections be pass through A and B, and, if possible, meeting at Λ. Let AB be bisected at Z, and let ΛZ be joined. Therefore [according to Proposition II.29] ΛZ is a diameter of the sections. If possible, let the center be Δ. Therefore pl.ΛΔZ will be equal to sq.ΔH according to the properties of one section, but to sq.MΔ according to the properties of other section, so that [according to Proposition I.37] sq.HΔ is equal to sq.ΔM, but this is impossible. Therefore the straight lines from A and B touching the sections do not meet. Therefore they are parallel, and for the same reason AB is a diameter [according to Proposition II.27], so that it passes through the center, what was to prove [40] .

[Proposition] 35

*A section of a cone or the circumference of a circle will not meet a section of a cone or the circumference of a circle not having its convexity in the same direction at more than two points* [41].

[Proof]. For let, if possible, a section of a cone or the circumference of a circle ΑΒΓ meet a section of a cone or the circumference of a circle ΑΔΒΕΓ not having its convexity in the same direction at more points than two, Α, Β, Γ.

Since three points Α, Β, Γ have been taken on the line ΑΒΓ, if ΑΒ and ΒΓ are joined, they will contain an angle having concavity in the same direction as the line ΑΒΓ. For the same reason ΑΒΓ contain an angle whose concavity is in the same direction as the line ΑΔΒΕΓ. Therefore the lines we have been speaking of have both their concave and convex parts in the same direction, but this is impossible.

## [Proposition] 36

*If a section of a cone or the circumference of a circle meets one of opposite hyperbolas at two points and the lines between the points of meeting have their concavity in the same direction, then the line drawn at the points of meeting will not meet the other opposite hyperbola* [42].

Let there be the opposite hyperbolas Δ and ΑΕΓΖ, and let there be a section of a cone or the circumference of a circle ΑΒΖ meeting one of two opposite hyperbolas at two points Α and Ζ, and let the sections ΑΒΖ and ΑΓΖ have their concavity in the same direction.

I say that continued ABZ will not meet the section Δ.

[Proof]. For let ΑΖ be joined. Since Δ and ΑΓΖ are opposite hyperbolas and ΑΖ cuts a hyperbola at two points, so continued it will not meet the opposite hyperbola Δ [according to Proposition II.33]. Neither therefore will the line ΑΒΖ meet the hyperbola Δ.

## [Proposition] 37

*If a section of a cone or the circumference of a circle meets one of the opposite hyperbolas it will not meet the remaining hyperbola at more points than two* [43] .

Let there be the opposite hyperbolas A and Β, and let a section of a cone or the circumference of a circle ΑΒΓ meet the hyperbola Α, and let ΑΒΓ cut the opposite hyperbola Β at Β and Γ.

I say that it will not meet ΒΓ at another point.

[Proof]. For let, if possible, it meet ΒΓ at Δ. Therefore ΒΓΔ meets the section ΒΓ not having its concavity in the same direction at more points than two, but [according to Proposition IV.35] it is impossible.

This is will be shown similarly if the line ΑΒΓ touches the opposite hyperbola.

[Proposition] 38

*A section of a cone or the circumference of a circle will not meet opposite hyperbolas at more points than four* [44].

This is evident from the fact that meeting one of the opposite hyperbolas it [according to Proposition IV.37] cannot meet the remaining hyperbola at more than two points.

[Proposition] 39

*If a section of a cone or the circumference of a circle touches one of the opposite hyperbolas in the concave part of the latter it will not meet the other opposite hyperbola* [45].

Let there be the opposite hyperbolas Α and Β, and let ΓΑΔ touch the hyperbola Α [from the direction of its concavity].

I say that ΓΑΔ will not meet the hyperbola Β.

[Proof]. For let ΕΑΖ be drawn from Α touching the hyperbola Α. Then it touches each of the sections [Α and ΓΑΔ] at Α, hence [according to Proposition II.30] it will not meet [the hyperbola] Β, so that neither will ΓΑΔ meet Β.

[Proposition] 40

*If a section of a cone or the circumference of a circle touches each of two opposite hyperbolas at one point, it will not meet the opposite hyperbolas at other point* [46].

Let there be the opposite hyperbolas Α and Β, and let a section of a cone or the circumference of a circle touch each of the hyperbolas Α and Β at the points A and B.

I say that the line ΑΒΓ will not meet the hyperbolas Α and Β at another point.

[Proof]. Indeed since the line ΑΒΓ touches the hyperbola A and meets [the hyperbola] Β at one point, therefore it will not touch Α in the direction of its concavity. Similarly it will be shown that neither will it touch Β in the direction of its concavity. Let ΑΔ and ΒΕ be drawn touching the hyperbolas Α and Β, then they will touch the line ΑΒΓ. For, if possible, let one of them cut the line [of the section] and let it be ΑΖ. Therefore between ΑΖ touching the hyperbola Α, and the hyperbola Α, a straight line ΑΗ is situated, but this is impossible. Therefore it touches ΑΒΓ, and because of this it is evident that ΑΒΓ does not meet the opposite hyperbolas at another point.

[Proposition] 41

*If a hyperbola meets one of the opposite hyperbolas at two points having its convexity in the opposite direction to the concavity of the touching hyperbola, then the opposite hyperbola of the mentioned hyperbola will not meet the other opposite hyperbola* [47] .

Let there be the opposite hyperbolas ΑΒΔ and Ζ, let the hyperbola ΑΒΓ meet ΑΒΔ at Α and Β, the former [of them] has its convexity in the opposite direction to the concavity of the latter, and let Ε be the opposite hyperbola of ΑΒΓ.

I say that Ε will not meet Ζ.

[Proof]. For let ΑΒ be joined and continued to Η. Since indeed the straight line ΑΒΗ cuts the hyperbola ΑΒΔ and continued it falls outside of each section, it [according to Proposition II.33] will not meet the hyperbola Ζ. Similarly because ΑΒΗ cuts the hyperbola ΑΒΓ, it will not meet the opposite hyperbola Ε, therefore neither will Ε meet Ζ.

[Proposition] 42

*If a hyperbola meets each of two opposite hyperbolas, its opposite hyperbola will meet neither of the opposite hyperbolas at two points* [48] .

Let there be the opposite hyperbolas A and B, and let the hyperbola ΑΓΒ meets each of the opposite hyperbolas Α and B.

I say that the opposite hyperbola of ΑΓΒ will not meet the hyperbolas Α and Β at two points.

[Proof]. For let, if possible, it meet one of the opposite hyperbola at Δ and Ε, and let ΔΕ be joined and continued. Because of the hyperbola ΔΕ the straight line ΔΕ [according to Proposition II.33] will not meet the hyperbola ΑΒ, and on the other hand because of the section ΑΕΔ [ the straight line] ΔΕ will not meet the hyperbola Β since it passed through the three places [according to Proposition II.33], but this is impossible. Similarly it will be shown that ΑΓΒ will not meet B at two points.

For the same reasons neither will it touch either of the opposite hyperbolas for drawing ΘΕ touching it will touch each of the hyperbolas, so that, because of the hyperbola ΔΕ it will not meet the hyperbola ΑΓ, whereas because of the hyperbola ΑΕ will it not meet the hyperbola B, so that neither will ΑΓ meet B, but this is contrary to what was assumed.

[Proposition] 43

*It a hyperbola cuts each of two opposite hyperbola at two points having its convexity in the opposite direction to each of them, the opposite hyperbola of the mentioned hyperbola will meet neither of the mentioned opposite hyperbolas* [49].

Let there be the opposite hyperbolas Α and Β, and let the hyperbola ΓΑΒΔ cut each of the hyperbolas Α and Β at two points containing convexities in the opposite directions.

I say that the opposite hyperbola ΕΖ [of ΓΑΒΔ] meets neither of the hyperbolas Α and Β.

[Proof]. For let, if possible, it meet the hyperbola Α at Ε, and let ΓΑ and ΔΒ be joined and continued, then these straight lines will meet one another [according to Proposition II.25]. Let them meet at Θ situated in the angle between the asymptotes of the hyperbola ΓΑΒΔ [according to PropositionII.25]. And ΕΖ is the opposite hyperbola of ΓΑΒΔ. Therefore the straight line joining Ε to Θ will fall in the angle ΑΘΒ. Again since ΓΑΕ is a hyperbola and ΓΑΘ and ΘΕ meet, and the points of meeting Γ and Α do not contain Ε, the point Θ will be between the asymptotes of the hyperbola ΓΑΕ. And ΒΔ is the opposite hyperbola of ΓΑΕ. Therefore the straight line from Β to Θ falls inside of the angle ΓΘΕ, but this is impossible for it also fall in the angle ΑΘΒ.
Therefore ΕΖ will not meet one of the opposite hyperbola Α and Β.

[Proposition] 44

*If a hyperbola cuts one of two opposite hyperbolas at four points, the opposite hyperbola of the hyperbola will not meet the other of the two opposite hyperbolas* [50].

Let there  be the opposite hyperbolas ΑΒΓΔ and Ε, and let a hyperbola cut ΑΒΓΔ at four points Α, Β, Γ, Δ, and let its  opposite hyperbola be Κ. I say that Κ will not meet Ε.

[Proof]. For let , if possible, it meet it at Κ. Let ΑΒ and  ΓΔ be joined and continued, then they will  meet one another. Let them meet at Λ, and let as ΑΠ be to ΠΒ, so ΑΛ be to ΛΒ, and let as  ΔΡ be to ΡΓ, so ΔΛ be to ΛΓ.

Therefore the straight line through Π and P will meet the hyperbolas on each side, and the straight lines from L to the points of meeting will touch the hyperbolas [according to Proposition IV.9]. Let ΚΛ be joined and continued. It will cut the angle ΒΛΓ and the hyperbolas at one and then another point. Let it cut them at Z and M [according to the properties of the opposite hyperbolas ΑΒΓΔ and E as NK is to ΚΛ, so NM is to ΜΛ, but this is impossible. Therefore E and K will not meet one another.

[Proposition] 45

*If a hyperbola meets one of two opposite hyperbolas at two points having its concavity in the same direction as the hyperbola, and it meets the other of two opposite hyperbolas at one point, then the opposite hyperbola of the mentioned hyperbolas will meet neither of the opposite hyperbolas*[51].

Let there be the opposite hyperbolas AB and Γ, and let the hyperbola ΑΓΒ meet AB at the points A and B and let it meet the hyperbola Γ at one point, and let Δ be the opposite hyperbola of ΑΓΒ.

I say that Δ will meet neither of the hyperbola AB and Γ.

[Proof]. For let ΑΓ and ΒΓ be joined and continued. Therefore ΑΓ and ΒΓ will not meet the hyperbola Δ [according to Proposition II.33]. Neither will they meet the hyperbola Γ at another point besides Γ for if they meet the hyperbola Γ at another point they will not meet the opposite hyperbola AB [according to Proposition II.33], where it is assumed that they do meet. Therefore the straight lines ΑΓ and ΒΓ meet the hyperbola Γ at one point Γ, and they do not meet Δ at all. Therefore Δ will be in the angle ΕΓΖ, so that the hyperbola Δ will not meet AB and Γ.

[Proposition] 46

*If a hyperbola meets one of two opposite hyperbolas at three points, the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola at more than one point*[52].

Let there be the opposite hyperbolas ΑΒΓ and ΔΕΖ, and let the hyperbola ΑΜΒΓ meet ΑΒΓ at three points A, B, and let ΔK be opposite hyperbola of ΑΜΓ.

I say that ΔK will not meet ΔΕΖ at more point than one.

[Proof]. For let, if possible, them meet at Δ and E , and let AB and ΔE be joined. Now they will either be parallel or not.

To start let them be parallel, and let AB and ΔE be bisected at H and Θ, and let HΘ be joined, therefore HΘ is a diameter for all these hyperbolas [according to Proposition II.36], and AB and ΔE are drawn as ordinates. Let ΓNΞO be drawn from Γ parallel to AB, then it will be drawn as an ordinate to the diameter, and it will cut the hyperbolas, one and then other for if it were to cut them at the same point, the hyperbolas would no longer meet at three points, but as four. In the hyperbola AMB then ΓN will be equal to NΞ, and in AΛB then ΓN will be equal to NO. And therefore ON is equal to NΞ, but this is impossible.

So let straight lines AB and ΔE not be parallel, but be continued. Let them meet at Π. Let ΓO be drawn parallel to AΠ and let it meet continued ΔΠ at P .And let AB and ΔE be bisected at H and Θ, through H and Θ let diameters HΣI and ΘΛM be drawn, and from I, Λ, and M let IYT, MY, and ΛT be drawn touching the hyperbola, then IT will be parallel to ΔΠ, and ΛT and MY will be parallel to AΠ and OP [according to Proposition II.5]. Since as sq.MY is to sq.YI, so pl.AΠB is to pl.ΔΠE [according to Proposition III.19], but as pl.AΠB is to pl.ΔΠE, so sq.ΛT is to sq.TI, and therefore as sq.MY is to sq.YI, so sq.ΛT is to sq.TI.

For the same reasons as sq.MY is to sq.YI, so pl.ΞPΓ is to pl.ΔPE, as sq.ΛT is to sq.TI, so pl.OPΓ is to pl.ΔPE. Therefore pl.OPΓ is equal to pl.ΞPΓ, but this is impossible.

[Proposition] 47

*If a hyperbola touches one of two opposite hyperbolas, and it cuts the other at two points, then the opposite hyperbola of the hyperbola will meet neither of the opposite hyperbolas.* [53]

Let there be the opposite hyperbolas ABΓ and Δ, and some hyperbola ABΔ cut ABΓ at A and B, and touch the hyperbola Δ at the point Δ, and let ΓE be the opposite hyperbola of ABΔ .

I say that ΓE meets neither of the opposite hyperbolas ABΓ and Δ.

[Proof]. For let, if possible, let ΓE meet ABΓ at Γ, and let AB be joined, and let a straight line be drawn through Δ touching the hyperbola ABΔ and meeting AB at Z.

Therefore Z [according to Proposition II.25] will be inside of the angle between the asymptotes of the hyperbola ABΔ. And ΓE is the opposite hyperbola of ABΔ. Therefore the straight line from Γ to Z falls inside of the angle BZΔ. Again since ABΓ is a hyperbola, and AB and ΓZ meet, and the points of meeting A and B do not contain Γ, the point Z is between the asymptotes of the hyperbola ABΓ. And Δ is the opposite hyperbola of ABΓ. Therefore the straight line from ΔZ falls inside of the angle AZΓ, but it is impossible for it fell in the angle BZΔ. Therefore ΓE does not meet one of the opposite hyperbolas ABΓ and Δ.

## [Proposition] 48

*If a hyperbola touches one of two opposite hyperbolas at one point, and it meets it at two points, then the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola* [54].

Let there be the opposite hyperbolas ABΓ and Δ, and let some hyperbola AHΓ touch ABΓ at A, and let it meet ABΓ at B and Γ, and let E be the opposite hyperbola of AHΓ.

I say that E will not meet Δ .

{Proof]. For let, if possible, E meet it at Δ, let BΓ be joined and continued to Z, and let AZ be drawn from A touching the hyperbola. As in the earlier proof it will be shown that Z is inside of the angle between the asymptotes [according to Proposition II.25]. Moreover AZ will touch both hyperbolas, and continued ΔZ will cut the sections at H and K between A and B. Let as ΓΛ is to ΛB, so ΓZ is to ZB, and let AΛ be joined and continued, it will cut the hyperbolas, one and then other [according to Proposition IV.1]. Let it cut them at N and M. Therefore the straight lines from Z to N and M will touch the hyperbolas [according to Proposition IV.1] ,and as in the earlier proof [according to the Proposition III.37] according to the properties of the one hyperbola as ΞK is to KZ, so ΞΔ is to ΔZ, and according to the properties of the other hyperbola as ΞH is to HZ, so ΞΔ is to ΔZ, but this is impossible. Therefore it does not meet the opposite hyperbola.

## [Proposition] 49

*If a hyperbola touching one of two opposite hyperbolas meets the same hyperbola at another point, then the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola at more points than one* [55].

Let there be the opposite hyperbolas ABΓ and EZH, and let some hyperbola ΔAΓ touch ABΓ at A, and let it cut ABΓ at Γ, and let EZΘ be the opposite hyperbola of ΔAΓ.

I say that it will not meet the other opposite hyperbola at more points than one.

[Proof]. For let, if possible, let it meet it at two points Ε and Ζ, and let ΕΖ be joined and through Α let ΑΚ be drawn touching the hyperbolas. Now ΕΖ and ΑΚ will be parallel or not parallel.

To start let them be parallel, and let the diameter bisecting ΕΖ be drawn, therefore it will pass through Α and it will be the diameter of two conjugate hyperbolas [according to Proposition II.34]. Let ΓΛΔΒ be drawn through Γ parallel to ΑΚ and ΕΖ. Therefore it will cut the hyperbolas at one and then at another point. Then in the one hyperbola ΓΛ will be equal to ΛΔ, and in the remaining hyperbola ΓΛ will be equal to LB, but this is impossible.

So, let ΑΚ and ΕΖ not be parallel, let them meet at Κ, and let ΓΔ drawn parallel to ΑΚ meet ΕΖ at Ν. Let ΑΜ bisecting ΕΖ cut the hyperbolas at Ξ and Ο, and let ΞΠ and ΟΡ be drawn from Ξ and Ο touching the hyperbolas. Therefore as sq.ΑΠ is to sq.ΠΞ, so sq.ΑΡ is to sq.ΡΟ, and for this reason as pl.ΔΝΓ is to pl.ΕΝΖ, and as pl.ΒΝΓ is to pl.ΕΝΖ. Therefore pl.ΔΝΓ is equal to pl.ΒΝΓ, but this is impossible.

[Proposition] 50

*If a hyperbola touches one of two opposite hyperbolas at one point, the opposite hyperbola of the hyperbola will not meet other opposite hyperbola at more points that two* [56] .

Let there be the opposite hyperbolas ΑΒ and ΕΔΗ, and let a hyperbola ΑΓ touch ΑΒ at Α, then let ΕΔΖ be the opposite hyperbola of ΑΓ.

I say that ΕΔΖ will not meet ΕΔΗ at more points than two.

[Proof]. For let, if possible, ΕΔΖ meet ΕΔΗ at three points Δ, Ε, and Θ, let ΑΚ be drawn touching hyperbolas ΑΒ and ΑΓ, let ΔΕ be joined and continued, and, start, let ΑΚ and ΔΕ be parallel. Let ΔΕ be bisected at Λ, and let ΑΛ be joined. Then ΑΛ be a diameter for two conjugate hyperbolas [according to Proposition II.34] , and will cut the hyperbola between Δ and Ε at Μ and Ζ. Let ΘΖΗ be drawn from Θ parallel to ΔΕ. Then in the one section ΘΞ will be equal to ΞΖ, and in the other section ΘΞ will be equal to ΞΗ, so that also ΞΖ is equal to ΞΗ, but this is impossible.

So let ΑΚ and ΔΕ not be parallel, but let them meet at Κ, and let the remaining constructions be the same. Let ΑΚ be continued and let it meet ΖΘ at Ρ . As before we will show that [according to Proposition III.19] in the hyperbola ΖΔΕ as pl.ΖΡΘ is to sq.ΡΑ, so pl.ΔΚΕ is to sq.ΑΚ, and in the hyperbola ΗΔΕ as pl.ΗΡΘ is to sq.ΡΑ, so pl.ΔΚΕ is to sq.ΑΚ. Therefore pl.ΗΡΘ is equal to pl.ΖΡΘ, but this is impossible. Therefore ΕΔΖ does not meet ΕΔΗ at more points than two.

[Proposition] 51

*If a hyperbola touches two opposite hyperbolas, the opposite hyperbola of the hyperbola will meet neither of the opposite hyperbolas* [57].

Let there be the opposite hyperbolas A and B, and let the hyperbola AB touch each of them at the points A and B, and let the opposite hyperbola of AB be E. I say that E will meet neither of the hyperbolas A and B.

[Proof]. For let, if possible, it meet A at Δ, and let straight lines be drawn from A and B touching the hyperbolas, they will meet one another hyperbola in the angle between the asymptotes of the hyperbola AB [according to Proposition II.25]. Let them meet at Γ, and let ΓΔ be joined. Therefore ΓΔ will be in the place between AΓ and ΓB. But it is between BΓ and ΓZ, it is impossible. Therefore E does not meet A and B.

[Proposition] 52

*If each of two opposite hyperbolas touch each of two opposite hyperbolas at one point, each having its concavity in the same direction, then they will not meet at another point* [58].

Let the opposite hyperbolas touch one another at A and Δ.

I say that they will not meet at another point.

[Proof]. For let , if possible, them meet at E. Since, indeed, a hyperbola touching one of the opposite hyperbolas meets at E, therefore the hyperbola AB will not meet the hyperbola AΓ at more points than one [according to Proposition IV.49]. Let AΘ and ΘΔ be drawn from A and Δ touching the hyperbolas, let AΔ be joined, let EBΓ be drawn through E parallel to AΔ, and let the second diameter ΘKΛ of the opposite hyperbolas be drawn from Θ [according to Proposition II.38]. Then it will bisect AΔ at K. And therefore EB and EΓ will be bisected at Λ [according to Proposition II.39]. Therefore BΛ is equal to ΛΓ, but it is impossible. Therefore the hyperbolas will not meet at another point.

[Proposition] 53

*If a hyperbola touches one of two opposite hyperbolas at two points, the opposite hyperbola of the hyperbola will not meet other opposite hyperbola* [59].

Let there be the opposite hyperbolas ΑΔΒ and Ε, and let the hyperbola ΑΓ touch ΑΔΒ at two points Α and Β, and let Ζ be the opposite hyperbola of ΑΓ.

I say that Ζ will not meet Ε.

[Proof]. For let, if possible, it meet it at Ε, and let ΑΗ and ΗΒ be drawn from Α and Β touching the hyperbolas, let ΑΒ and ΕΗ be joined, and let ΕΗ be continued, it will cut the hyperbolas at one and then at another point, let it be as ΕΗΓΔΘ. Since ΑΗ and ΗΒ indeed touch the hyperbola, and ΑΒ joins the points of contact in one of the conjugate hyperbolas as ΟΔ is to ΔΗ, so ΘΕ is to ΕΗ, and in other hyperbola as ΘΓ is to ΓΗ, so ΘΕ is to ΕΗ, but it is impossible. Therefore the hyperbola Ζ does not meet the hyperbola Ε.

## [Proposition] 54

*If a hyperbola touches one of two opposite hyperbolas with the convexities in the opposite directions, then the opposite hyperbola of the hyperbola will not meet other opposite hyperbola* [60].

Let there be the opposite hyperbolas Α and Β, and some hyperbola ΑΔ touch the hyperbola Α at the point Α, and let the opposite hyperbola of ΑΔ be Ζ. I say that Ζ will not meet Β.

[Proof]. For let ΑΓ be drawn from Α touching the hyperbolas, therefore because of the properties of the hyperbola ΑΔ [the straight line] ΑΓ will not meet Ζ, and because of the properties of the hyperbola A [according to Proposition II.33] it will not meet Β, so that ΑΓ falls between the hyperbolas Β and Ζ. Then it is evident that Β will not meet Ζ.

## [Proposition] 55

*Opposite hyperbolas will not meet opposite hyperbolas at more points than four* [61].

Let there be one pair of opposite hyperbolas ΑΒ and ΓΔ, and let another pair of opposite hyperbolas be ΑΒΓΔ and ΕΖ, and, to start let ΑΒΓΔ cut each of ΑΒ and ΓΔ at four points Α, Β, Γ, and Δ containing convexities in opposite directions, as in the first diagram. Therefore the opposite hyperbola of ΑΒΓΔ, that is ΕΖ, will not meet ΑΒ and ΓΔ [according to Proposition IV.43].

But let ΑΒΓΔ cut ΑΒ at Α and Β and Γ at one point Γ, as in the second diagram. Therefore ΕΖ does not meet the hyperbola Γ [according to Proposition IV.41]. If ΕΖ meets ΑΒ, it will meet it at one point only for if it meets it at two points, its opposite hyperbola ΑΒΓ will not meet other opposite hyperbola Γ [according to Proposition IV.43]. But it has been assumed that it meets it at one point Γ.

If, as in the third diagram, ΑΒΓ cuts ΑΒΕ at two points Α and Β, and ΕΖ meets ΑΒΕ at one point, ΕΖ will not meet the hyperbola Δ [according to Proposition IV.41], where as meeting ΑΒΕ it will not meet ΑΒΕ at more points than two.

If, as in the fourth diagram, ΑΒΓΔ cuts each of two opposite hyperbolas at one point, ΕΖ will meet neither at two points [according to Proposition IV.42]. [So that according to already said and its converse, ΑΒΓΔ and ΓΖ will not meet the opposite hyperbolas ΒΕ and ΕΖ at more points than four] [62]. If the hyperbolas have their concavities in the same direction and one cuts other at four points Α, Β, Γ, and Δ, has in the fifth diagram, ΕΖ will not meet other opposite hyperbola [according to Proposition IV.44]. Of course, ΕΖ will not meet ΑΒ for again ΑΒ will not meet the opposite hyperbolas ΑΒΓΔ and ΕΖ at more points than four [according to Proposition IV.38], neither will ΓΔ meet ΕΖ.

If, as in the sixth diagram, ΑΒΓΔ meets other hyperbola at three points, ΕΖ will meet other hyperbola at one point only [according to Proposition IV.46].

And we will say the same as before for the remaining cases.

So, since what was proposed is clear in all possible configurations, opposite hyperbolas will not meet opposite hyperbolas at more points than four.

[Proposition] 56

*If opposite hyperbolas touch opposite hyperbolas at one point, they will not meet at more than two other points* [63].

Let there be the opposite hyperbolas ΑΒ and ΓΔ and others Δ and ΕΖ, let ΒΓΔ touch ΑΒ at Β, let their convexities in opposite directions, and, first, let ΒΓΔ meet ΓΔ at two points Γ and Δ, as in the first diagram.

Indeed since ΒΓΔ cuts ΓΔ at two points having their convexities in opposite directions, ΕΖ will not meet ΑΒ [according to Proposition IV.41]. Again since ΒΓΔ touches ΑΒ at Β, and their convexities are in opposite directions, ΕΖ will not meet ΓΔ [according to Proposition IV.54]. Therefore ΕΖ will not meet either the hyperbolas ΑΒ and ΓΔ, therefore these hyperbolas will meet at two points Γ and Δ only.

But let ΒΓ cut ΓΔ at one point Γ, as in the second diagram. Therefore ΕΖ will not meet ΓΔ [according to Proposition IV.54], whereas it will meet ΑΒ at one point only for if EZ meets AB at two points, BΓ will not meet ΓΔ [according to Proposition IV.41]. But it was assumed that they meet at one point.

If BΓ does not meet the hyperbola Δ, as in the third diagram, then according to what has been said above, EZ will not meet Δ [according to Proposition IV.54], whereas EZ will not meet AB at more points than two [according to Proposition IV.37].

If the hyperbolas have their concavities in the same direction, the same proof will applied.

So, from that proof, what was proposed is clear in all possible configurations.

## [Proposition] 57

*If opposite hyperbolas touch opposite hyperbolas at two points, they will not meet at another point* [64].

Let there be the opposite hyperbolas AB and ΓΔ, and others AΓ and EZ, and first, let them touch at A and Γ, as in the first diagram.

Indeed since AΓ touches each of the hyperbolas AB and ΓΔ at A and Γ, therefore EZ will meet neither on the hyperbolas AB and ΓΔ [according to Proposition IV.51].

So, let them touch as in the second diagram. It will be proved similarly that ΓΔ will not meet EZ [according to Proposition IV.53].

So, let ΓA touch AB at A and let Δ touch EZ at Z, as in the third diagram. Indeed, since AΓ touches AB having their convexities in opposite directions, EZ will not meet AB. Again, since ZΔ touches EZ, ΓA will not meet ΔZ.

If AΓ touches AB at A, and EΓ touches ΓΔ at Γ, and their concavities are in the same direction, as in the fourth diagram, they will not meet at another point [according to Proposition IV.52]. EZ will not even meet AB.

So, from the proposed proof it is clear in all possible configurations [65] .

## BOOK FIVE

### Apollonius greets Attalus

In fifth book I have composed propositions on the maximal and minimal straight lines. You should realize that our predecessors and contemporaries paid (a little) attention only to the minimal straight lines : they proved thereby which straight lines are tangent to the section and also the reverse, that is what properties are possessed by the tangents to the section[1] such that when those properties are possessed by straight lines they are tangents. But as for us, we have proven those things in Book 1 without making use, in our proof of that, of the topic of minimal straight lines, for we wanted to make the place where those [things] were put near to our discussion of the derivation of the three sections, in order to show in this way that in each of the sections there may occur an indefinite number [2] of properties and necessities of these things, as is the case with the original diameters. As for the propositions in which we speak of the minimal straight lines, we have separated them out and treated them individually, after much investigation, and have attached the discussion of them to the discussion of the maximal straight lines which we mentioned above, because of our opinion that students of this science need them for the knowledge of analysis and determination of problems and their synthesis, not to speak of the fact that they are one of the subjects which deserve investigation in their own right. Farewell.

### [Proposition] 1

*If there is a hyperbola or an ellipse, and there is erected at the end of one of its diameters the half of the* latus rectum *to that diameter at right angles, and a straight line is drawn from its end to the center of the section, and from a place on the section is drawn a straight line as an ordinate to the diameter, then that straight line will be equal in the square to the double quadrangle formed on the half of the* latus rectum *as it is described in the example [3].*

Let there be the hyperbola or the ellipse ΑΒ whose the diameter ΒΓ and the center Δ and the *latus rectum* for the section ΒΕ, and the half of ΒΕ is ΒΗ. Let ΔΗ be joined, and the ordinate ΑΖ be drawn, and from Ζ the straight line ΖΘ parallel to BE be drawn.

I say that sq.ΑΖ is equal to the double quadrangle ΒΖΘΗ.

[Proof], For let EΓ be drawn from E. Then ΔH is parallel to ΓE, because ΓB and BE are bisected at Δ and H [respectively]. Let ZΘ be continued to [meet ΓE at] K. Then ΘK is parallel to HE,and ΘK is equal to HE.

But HE is equal to BH, therefore BH is equal to ΘK.

We make ZΘ common, then ZK is equal to the sum of BH and ZΘ. Therefore pl.BZK is equal to pl.BZ, the sum of BH and ZΘ.

But pl.BZK is equal to sq.AZ, therefore pl.BZ, the sum BH and ZΘ is equal to sq.AZ, as is proved in Theorems 12 and 13 of Book I.

And pl.BZ, the sum BH and ZΘ is equal to the double quadrangle BZΘH. Therefore sq.AZ is equal to the double quadrangle BZΘH[4] .

[Proposition] 2

But if the straight line drawn as an ordinate falls on Δ which is the center in the ellipse, and BE is made double BZ, and ΔZ is joined, then sq.AΔ is equal to the double triangle BZΔ [5].

[Proof]. For let ΓE be joined, then BZ is equal to ZE.

But ZE is equal to ΔH, which is parallel to BE. Therefore pl.BΔH is equal to the double triangle ΔZB.

But pl.BΔH is equal to sq.AΔ, as is proved in Theorem 13 of Book I. Therefore sq.AΔ is equal to the double triangle ZBΔ..

[Proposition] 3

But if the straight line drawn as an ordinate in the ellipse falls on the other side of Δ which is the center as AZ, and BH is made the half of BE which is the *latus rectum*, and HΔ is joined and continued in a straight line, and there is drawn from Z a straight line ZΘ parallel to BE, to meet HΔ , then sq.AZ is equal to the double triangle BΔH without the double triangle ΔZΘ [6].

[Proof]. For let from Γ be drawn a straight line ΓK parallel to BE, and HΔ be continued until meets ΓK at K, and the section AB be completed, and AZ be continued in a straight line to [meet it at] L. Then sq.ZA is equal to the double quadrangle ΓKΘZ, as is proved in Theorem I of this Book.

But ZΛ is equal to AZ, so sq.AZ is equal to the double quadrangle ΓKΘZ. And the quadrangle ΓKΘZ is equal to the triangle ΓKΔ without the triangle ΔZΘ. But the triangle ΓKΔ is equal to the triangle ΔBH because BΔ is equal to ΔΓ. Therefore sq.AZ is equal to the double triangle ΔBH without the double triangle ΔZΘ.

## [Proposition] 4

*If a point is taken on the axis of a parabola, the distance of which from the vertex of the section is equal to the half of the* latus rectum*, and the straight lines are drawn from that point to the section, then the minimal of these [straight lines] if the straight line drawn to the vertex of the section, and those closer to this [straight line] will be smaller than those farther [from it], and their squares will greater than the square on it by the equal to the square on the segment cut off on the axis towards the vertex by the perpendiculars [drawn] to the axis from the end of each of them* [7] .

Let the axis of the parabola be ΓE and let ΓZ be equal to the half of the *latus rectum*, and let from Z to the section ABΓ be drawn ZH, ZΘ, ZB, and ZA.

I say that the least of the straight lines drawn from Z to the section ABΓ is ΓZ, and that those [straight lines] which are nearer to it are smaller than those which are farther [from it], and that the square on the segment between Γ and the foot of the perpendicular from it [the end of the straight line].

{Proof]. For let the perpendiculars HK, ΘΛ and AE be drawn. Let the half of the *latus rectum* be ΓM, then ΓZ is equal to ΓM.

And the double pl.MΓK is equal to sq.KH, as is proved in Theorem 11 of Book I. But the double pl.MΓK is equal to the double pl.ZΓK. Therefore the sum of the double pl.ZΓK and sq.KZ is equal to the sum of sq.KZ and sq.KH.
But these two squares are equal to sq.ZH. Therefore the sum of the double pl.ZΓK and sq.ZK is equal to sq.ZH. Therefore sq.ZH is greater than sq.ZΓ by sq.ΓK. And it will be proved from this that ΘZ is greater than ZH and ZH is greater than ZΓ.

So ZΓ is the shortest and those [straight lines] that are closer to it are shorter than those which farther. And it is proved that the excess of the square on each of them over the square on the shortest straight line is of the another of the square on the segment cut off from the axis towards the vertex of the section by the perpendiculars from the ends of the straight lines.

## [Proposition] 5

*But is taken on the axis of a hyperbola such that its distance from the vertex of the section is equal to the half of the latus rectum, then in this case the same result will obtain as happened in the parabola, except that the increments of the square on the straight lines over the square on the minimal straight line will be equal to the rectangular plane on the straight line joining the foot of [each of] the perpendiculars to the vertex of the section which is similar* *to the rectangular plane under the transverse diameter and a straight line equal to the sum of the transverse diameter and the latus rectum where the transverse diameter corresponds to straight line joining [the foot of] each of the perpendicular and the vertex of the section* [8].

Let there be the hyperbola ΑΒΓ whose axis be ΓΕ, and let the half of the *latus rectum* be ΓΖ. From Ζ the straight lines ΖΑ, ΖΒ, ΖΓ, ΖΗ, and ΖΘ.
To the section ΑΒΓ, as many as we please.

I say that ΖΓ is the least of the straight lines drawn from Ζ to the section, and that those which are closer to it are shorter than those farther, and that for each of the straight lines ΖΘ, ΖΗ, ΖΒ, and ΖΑ the square on ΓΖ is smaller than the square on it by an amount equal the rectangular plane on the segment between the foot of the corresponding perpendicular and Γ which similar to the rectangular plane under ΔΓ which is the transverse diameter of the section and a straight line equal to the sum of ΔΓ and the *latus rectum*. So let the *latus rectum*. So let the *latus rectum* be ΓΧ, and the half of it be ΓΚ, and the center of the section be Φ.

[Proof]. For let the perpendiculars ΘΜΝ, ΗΛΞ, and ΑΕΠ, to ΓΕ be drawn and continued, and the perpendicular ΒΖ be continued to Ο, and ΚΤ and ΣΝ parallel to ΓΜ be drawn. Then sq.ΘΜ is equal to the double quadrangle ΓΚΝΜ, as is proved in Theorem I of this Book. And sq.ΖΜ is equal to the double the triangle ΖΜΙ because ΖΜ is equal to ΜΙ for ΓΚ is equal to ΓΖ. Therefore sq.ΘΖ is equal to the sum of double triangles ΓΚΖ and ΚΝΙ for sq.ΘΖ is equal to the sum of sq.ΘΜ and sq.ΜΖ. But sq.ΓΖ is equal to the double triangle ΓΚΖ because ΓΖ is equal to ΓΚ. And the quadrangle ΣΝΙΥ is equal to the double triangle ΙΚΝ.Therefore sq.ΓΖ is less than sq.ΘΖ by the quadrangle ΥΣΝΙ. And pl.ΔΓΧ is equal to pl.ΦΓΚ and as ΦΓ is to ΓΚ, so ΚΤ is to ΤΝ. But ΚΤ is equal to ΤΙ because ΙΜ is equal to ΜΖ [for ΓΚ is equal to ΓΖ]. Therefore pl.ΔΓΧ is equal to ΙΤΝ, and *invertendo* as ΧΓ is ΓΔ, so ΤΝ is to ΤΙ. And *componendo* as the sum of ΧΓ and ΓΔ is to ΓΔ, so ΝΙ is to ΤΙ.

But ΤΙ is equal to ΥΙ, therefore as ΝΙ is to ΥΙ, so the sum ΧΓ and ΓΔ is to ΓΔ. Let ΧΓ be continued to Ψ, and let ΓΨ be equal to ΓΔ. Then as ΝΙ is to ΥΙ , so ΧΨ is to ΨϘ, and these sides that are in the same ratio and close the equal angles. Therefore the rectangular planes ΥΝ and Χo are similar, and ΥΙ, which is equal το ΓΜ, corresponds to ΨϘ, which is equal to ΓΔ. Therefore the rectangular plane on ΓΜ similar to the rectangular plane under ΔΓ and a straight line equal to the sum of ΔΓ and the *latus rectum* is the quadrangle ΥΝ. Therefore sq.ΘΖ is greater than sq.ΓΖ by an amount equal to the rectangular plane on ΓΜ similar to the rectangular plane under ΓΔ and the segment equal to the sum of ΓΔ and the *latus rectum*.

Similarly too it will be proved that sq.ZH is greater than sq.ZΓ by an amount equal to rectangular plane on ΓΛ similar to the mentioned plane.

And I say that sq.BZ is greater than sq.ΓZ by an amount corresponding to the mentioned plane for sq.BZ is equal to the double area ΓKOZI, as is proved in Theorem I of this Book.

But sq.ΓZ is equal to the double triangle ΓKZ. Therefore sq.BZ is greater than sq.ΓZ by the double triangle ZKO.

And similarly we will prove that the rectangular plane that the double triangle ZKO is the rectangular plane on ΓZ similar to the mentioned plane. Therefore sq.BZ is greater than sq.ΓZ by an amount equal to the double rectangular plane on ΓZ similar to the mentioned plane.

But I also say that sq.AZ is in the same case as we mentioned for sq.AE is equal to the double quadrangle ΓKΠE, as is proved in Theorem I of this Book. But sq.ZE is equal to the double triangle PZE.

Therefore sq.AZ is equal to the sum of the double triangles PKΠ and ΓKZ, for sq.AZ is equal to the sum of sq.AE and sq.EZ. But the double triangle ΓKZ is sq.ΓZ. Therefore sq.AZ without sq. ΓZ is equal to the double triangle PKΠ.

And similarly too we will prove that the rectangular plane equal to the double triangle PKΠ is the rectangular plane on GE similar to the mentioned t plane.

And because the increments of the squares on these straight lines over the square on ΓZ are the rectangular planes on ΓE, ΓZ, ΓΛ, and ΓM, and these rectangular planes differ from each other, the rectangular plane on ΓE is greater than that on ΓZ, and that on ΓZ is greater than that on ΓΛ, and that on ΓΛ than that on ΓM, and ΓZ is the least of the straight lines [so] drawn, and those of the other straight lines which are closer to it are smaller than those which are farther.

And the square on each of straight lines [so] drawn is equal to the square on the least of these straight lines together with the rectangular plane on the segment between the foot of the perpendicular and Γ similar to the rectangular plane under ΓΔ and a segment equal to the sum of ΓΔ and the *latus rectum* [9-10].

[Proposition] 6

*But if the same conditions as we mentioned hold, except that the section is an ellipse, and the axis is its major axis, then least of the straight lines drawn from that point is the one equal to the half of the* latus rectum*, and the great-*

*est of them is the remainder of the axis. As for the other straight lines, those of them that are closer to the minimal straight line are less than those that are farther from it. And each of them is greater than it by an amount equal to rectangular plane on the segment between the foot of the perpendicular from it and the vertex of the section similar to the rectangular plane under the transverse diameter and the difference between the transverse diameter and the* latus rectum*, where the transverse diameter corresponds to the segment between the foot of the perpendicular and the vertex of the section* .

Let there be the ellipse ΑΒΓ whose major axis be ΑΓ, and let ΧΔ be equal to the half of the *latus rectum*. And let from Δ to the section ΔΖ, ΔΕ, ΔΒ and ΔΗ are drawn.

I say that ΔΓ is the shortest of the straight lines drawn from Δ, and that ΔΑ is the longest of them, and that of the remaining straight lines those which are closer to ΔΓ are shorter than those which are farther, and that the square on each of them is greater than sq.ΔΓ by an amount equal to the rectangular plane on the segment between the foot of its perpendicular and Γ similar to the rectangular plane under ΓΑ to together with excess of it over the *latus rectum*.

[Proof].For let ΓΘ be made the half of the *latus rectum*, and the center be Ι, and the perpendiculars ΖΚΣ, ΕΛ, and ΒΔΡ [to the major axis] be drawn, and [from Α] a straight line ΑΞ parallel to the ordinates is drawn , and ΤΥ and ΣΦ parallel to ΓΑ are drawn. Than sq.ΖΚ is equal to the double quadrangle ΓΘΣΚ, as is proved in Theorem I of this Book.

And sq.ΔΚ is equal to the double triangle ΚΤΔ for ΚΔ is equal to ΚΤ [because ΔΓ is equal to ΓΘ]. Therefore sq.ΔΖ is equal to the sum of the double triangles ΔΓΘ and ΤΘΣ.

But sq.ΔΓ is equal to the double triangle ΔΓΘ. And the quadrangle ΤΥΦΣ is equal to the double triangle ΤΘΣ, therefore sq.ΔΖ is greater than sq.ΓΔ by an amount equal to the quadrangle ΤΣΦΥ. And as ΙΓ is to ΓΔ, so ΑΓ is to *latus rectum*, which is ΣΦ is to ΦΘ. Therefore as ΑΓ is to the *latus rectum*, so ΣΦ is to ΦΘ.

But ΣΦ is equal to ΥΘ therefore as ΑΓ is to the *latus rectum*, so ΥΘ is to ΘΦ. And convertendo as ΓΑ is to ΓΑ without the *latus rectum*, so ΘΥ is to ΥΦ.

But ΘΥ is equal to UT because ΓΔ is equal to ΓΘ. Therefore as ΥΤ is to ΤΣ, so ΑΓ is to ΑΓ without the *latus rectum*.

And ΑΓ corresponds to ΥΤ, which is equal to ΓΚ. Therefore the rectangular plane ΥΣ is equal to the rectangular plane on ΚΓ similar to the rectangular plane under ΑΓ and its excess over the *latus rectum*.

But sq.ZΔ is greater than sq.ΔΓ by an amount equal to the rectangular plane ΥΣ. Therefore sq.ZΔ is greater than sq.ΔΓ by an amount equal to the rectangular plane on ΓK similar to the mentioned plane.

I also say that sq.BΔ is in the same case as the mentioned straight line [ZΔ] for sq.BΔ is equal to the double quadrangle ΔΓΘP. And sq.ΓΔ is equal to the double triangle ΔΓΘ. Therefore sq.ΔB without sq.ΔΓ is equal to the double triangle ΔΘP.

But the rectangular plane on ΓΔ similar to the mentioned plane is equal to the double triangle ΔΘP. Therefore the difference between sq.ΔB and sq.ΔΓ is equal to the rectangular plane on ΓΔ similar to the mentioned plane.

I also say that sq.ΔH is grater than sq.ΔΓ by an amount equal to the rectangular plane on MΓ similar to the mentioned plane for sq.HM is equal to the double area MAOΨ, as is proved in Theorem I of this Book. And sq.MΔ is equal to the double triangle ΔMN because ΔM is equal to MN [for ΔΓ is equal to ΓΘ]. Therefore sq.ΔH is equal to the sum of the double triangle AIO and the double area IΨNΔ.

But the triangle OAI is equal to the triangle ΓΘI. Therefore sq.ΔH is equal to the sum of the double triangle ΓΘI and the double area IΨNΔ. And these [latter] are equal to the sum of the double triangles ΔΓΘ and NΘΨ.

But sq.ΓΔ is equal to the double triangle ΓΔΘ. Therefore sq.ΔH without sq.ΓΔ is equal to the double triangle NΘΨ. And the rectangular plane on ΓM similar to the mentioned plane is equal to the double triangle NΘΨ. Therefore sq.ΔH without sq.ΔΓ is equal to the rectangular plane on ΓM similar to the mentioned plane.

Furthermore sq.AΔ is equal to the double triangle ΞΔA. But the triangle OIA is equal to the triangle ΘΓI, so sq.AΔ is equal to the sum of the double triangles ΞΘO and ΔΓΘ. But sq.ΓΔ is equal to the double triangle ΞΘO. And the rectangular plane on ΓA similar to the mentioned rectangular plane is equal to the double triangle ΘOΞ. Therefore sq.AΔ is greater than sq.ΔΓ by an amount equal to the rectangular plane on ΓA together with the excess of it over the *latus rectum*. And the rectangular plane on ΓA is greater than that on ΓM, and that on ΓM is greater than that on ΓΔ, [and that ΓΔ is greater than that on ΓΛ, and that on ΓΛ is greater than that on ΓK].

Therefore ΓΔ is the smallest of the straight lines drawn from Δ to the section, and ΔA is the greatest of them. And as for the other straight lines those of them drawn closes to shortest straight line are smaller than those drawn farther from it. And the square of each of them is greater than the square on the shortest straight line by an amount equal to the mentioned plane.

[Proposition] 7

*If a point is taken on the  mentioned minimal straight lines in one of three section, and straight lines are drawn from it to the section, then the shortest of them is the straight line between the point and the vertex of the section, and those of other straight lines drawn in that half of the section closer to it are shorter than those drawn farther* [11].

Let there be of a cone ΑΒΓΔ whose axis be ΔΗ. Let the minimal straight line be ΔΕ. Let there be an arbitrary point Ζ on ΔΕ. From it to the section straight lines ΖΓ, ΖΒ, and ΖΑ are drawn.

I say that ΔΖ is the shortest of them, and that those [of them] drawn closer to it are smaller than those drawn farther.

[Proof]. For let ΓΕ be drawn . Then ΓΕ is greater than ΕΔ. Therefore the angle ΓΔΕ is greater than the angle ΔΓΕ. By how much the more is the angle ΖΔΓ greater than  the angle ΔΓΖ, so ΓΖ is greater than ΖΔ.

Furthermore ΒΕ is greater than ΕΓ, so the angle ΒΓΕ is greater than the angle ΓΒΕ. So by how much the more is the angle ΓΒΖ less than the angle ΒΓΖ, therefore ΒΖ is greater than ΖΓ.

Similarly also it will be proved that ΑΖ is greater than ΒΖ. So ΔΖ is the shortest of the straight lines drawn from Ζ to the section, and as for other straight lines those of them drawn closer to ΔΖ are shorter than those drawn farther.

[Proposition] 8

*If a point is taken on the axis of a parabola, the distance of which from the vertex of the section is greater than the half of the* latus rectum*, and there is cut off on the axis from the point witch was taken on it towards the vertex of the section a straight line equal to the half of the* latus rectum*, and from the [other] end of that straight line which was cut off there is drawn a perpendicular to the axis, and that perpendicular is continued to meet the section, and there is drawn from the place there it meets the section a straight line to the taken point, then that straight line is the shortest of the straight lines drawn from the taken point on the axis to the section, and of all other straight lines on both sides [of it] those drawn closer to it are shorter than those drawn farther, and the square on each of them is greater than thee square on the shortest straight line by an amount equal to the square on the segment between the feet of the perpendiculars to the axis from two of them.* [12]

Let there be the parabola ΑΒΓ whose axis ΓΔ, and let ΓΕ be longer than the half of the *latus rectum*, and let the half of the *latus rectum* be ΖΕ. The perpendicular ΖΗ to ΓΕ is drawn and ΕΗ is joined.

I say that ΕΗ is the shortest of the straight lines drawn from Ε to the section, and as for other straight lines drawn from [Ε to] ΑΒΓ those of them drawn closer to ΕΗ are shorter than drawn farther on both sides. From Ε to the section ΕΚ, ΕΛ, ΕΘ, and ΕΑ are drawn.

I say also that the square on each of these straight lines are greater than sq.ΕΗ be an amount equal to the square on the segment between the foot of the perpendicular from it and Ζ.

[Proof]. For let the perpendiculars [ΚΞ, ΛΜ, ΘΧ, and ΑΔ] be drawn and let ΒΕ be a perpendicular [to the axis],and let ΓΝ be the half of the *latus rectum*. Then the double pl.ΝΓΞ is equal to sq.ΚΞ, as is proved in Theorem 11 of Book I, and the double pl.ΝΓΞ is equal to the double pl.ΕΖ, ΓΞ.

We make the sum of the double pl.ΕΖΞ, sq.ΕΖ, and sq.ΖΞ common. Then the sum of the double pl.ΕΖ,ΓΞ ,the double pl.ΕΖΞ, sq.ΕΖ, and sq.ΖΞ is equal to the sum sq.ΚΞ and sq.ΞΕ which sq.ΚΕ .But the sum of the double pl.ΕΖ,ΓΞ and the double pl.ΕΖΞ is equal to the double pl.ΓΖΕ. Therefore sq.KE is equal to the sum of the double pl.ΓΕΖ, sq.ΖΞ, and sq.ΕΖ. But the double pl.ΓΖΕ is equal to sq.ΖΗ because ΖΕ is equal to ΓΝ. Therefore the sum of sq.ΖΗ, sq.ΖΕ, and sq.ΖΞ is equal to sq.ΕΚ. But the sum of sq.ΖΗ and sq.ΖΕ is equal to sq.ΕΗ. Therefore sq.ΚΕ is equal to the sum of sq.ΕΗ and sq.ΖΞ. Therefore the amount by which sq.ΚΕ is greater than sq.ΕΗ is equal to sq.ΖΞ .

Similarly also it will be proved that the difference between sq.ΕΛ and sq.ΕΗ is equal to sq.ΜΖ. And since the double pl.ΓΖΕ is equal to sq.ΖΗ [because ΖΕ is equal to ΓΝ], therefore the difference between sq.ΓΕ and sq.ΕΗ is equal to sq.ΓΖ. And ΖΞ is smaller than ΖΜ, which is smaller than ΖΓ.

Therefore EH is the least of the straight lines drawn from Ε to the section on the side of Γ.

Furthermore sq.ΒΕ is equal to the double pl.ΝΓΕ and is equal to the double pl.ΓΕΖ. And the double pl.ΓΖΕ is equal to sq.ΖΗ. Therefore sq.ΒΕ is equal to the sum of sq.ΗΕ and sq.ΕΖ. Therefore amount by which sq.ΒΕ is greater than sq.ΕΗ is equal to sq.ΖΕ.

Furthermore sq.ΧΘ is equal to the double pl.ΓΧ,ΖΕ because ΖΕ is equal to ΓΝ. We make sq.ΧΕ common. Then the sum of the double pl.ΓΖΕ, the double sq.ΖΕ, and the double sq.ΖΧ is equal to sq.ΕΘ. But the sum of the double pl.ΧΖΕ, and the double sq.ΖΕ is equal to sq.ΕΗ. Therefore sq.ΕΘ without sq.ΕΗ is equal to sq.ΖΧ.

Similarly also it will be proved that sq.AE without sq.EH is equal to sq.DZ. But ΔZ is greater than ZX, which is greater than ZE.

Therefore EH is the least of the straight lines drawn from E to the section, and those drawn closer to it are smaller than those drawn farther, and the difference between them and it is equal to the square on the segment between the foot of the perpendicular from it and Z.

[Proposition] 9

*If a point is taken on the axis of a hyperbola such that the distance between it and the vertex of the section is greater than the half of the* latus rectum*, and the segment between the taken point and the center is cut in two parts such that as one is to other, so the transverse diameter is to the* latus rectum*, and the segment next to the center is one corresponding to the transverse diameter, and there is drawn from the point at which that segment was cut a perpendicular to the axis so as to meet the section and the segment between the point of its meeting and the taken point is joined, then that joined straight line is the least of thee straight lines drawn from the taken point to the section, and as for the other straight lines on either side of it those of them drawn closer [to it] are smaller than those drawn farther, and the amount by which the square on each of them is greater than the square on it is equal to the rectangular plane on the segment between the foot the perpendiculars from two of them similar to the rectangular plane under the transverse diameter and a segment equal to the sum of the transverse diameter and the* latus rectum *when the side corresponding to the transverse diameter is the segment between two perpendiculars* [13].

Let there be the hyperbola ΑΒΓ whose external axis ΩΔ and center H. Let ΓE be greater than the half of the *latus rectum*. Let as HB is to ZE, so transverse diameter is to the *latus rectum* [Then Z falls between Γ and E] from Z a perpendicular ZΘ to the axis is drawn, and ΘE is joined.

I say that EΘ is the smallest of the straight lines drawn from E to the section, and that [other straight lines] on both sides those drawn closer to it are smaller than those drawn farther, and that the difference between the square on each of them and the square on it is equal to rectangular plane on the segment between the feet of their two perpendiculars similar to the rectangular plane under the transverse diameter and a segment equal to the sum of the transverse diameter and the *latus rectum*, when the transverse diameter corresponds to the segment between two perpendicular.

[Proof]. For let the half of the *latus rectum* be made ΓΙ, and let the perpendicular ΛΝ and ΚΞ and other perpendicular [ΒΕ, ΜΤ, and ΑΔ], be drawn and continued in a straight line. Let ΗΙΨ be joined [to meet the perpendicular at Ο, Ρ, Φ, Χ, and Ψ] and ΡΕ be joined and continued in both directions [to meet ΜΧ at Γ, ΚΟ□, and ΓΙ at Υ] . Then as ΓΗ is to ΓΙ, so the transverse diameter is to the *latus rectum*. But as ΓΗ is to ΓΙ, so ΗΖ is to ΖΡ ,and as ΗΖ is to ΖΕ. Therefore ΖΕ is equal to ΖΡ.

But sq.ΖΘ is equal to the double area ΓΙΡΖ, as is proved in Theorem 1 of this Book, and sq.ΖΕ is equal to the double triangle ΖΕΡ. Therefore sq.ΘΕ is equal to the double area ΓΕΡΙ.

Furthermore sq.ΚΞ is equal to the double area ΟΞΓΙ, as is proved in Theorem 1 of this Book, and sq.ΕΞ is equal to thee double triangle ΕΞ□.
Therefore sq.ΚΕ is equal to the sum of the double area ΡΕΓΙ and the double triangle ΡΟ□.

But it was proved that sq.ΘΕ is equal to the double area ΡΕΓΙ. Therefore sq.ΕΚ without sq.ΘΕ is equal to the double triangle ΡΟ□.

Let the straight lines ΟΣ, ΡΠ, and □Ϙ be drawn parallel to ΓΔ. Then as HG is to ΓΙ, so □Π is to ΠΟ because ΡΠ is equal to Π□.

So as □Π is to ΠΟ so the transverse diameter is to the *latus rectum*.

Therefore as Π□ is to □Ο, so transverse diameter is to a segment equal to the sum of the transverse diameter and the *latus rectum*.

But Π□ is equal to □Ϙ. Therefore the rectangular plane ΣΟ□Ϙ is similar to the rectangular plane under the transverse diameter and a segment equal to the sum of the transverse diameter and the *latus rectum*.
And the quadrangle ΣΟ□Ϙ is equal to the double triangle ΟΡ□, which is the difference between sq.ΕΚ and sq.ΕΘ.

And ΣΟ is equal to ΖΞ. Therefore sq.ΚΕ without sq.ΘΕ is equal to the rectangular plane on ΖΞ similar to the mentioned plane when the transverse diameter corresponds to ΖΞ.

Similarly also it will be proved that sq.ΕΛ without sq.ΕΘ is equal to rectangular plane on ΖΝ similar to the mentioned plane when again the transverse diameter corresponds to ΖΝ.

Furthermore sq.ΓΕ is equal to the double triangle ΓΥΕ, and sq.ΕΘ is equal to the double quadrangle ΓΕΡΙ, as is proved in Theorem 1 of this Book.

Therefore sq.ΓΕ without sq.ΕΘ is equal to the double triangle ΥΡΙ.

But the double triangle ΥΡΙ is equal to the rectangular plane on ΓΖ similar to the mentioned . Therefore sq.ΓΕ without sq.ΕΘ is equal to the rectangular plane on ΓΖ similar to the mentioned plane.

And ZΞ is smaller than ZN, which is smaller than ZΓ. Therefore ΘE is smaller than EK, which is smaller than EΛ, which is smaller than EΓ.

Therefore EΘ is the least of the straight lines drawn from E to the section on the one side that towards Γ.

Furthermore sq.BE is equal to the double quadrangle ΓIΦE, as is proved in Theorem 1 of this Book, and it was proved that sq.ΘE is equal to the double quadrangle ΓIPE. Therefore sq.EB without sq.EΘ is equal to the double triangle ΦEP, and the rectangular plane on ZE similar to the mentioned plane is equal to the double that triangle.

Furthermore sq.MT is equal to the double quadrangle TXIΓ, as is proved in Theorem 1 of this Book, and sq.TE is equal to the double triangle TEϛ. Therefore sq.ME is equal to the sum of the double triangle ϛXP and the double quadrangle ΓIPE.

But it was proved that sq.ΘE is equal to the double quadrangle ΓIPE. And the rectangular plane on ET similar to the mentioned plane is equal to the double triangle ϛPX.

Similarly also it can be proved that sq.EΛ without sq.ΘE is equal to the rectangular plane on ZΔ similar to the mentioned plane. And EZ is smaller than ZT which is smaller than EΛ. Therefore ΘE is smaller than EB which is smaller than EM which is smaller EΛ. Therefore EΘ is the least of the straight lines drawn from E to the section, and of the straight lines on either side of ΘE those of them drawn closer to ΘE are smaller than those drawn farther, and the square on each of them is greater than the square on ΘE by an amount equal to the rectangular plane on the segment between the feet of their perpendiculars and the foot of its perpendicular similar to the mentioned rectangular plane.

[Proposition] 10

*If a point is taken on the major axis of an ellipse such that the distance between that point and the vertex of the section is longer than the half of the* latus rectum*, and as the segment between the vertex of the section and the taken point on the axis is cut at a point such that the segment between the center of the section and the point at which the cut was made is to the segment between that [latter] point and the first taken point, so the transverse diameter is to the* latus rectum*, and from the point at which the cut was made a perpendicular is drawn to the axis to meet the section, and from the point where it meets [the sections] a straight line is drawn to the first taken point, then this straight line is the smallest of the straight lines drawn from the taken point to the section, and of the remaining straight lines [drawn from that point* *to the section] those of them drawn closer to that straight line are smaller than those drawn farther, and the amount by which [each of] the squares on them is greater than the square on it is equal to the rectangular plane on the segment between feet of the perpendiculars from them and the foot of the perpendicular from it which is similar to the rectangular plane under the transverse diameter and the excess of the transverse diameter over the* latus rectum *when the transverse diameter corresponds to that segment* [14].

Let there be the ellipse ΑΒΓ whose major axis be ΑΓ, and center Δ. Let ΕΓ be greater than the half of the *latus rectum*, and as ΔΖ is to ΖΕ, so ΑΓ is to the *latus rectum*. From Ζ a perpendicular to the major axis is drawn, namely ΖΗ, it is continued to Τ, and ΕΗ is joined.

I say that ΕΗ is the smallest of the straight lines, drawn from Ε to the section, and that of thee other straight lines [drawn from Ε to the section] those of them drawn closer to that straight line are smaller than those drawn farther and that the amount by which their are squares are greater than its square is equal to the rectangular plane on the segment between the feet of the perpendiculars from them and Ζ similar to the rectangular plane under the diameter ΑΓ and the excess of that diameter over the *latus rectum* then the diameter ΑΓ corresponds to the segment between Ζ and the foot of the perpendicular.

[Proof]. For let the straight lines [ΚΕ, ΘΕ, ΛΕ, and ΜΕ] and the perpendiculars [ΚΣ, ΘΡ, ΛΔ, ΜΠ, and ιΑ] be drawn as in the diagram, and let ΒΕ be perpendicular to ΑΓ, and let ΓΝ be the half of the *latus rectum*. ΝΔ, ΤΕ are joined and continued [and ΘΡ is continued to meet them at Χ and Ψ, and ΒΕ is continued ΝΔ at Ϙ].

Then as ΔΓ is to ΓΝ, so the transverse diameter is to the *latus rectum* therefore as ΔΖ is to ΖΕ, so ΔΓ is to ΓΝ. But as ΔΓ is to ΓΝ, so ΔΖ is to ΖΤ, therefore as ΔΖ is to ΖΕ so ΔΖ is to ΖΤ. Therefore ΖΕ is equal to ΖΤ.

Let Τ□, ΧΥ, and ΨΦ be drawn parallel to ΑΓ. Then sq.ΖΕ is equal to the double triangle ΖΕΤ, and sq.ΖΗ is equal to the double quadrangle ΖΓΝΤ, as is proved in Theorem 1 of this Book. Therefore sq.ΕΗ is equal to the double quadrangle ΝΓΕΤ.

Furthermore sq.ΘΡ is equal to the double quadrangle ΓΡΧΝ, as is proved in Theorem 1 of this Book, and esq. is equal to the double triangle ΡΨΕ. Therefore sq.ΕΘ is equal to the sum of the double quadrangle ΓΝΤΕ and the double triangle ΨΤΧ.

But sq.ΕΗ was shown to be equal to the double quadrangle ΓΝΤΕ.

Therefore sq.EΘ without sq.EH is equal to the double triangle TΨX. But the double triangle TΨX is equal to the quadrangle ΨΦYX.

Furthermore as EZ is to ZT, so T□ is to □Ψ. But EZ is to ZT. Therefore T□ is equal to □Ψ. And as T□ is to □X , so ΔΓ is to ΓN.
Therefore as □Ψ is to □X ,so ΔΓ is to ΓN.

But as ΔΓ is to ΓN, so the transverse diameter is to the *latus rectum*.
Therefore as □Ψ is to □X , so the transverse diameter is to the *latus rectum*.

Convertendo as □Ψ is to ΨX, so the transverse diameter is to the excess of the transverse diameter over the *latus rectum*;

But □Ψ is equal to ΦΨ, so the quadrangle XΨΦY is similar to the rectangular plane under the transverse diameter and its excess over the *latus rectum*. Therefore sq.EΘ without sq.EH is equal to the rectangular plane on ZP similar to the mentioned one where ZP corresponds to the transverse diameter.

Similarly also it will be proved that sq.KE without sq.EH is equal to the rectangular plane on ZΣ similar to the mentioned plane, and that sq.EΓ without sq.EH is equal to the rectangular plane on ZΓ similar to the mentioned plane.

But ZP is smaller than ZΣ, which is smaller than ZΓ. Therefore EH is smaller EΘ, which is smaller than EK, which is smaller than EΓ.

Furthermore sq.BE is equal to the double quadrangle EΓNϘ, as is proved in Theorem 1 of this Book. And sq.EH is equal to the double quadrangle EΓNT, as we moved above. Therefore sq.BE without sq.EH is equal to the double triangle ETϘ.

But the double triangle ETϘ is equal to the rectangular plane on ZE similar to the mentioned plane, and that will proved in the way described previously.

Furthermore sq.ΔΛ is equal to the double triangle ΔΓN, as is proved in Theorem 2 of this Book. And sq.ΔE is equal to the double triangle ΔEς. Therefore sq. ΛE is equal to the sum of the double triangle ΔςT and the double quadrangle ΓNTE. Therefore sq.ΛE without sq.EH is equal to the double triangle ΔςT.

But the double triangle ΔςT is equal to the rectangular plane on ΔZ similar to the mentioned plane.

Furthermore sq.MΠ is equal to the double quadrangle ΞOΠΛ, as is proved in Theorem 3 of this Book.

And sq.ΠE is equal to the double triangle ΠEΩ. Therefore sq.ME is equal to the sum of the double triangle ΞΔΛ and the double quadrangle ΩEΔO.

But the triangle ΞΔΛ is equal to the triangle ΓΔN. Therefore sq.ME is equal to the sum of the double quadrangle ΓETN and the triangle OTΩ. Therefore sq.ME without sq.EH is equal to the double triangle ΩTO. But the double triangle ΩTO is equal to the rectangular plane on ZΠ similar to the mentioned plane.

Furthermore sq.EA is equal to the double triangle AEιι, and the triangle ΔΓN is equal to the triangle AΔΞ. Therefore sq.EA is equal to the sum of the double triangle TΞιι and the quadrangle ΓETN. Therefore sq.AE without sq.EH is equal to the double triangle TΞι. But the double triangle Ξι is equal to the rectangular plane on AZ similar to the mentioned plane.

And EZ is smaller than ZΔ which ZΠ, which is smaller than ZA. Therefore BE is smaller than EΛ which is smaller than EM which is smaller than EA.

Therefore EH is the least of the straight lines drawn from E to section ABΓ, and as for the rest of the straight lines on both sides [of EH] those drawn closer to EH are smaller than those drawn farther, and the amounts by which the squares on them are greater than the square on it are equal to the rectangular planes on the segments between the feet of their perpendiculars and the foot of its perpendicular similar to the mentioned plane [15].

[Proposition] 11

*The smallest of the straight lines drawn from the center of an ellipse to the boundary of the section is the half of the minor axis, and the graters of them is the half on the major axis, and those straight lines drawn [from the center] closer to the longest straight line are greater than those drawn farther, and the amount by which the square on each of those straight lines is greater than the square on the shortest straight line is equal to the rectangular plane on the segment between the foot of the perpendicular [from that straight line] and the center similar to the rectangular plane under the transverse diameter and the excess of it and over the* latus rectum [16].

Let there be the ellipse ABΓ whose major axis be AΓ and minor axis BΔ.

I say that the longest of the straight lines drawn from the center E to the section is EΓ, and the shortest of them is EB, and that of the other the straight lines between EB and EΓ those of them drawn closer to ΓE are greater than those drawn farther from it, and that the amounts by which the squares on them are greater the square on BE are equal to the rectangular planes on the segments between the feet of the perpendiculars from them onto AΓ and E similar to the rectangular plane under AΓ and the excess of AΓ over the *latus rectum*.

[Proof]. For let EZ and EH be drawn, and the perpendiculars ZI and HΠ are dropped. Let the half of the *latus rectum* be ΓΘ. Then ΓΘ is smaller than

ΓΕ. So let ΓΚ be equal to ΓΕ. Let ΘΕ and ΕΚ be joined, and ΗΠ and ΖΙ are continued to Ο and Ξ, and ΜΛ and ΝΞ be drawn parallel to ΑΓ. Then pl.ΕΓΚ is equal to ΕΙΞ. But ΕΓ is equal to ΓΚ, therefore ΕΙ is equal to ΞΙ. And sq.ΓΖ is equal to the double quadrangle ΓΘΛΙ, as is proved in Theorem 1 of this Book.

And sq.IE is equal to the double triangle ΙΕΞ. Therefore sq. ΖΕ is equal to the sum of the double triangles ΕΓΘ and ΕΛΞ. And sq.ΕΒ is equal to the double triangle ΕΓΘ, as is proved in Theorem 2 of this Book.

And the double triangle ΕΛΞ is equal to the quadrangle ΛΞΜΝ. Therefore sq.ΕΖ without sq.ΕΒ is equal to the quadrangle ΛΝ. And as ΚΓ is to ΓΘ, so the transverse diameter is to the *latus rectum*, and as ΚΓ is to ΓΘ, so ΞΙ is to ΙΛ, and convertendo as ΞΙ is to ΞΛ so the transverse diameter is to the excess of the transverse diameter over the *latus rectum*.

But ΞΙ is equal to ΞΝ. Therefore thee quadrangle ΛΞΝΜ is similar to the rectangular plane under the transverse diameter and its excess over the *latus rectum*. But ΛΜ is equal to ΙΕ. Therefore sq.ΕΖ without sq.ΕΒ is equal to the rectangular plane on ΙΕ similar to the mentioned plane.

Similarly also it will be proved that sq.ΕΗ without ΕΒ is equal to the rectangular plane on ΕΠ similar to the plane.

Furthermore sq.ΓΕ is equal to the double triangle ΓΕΚ, and sq.ΒΕ is equal to the double triangle ΓΕΘ. Therefore sq.ΓΕ without sq.ΒΕ is equal to the double triangle ΕΚΘ. But the double triangle ΕΚΘ is equal to the rectangular plane on ΓΕ similar to the mentioned plane.

And ΕΓ is greater than ΕΠ which is greater than ΕΙ. Therefore ΕΓ is greater than ΕΗ which is greater than ΕΖ, which is greater than ΕΒ.

Therefore the longest on the straight lines drawn from Ε is ΕΓ, and the shortest of them is ΕΒ, and as for the other straight lines [from Ε] between ΕΒ and ΕΓ those of them drawn closer to ΕΓ are longer than those drawn farther, the amount by which the square on each of then is greater than the square on ΕΒ is equal to the rectangular plane on the segment between the foot the perpendicular from it onto ΑΓ and the center similar to the mentioned plane.

[Proposition] 12

*If a point is taken on one of the straight lines which has been proved to be minimal on straight lines drawn from some point on the axis to one of the [three] sections and straight lines are drawn from that [first] point to the section on one side, then the shortest of them is the segment of the minimal line* *adjoining the section, and those drawn closer to it are shorter than those drawn farther* [17].

Let there be the conic section AB whose axis BΓ and the minimal straight line drawn from some point on it be ΓA. On it an arbitrary point Δ is taken. I say that ΔA is the shortest of the straight lines drawn from Δ in that part of the section.

[Proof]. For let ΔE, ΔZ, and ΔB be drawn, and ZΓ, ΓE, AE, EZ , and ZB be joined then EΓ is greater than ΓA, so the angle ΓAE is greater than the angle ΓEA. But the angle ΓEA is greater than the angle AEΔ, therefore the angle EAΔ is mush greater than the angle AEΔ. Therefore EΔ is greater than ΔA.

Furthermore ZΓ is greater than ΓE, therefore the angle ZEΓ is greater than the angle EZΓ. Therefore the angle ZEΔ is much greater than the angle EZΔ. Therefore ZΔ is greater than ΔE.

Similarly also it will be proved that BΔ is greater than ΔZ. Therefore AΔ is the smallest of the straight lines drawn in this part of the section, and those drawn closer to it are smaller than those drawn farther.

Similarly also it will proved concerning those straight lines where they are drawn in the other part of the section.

[Proposition] 13

*If there is drawn from a point from the axis of a parabola the minimal of the straight lines drawn from that point to the section, so as to form an angle with the axis, then that angle which it forms with the axis will be acute, and if a perpendicular is dropped from its [other] end to the axis, then [that perpendicular] cuts off from it segment equal to the half of the* latus rectum[18].

Let there be the parabola AB whose axis BΓ, and the minimal straight line drawn [from Γ] in the parabola, AΓ.

I say that the angle at Γ is acute, and that the perpendicular drawn from A to BΓ cuts off from it a segment equal to the half of the *latus rectum*.

[Proof]. For AΓ is minimal, so BΓ is greater than the half of the *latus rectum*. For if it were not greater than it, would be either equal to it or less than it.

But if it were equal to it, BΓ would minimal, as is proved in Theorem 4 of this Book. But that is not so for the minimal is AΓ. And if BΓ were less than the half of the *latus rectum*, then where a straight line equal to the half of the *latus rectum* was cut off from the axis the point at which the cut was made would be beyond Γ. Therefore it could be proved from Theorem 4 of this Book that BΓ is smaller than ΓA. Therefore BΓ is not smaller than the half of the *latus rectum*.

And we have proved that it is not equal to it. Therefore it is greater than it. Therefore let the [straight line] equal to the half of the *latus rectum* be ΓΔ. Then I say that the perpendicular drawn from Δ meets A.

[Proof]. For let if that is not so the perpendicular be ΔE. Then EΓ is the shortest of the straight lines drawn from Γ to the section, as is proved in Theorem 8 of this Book. But AΓ was the minimal. That is impossible.

Therefore the perpendicular drawn from Δ meets A, and ΔΓ is equal to the half of the *latus rectum*, and the angle AΓB is acute.

[Propositions] 14

*If there is drawn from the axis of a hyperbola a straight line which is minimal of the straight lines drawn from that point, so as to form with the axis two angles, then that angle of two which is towards the vertex of the section is acute, and if there is drawn from the [other] end of the minimal straight line a perpendicular to the axis, it cuts the straight line between the center of the section and the point on the axis from which the minimal line is drawn into two parts such that as that part adjacent to the center is to the other part, so the transverse diameter is to the* latus rectum [19].

Let there be the hyperbola AB whose axis BΓ, and the minimal straight line AΓ drawn from Γ, and the center Δ.

I say that the angle AΓB is acute, and that the perpendicular falling from A onto axis BΓ cuts ΓΔ into two parts such that as one part of two is to the other, so the transverse diameter is to the *latus rectum*.

[Proof]. For BΓ is longer than the half of the *latus rectum*, as is proved from Theorem 4 of this Book. And BΔ is the half of the transverse diameter. Therefore the ratio ΔB to BΓ is less than the ratio of the transverse diameter to the *latus rectum*.

Therefore we cut ΔΓ into two parts at E such that as one of them is to the other, so the transverse diameter is to the *latus rectum*.

Then I say that the perpendicular drawn from E to ΔΓ reaches A for if that is not so, let it be as perpendicular EZ let ΓZ be joined then GZ is the minimal straight line drawn from Γ, as is proved in Theorem 9 of this Book.

But the minimal straight line was AG, that impossible. Therefore the perpendicular drawn from E reaches A, therefore the angle AΓB is acute, and the perpendicular drawn from A cuts ΓΔ into two parts such that as one of them is to the other, so the transverse diameter is to the *latus rectum*.

[Proposition] 15

*If there is drawn from a point on the major of two axes of an ellipse a straight line that is minimal of the straight lines drawn from that point, then that minimal straight line, if it was drawn from the center, is a perpendicular to the major axis* [20].

Let there be the ellipse ΑΒΓ whose the major axis is ΑΓ and the center Ι. Let first from Ι the minimal straight line ΙΒ be drawn to the section.

I say that ΙΒ is perpendicular to ΑΓ .

[Proof]. For let it be not so, let ΙΔ be perpendicular to ΑΓ . Then, as is proved in Theorem 11 of this Book, ΙΔ is minimal straight line drawn from I to the section. But this straight line is ΙΒ, and this impossible, therefore ΙΒ is perpendicular to ΑΓ.

Furthermore let other point Η is taken on the major axis. Then the minimal straight line drawn from Η to the section is ΗΖ.

I say that the angle ΖΗΙ is obtuse, and that the perpendicular dropped from Ζ to ΑΓ is such that as the segment between the foot of the perpendicular and Ι is to the segment between the foot of the perpendicular and Η, so the transverse diameter is to the *latus rectum*. If ΖΗ is the minimal straight line drawn from Η [to the section] then as is proved in Theorem 10 of this Book, then the ratio of ΓΙ to ΓΗ is less than the ratio of the transverse diameter to the *latus rectum*.

Let ΓΗ be divided at Κ so that as ΙΚ is to ΗΚ, so the transverse diameter is to the *latus rectum*. I say that the perpendicular drawn from Κ passes through Ζ for if that is not so, let it be as ΚΛ, then ΛΗ is minimal of the straight lines drawn from Η, as is proved in Theorem 10 of this Book. But the minimal of those straight lines was ΖΗ, and that is impossible. Therefore the perpendicular drawn from Κ passes through Ζ, and the angle ΙΗΖ is obtuse. So the perpendicular drawn from Ζ to ΑΓ is ΖΚ, and as ΙΚ is to ΚΗ, so the transverse diameter is to the *latus rectum*.

[Proposition] 16

*If a point is taken on the minor of two axes of an ellipse such that the segment of the minor axis between it and the vertex of the section is equal to the half of the* latus rectum*, then of the straight lines drawn from the point to the section the greatest is the part of the minor axis which is equal to the half of the* latus rectum*, and the smallest is the complement of the minor axis and of the other straight lines [so drawn] those of them drawn closer to the maximal straight line are longer than those drawn farther, and the excess of the* *square on it over the square on each of them is equal to rectangular plane on the segment between the foot of the perpendicular from it and the end of the minor axis similar to the rectangular plane under the minor axis and the excess of the* latus rectum *over it* [21].

Let there be the ellipse ΑΒΓ whose minor axis ΑΓ and center Π, let on the axis be taken Δ such that ΓΔ is equal to the half of the *latus rectum*.

I say that the greatest of the straight lines drawn from Δ to the section ΑΒΓ is ΔΓ, and the smallest of them is ΔΑ, and that of the remaining straight lines those drawn nearer to ΔΓ are longer than those farther, and that sq.ΓΔ is greater than the square on each of them by an amount equal to the rectangular plane on the segment between the foot of the perpendicular from it and Γ similar to the mentioned plane.

[Proof]. For let ΔΖ, ΔΕ, ΔΒ, and ΔΗ be drawn. Let ΔΒ be perpendicular to ΑΓ, and let the half of the *latus rectum* be ΓΞ, and ΞΠ and ΞΔ be joined and continued, and let the perpendiculars ΖΘ, ΕΚ, and ΗΛ be dropped, and ΑΡ parallel to the ordinates be drawn, and ΜΤ, [Ψ]ΥΦ parallel to ΑΓ be drawn. Then ΓΔ is equal to ΓΞ. Therefore sq.ΓΔ is equal to the double triangle ΓΔΞ.

But sq.ΘΔ is equal to the double triangle ΔΘΜ, and sq.ΖΘ is equal to the double quadrangle ΓΞΥΘ, as is proved in Theorem 1 in this Book. Therefore sq.ΓΔ without sq.ΔΖ is equal to the double triangle ΥΜΞ.

But the double this triangle is the quadrangle ΤΜΥΦ, and as ΠΓ is to ΠΔ, so the transverse diameter is to the excess of the *latus rectum* over it [because as the half of the transverse diameter is to the half of the *latus rectum*, so the transverse diameter is to the *latus rectum*], and as ΠΓ is to ΠΔ, so ΥΦ is to ΥΨ, that is ΥΦ to ΥΜ. Therefore as ΥΦ is to ΥΜ, so the transverse diameter is to the excess of the *latus rectum* over it.

And ΥΦ is equal to ΓΘ. Therefore sq.ΓΔ without sq.ΔΖ is equal to the rectangular plane on ΓΘ similar to the mentioned plane.

Similarly also it will be proved that sq.ΓΔ without sq.ΔΕ is equal to the rectangular plane on ΓΚ similar to the mentioned plane.

Furthermore sq.ΒΔ is equal to the double quadrangle ΡϘΔΑ, as is proved in Theorem 3 of this Book, and sq.ΔΓ is equal to the double triangle ΔΓΞ, and the triangle ΡΠΑ is equal to the triangle ΓΞΠ. Therefore sq.ΓΔ without sq.ΔΒ is equal to the double triangle ΔϘΞ.

But the double this triangle is equal to the rectangular plane on ΓΔ similar to the mentioned plane.

Therefore ΓΔ is greater than ΔΖ, which is greater than ΔΕ, which is greater than ΔΒ.

Furthermore sq.ΛH is equal to the double quadrangle PΓΛA, as is proved in Theorem 3 of this Book.

And sq.ΛΔ is equal to the double triangle ΛXΔ. Therefore sq.ΔH is equal to the sum of the double quadrangle PϚΛA and the double triangle XΔΛ.

But sq.ΓΔ is equal to the double triangle ΓΞΔ, and the triangle ΓΞΠ is equal to the triangle ΠPA. Therefore sq.ΓΔ without sq.ΔH is equal to the double triangle ϚΞX.

But the double this triangle is equal to the rectangular plane on ΓΛ similar to the mentioned plane.

Furthermore sq.ΔA is equal to the double triangle ΔAΣ, and the triangle ΓΠΞ is equal to the triangle AΠP. Therefore sq.ΔΓ without sq.ΔA is equal to the double triangle PΞΣ.

But the double this triangle is equal to the rectangular plane on AΓ similar to the mentioned plane.

Therefore ΓΔ is the greatest of the straight lines drawn from Δ to the section, and ΔA is the shortest of them, and of the other straight lines those drawn nearer to ΓΔ are greater than those drawn farther, and the excess of sq.ΓΔ over the squares on the other straight lines is equal to the rectangular plane on the segment between the foot of the perpendicular from [each of] them and Γ similar to the mentioned plane.

[Proposition] 17

Furthermore if AΓ [which is the minor axis of the ellipse] equal to the half of the *latus rectum* and the center be made O, then I say that ΓA is the greatest of the straight lines drawn from A to the section, and those [straight lines drawn closer to it are greater than those drawn farther, and the difference between the square on it and the square on each of them is equal to the rectangular plane on the segment between the feet of the perpendiculars from [each of] them and Γ similar to the mentioned plane in the previous theorem [22].

[Proof]. For let the straight lines set up this diagram like the set up of the previous diagram be drawn. Then it will proved in the way proved there that sq.AΓ is greater than sq.AE by an amount equal to the rectangular plane on ΓΘ similar to the mentioned plane.

Similarly also it will be proved that sq.AΓ is greater than sq.AΛ by an amount equal to the rectangular plane on ΓH.

Furthermore sq.BZ is equal to the double quadrangle KPZA, as is proved in Theorem 3 of this Book. And sq.ZA is equal to the double triangle AΞZ.

Therefore sq.ΑΒ is equal to the double quadrangle ΚΡΞΑ. And sq.ΓΑ is equal to the double triangle ΑΓΔ, because ΑΓ is equal to ΓΔ, and the triangle ΓΟΔ is equal to the triangle ΚΟΑ. Therefore sq.ΓΑ without sq.ΒΑ is equal to the double triangle ΡΞΔ. And the double this triangle is equal to the rectangular plane on ΓΖ similar to the mentioned plane, that will be proved as in the preceding theorem. Therefore ΑΓ is greater than ΑΕ, which is greater than ΑΛ, which is greater than ΑΒ.

Therefore the greatest of the straight lines drawn from Α [to the section] is ΑΓ, and of the remaining straight lines those drawn closer to it are greater than those drawn farther, and the excess of sq.ΑΓ over the square on [each of] them is equal to the rectangular plane under the segment between the foot of the perpendicular from [each of] them and Γ similar to the mentioned plane.

[Proposition] 18

Furthermore if the minor axis of the ellipse is made ΑΓ, the center Ν, and the straight line equal to the half of the *latus rectum* ΓΔ [which is greater than ΑΓ], then I say that ΓΔ is the greatest of the straight lines drawn from Δ to the section, and the smallest of them is ΔΑ, and that of the others straight lines which cut the section those drawn closer to ΓΔ are greater than those drawn farther, and for those straight lines which fall outside [the section] those drawn closer to ΑΔ are smaller than those drawn farther, and that sq.ΓΔ is greater than the square on each of them by the amount of the rectangular plane under the segment between Γ and the foot of the perpendicular [from the end of the segment] similar to the plane mentioned in two preceding theorems[23].

[Proof] . For let ΔΖ, ΔΕ, ΔΒ be drawn and set up like in the preceding diagram. Then it will also be proved that sq.ΓΔ is greater than sq.ΔΖ by an amount equal to the rectangular plane under ΓΛ similar to the mentioned plane, and that sq.ΔΓ is greater than sq.ΔΕ by an amount equal to the rectangular plane on ΓΘ similar to the mentioned plane , and that sq. ΓΔ is greater than sq.ΔΒ by an amount equal to the rectangular plane on ΓΚ similar to the mentioned plane.

Furthermore sq.ΑΔ is equal to the double triangle ΑΔΣ [because ΔΓ is equal to ΓΜ], and sq.ΓΔ is equal to the double triangle ΔΓΜ, and the triangle ΓΜΝ is equal to the triangle ΞΑΝ, therefore sq.ΓΔ without sq.ΔΑ is equal to the double triangle ΞΜΣ. But the double triangle ΞΜΣ is equal to the rectangular plane on ΛΓ similar to the mentioned plane.

Therefore ΔΓ is greater than ΔZ, which is greater than ΔE, which is greater than ΔB, which is greater than ΔA.

Furthermore sq.ΠT is equal to double quadrangle ΞOΠA, as is proved in Theorem 3 of this Book, and sq.ΔΠ is equal to the double triangle ΔΠP.

Therefore sq.TΔ s equal to the sum of the double quadrangle ΞOΠA and the double triangle ΠΔP. And sq.ΓΔ is equal to the double triangle ΓMΔ, and the triangle ΓMN is equal to the triangle NΞA. Therefore sq.ΓΔ without sq.TΔ is equal to the double triangle OMP. But the double triangle OMP is equal to the rectangular plane on ΓΠ similar to the plane mentioned in two preceding theorems.

Similarly too it will be proved that sq.ΓΔ is greater than sq.ΔΦ by an amount equal to the rectangular plane on ΓY similar to the mentioned plane, and that the difference between sq.ΓΔ and sq.Δ Ϙ is equal to the rectangular plane on ΓH similar to the mentioned plane.

And it has been shown that the difference between sq.ΓΔ and sq.ΔA is equal to the rectangular plane on ΓA similar to the mentioned plane. Therefore AΔ is smaller than ΔT which is smaller than ΔΦ which the smaller than ΔϘ.

Therefore ΓΔ is the greatest of the straight lines drawn from Δ [to the section] and ΔA is the least of them, and of the other straight lines which cut the section those of them  drawn closer to ΔΓ are grater than those drawn farther, and for those [straight lines] which do not cut  the section, those of them drawn closer to AΔ are smaller than those farther, and the difference between the square  on [one of those] straight lines and sq.ΔΓ or sq.ΔA is equal to the rectangular plane on the segment between Γ [or A] and the foot of the perpendicular [from the other end of the segment] similar to the mentioned plane.

[Proposition] 19

*If a point is taken on the minor of two axes on a ellipse such that its difference from the vertices of the section is a distance greater than the half of the* latus rectum*, then the greatest of the straight lines drawn from that point to the section is the straight line drawn to the vertex of the section and of the others straight lines those drawn closer to it are greater than those drawn farther*[24].

Let there be the ellipse AB whose minor axis AΓ, and let for it Δ is taken and let Δ be greater than the half of the *latus rectum*, I say that ΓΔ is the greatest of the straight lines drawn from Δ to the section, and that of the other straight lines those drawn closer to ΓΔ are greater than those drawn farther.

[Proof]. For let the half of the *latus rectum* be ΓH, from ΔE, ΔZ, and ΔB are drawn and HZ, HE and HB are joined, and ΓZ, ZE, EB, and BA are joined. Then ΓH is greater than ZH, because it was proved in three preceding theorems. Therefore the angle ΓZΔ is greater than the angle ZΓΔ, and ΓΔ is greater than ΔZ.

Furthermore HZ is greater than EH. Therefore the angle ZEH is greater than the angle EZH. Therefore the angle ZEΔ is much greater than the angle EZΔ. Therefore ΔZ is greater than ΔE.

Similarly it will be proved that ΔE is greater than ΔB.

Therefore ΔΓ is the greatest of the straight lines drawn from Δ to the section, and the remaining straight lines those drawn closer to it are greater than those drawn farther.

[Proposition] 20

*If a point is taken on the minor of two axes on a ellipse such that the segment between that point and the vertex of the section is smaller than the half of the* latus rectum*, but greater than the half of the [transverse] diameter, and the segment between the vertex of the section and its center is divided at a point such that as the segment between the center and that point at which the segment was divided is to the segment between that point and the first taken point, so the transverse diameter is to the* latus rectum*, and there is drawn from this last point which was taken a perpendicular to the axis to meet the section, and a straight line id drawn from the point where it reaches [the section] to the first taken point, then the greatest of the straight lines drawn to the section from that first taken point is the straight line which was joined, and of the other straight lines those drawn closer to it are greater than those drawn farther, and the amount by which the square on it is greater than the square on each of them is equal to the rectangular plane on the segment between the second taken point and the foot of the perpendicular from [the end of] the segment similar to the rectangular plane under the transverse diameter and the amount by which the* latus rectum *is greater than it* [25].

Let there be the ellipse ABΓ whose minor axis AΓ, and let there be on it a point Δ such that ΓΔ is greater than the half of the transverse diameter which is AΓ, but smaller than the half of the *latus rectum*. Let the center be E, and let EΓ be divided at M such that as EM is to MΔ, so the transverse diameter which is ΑΓ is to the *latus rectum*. [that is possible because the half of the *latus rectum* is greater than ΓΔ]. Let from Μ a perpendicular to ΑΓ is drawn, namely ΖΜ, and let ΖΔ be joined.

I say that ΖΔ is the greatest of the straight lines drawn from Δ to the section, and that of the straight lines drawn on both sides [of ΖΔ] those drawn nearer to it are greater than those drawn farther, and that the amount by which sq.ΖΔ is greater than the square on each of them is equal to the rectangular plane under the segment between Μ and the foot of the perpendicular from it similar to the mentioned plane.

[Proof]. For let ΔΓ, ΔΗ, ΔΖ, and ΔΑ arbitrary positions be drawn, let ΔΒ be a perpendicular to the axis, and let the half of the *latus rectum* be ΓΥ, and let perpendiculars ΘΝ, ΗΚ, ΖΜ, ΛΞ be drawn and, ΥΕ be joined and continued, and the perpendiculars and the straight lines parallel to ΑΓ, as we did in the preceding theorems, be drawn. Then as ΜΕ is to ΔΜ, so the transverse diameter is to the *latus rectum*, that is ΕΓ is to ΓΥ. But as ΕΓ is to ΓΥ, so ΜΕ is to ΜΦ. Therefore ΜΔ is equal to ΜΦ, and sq.ΜΔ is equal to the double be triangle ΜΔΦ. And sq.ΜΖ is equal to the double quadrangle ΜΦΥΓ, as is proved in Theorem 1 of this Book. Therefore sq.ΖΔ is equal to the sum of the double triangle ΔΜΦ and the double quadrangle ΜΦΥΓ.

Furthermore sq.ΗΚ is equal to the double quadrangle ΚΓΥΡ, and sq.ΔΚ is equal to the double triangle ΚΙΔ. Therefore sq.ΔΗ is equal to the sum of the double triangle ΚΙΔ and the double quadrangle ΚΓΥΡ, and sq.ΔΖ without sq.ΔΗ is equal to the double triangle ΡΙΦ.

But this double triangle is equal to the rectangular plane on ΚΜ, which is equal to the mentioned plane [that will be proved in a way similar to that described in the proof of Theorem 16 of this Book].

Similarly also it will be proved that sq.ΔΖ without sq.ΔΘ is equal to the rectangular plane on ΜΝ similar to the mentioned plane.

Furthermore sq.ΓΔ is equal to the double triangle ΔΓΤ. Therefore sq.ΔΖ without sq.ΔΓ is equal to the double triangle ΤΥΦ, which is equal to the rectangular plane on ΓΜ similar to the mentioned plane.

Therefore ΔΖ is greater than ΔΗ which greater than ΔΘ which is greater than ΔΓ.

Furthermore sq.ΔΒ is equal to the double quadrangle ΠΛΔΨ, as is proved in Theorem 3 of this Book. And it has already been shown that sq.ΔΖ is equal to the sum of the double triangles ΕΓΥ and ΔΕΦ. But the triangle ΕΓΥ is equal to the triangle ΠΕΑ. Therefore sq.ΔΖ without sq.ΔΒ is equal to the double triangle ΦΔΨ. And the double triangle ΦΔΨ is equal to the rectangular plane on ΜΔ similar to the mentioned plane [that will be proved in a way similar to the way which was in the proof of Theorem 16 of this Book].

Similarly also it will be proved that sq.ΔZ without sq.ΔA is equal to the rectangular plane on ΜΞ similar to the mentioned plane.

Therefore ΔZ is the longest of the straight lines drawn from Δ to the section, and for the others straight lines those of them drawn closer to ΔZ are longer than those drawn farther, and the amount by which sq.ΔZ is greater than the square on each of them is equal to the rectangular plane on the segment between M and the foot of the perpendicular from it [the other end of the segment] similar to the mentioned plane.

Similarly also it will be proved that the half of the *latus rectum* is greater than the [transverse] diameter is equal to the minor axis, or if it is greater than it, then of the straight lines drawn from the point Δ of first diagram, or from the point A of the second diagram, or from a point such as the point Δ outside the point A of the third diagram, the greatest is the mentioned straight line. That will be proved in the second and third diagrams by a method similar to the one stated for the first diagram.

[Proposition] 21

*If a point is taken on the maximal straight line mentioned in the preceding theorem in the ellipse such that the distance between it and that end of the maximal straight line which lies on the section is greater than the maximal straight line, then the greatest of the straight lines drawn from that point [to the section] in one part of the section is the straight line of which the maximal is a part, and as for the straight line on either side of it, those of them nearer to the straight line are greater than those drawn farther* [26].

Let there be the ellipse ΑΒΓ whose [minor] axis ΑΓ, and let ΔΒ be the maximal straight line drawn from Δ, that is one mentioned in the theorem preceding this. Let ΒΔ be drawn and Ε be taken on it in such a way that ΒΕ is greater than the maximal straight line ΔΒ.

I say that the greatest of the straight lines drawn from Ε to the section is ΕΒ, and that of the other straight lines those drawn closer to it are greater than those drawn farther.

[Proof]. For let ΕΖ and ΕΗ be drawn, and ΔΖ, ΗΔ, and [also] ΓΕ, ΓΗ, ΗΖ, and ΖΒ be joined.

Then ΔΒ is greater than ΔΖ. Therefore the angle ΔΖΒ is greater than the angle ΖΒΔ. Therefore the angle ΕΖΒ is much greater than the single ΖΒΕ, and ΒΕ is greater than ΕΖ.

Furthermore ΔZ is greater than ΔH. Therefore the angle ΔHZ is greater than the angle ΔZH. Therefore the angle EHZ is much greater than the angle EZH, and therefore ZE is greater than EH.

Similarly also it will be proved that EH is greater than EΓ.

Therefore EB is the longest of the straight lines drawn from E to the section in this part of the section, and of the others straight lines those drawn closer to EB are greater than those drawn farther.

Similarly also what we asserted will be proved if the maximal straight line proceeds from A or from one of the other points which lie on the continued axis ΓA.

[Proposition] 22.

*If there is drawn from a point on the minor of two axes on an ellipse a straight line such that it encloses together with the axis an angle, and that the straight line is maximal of the straight lines drawn from that point to the section, then, if that point is the center of the section, the maximal straight line is perpendicular to the minor axis, but if it is not the center, then the angle enclosed between it and that part of the axis towards the center is acute, and if there is drawn from the [other] end of the straight line a perpendicular to the axis, then as the segment between the foot of its perpendicular and the center of the section is to the segment between the foot and the taken point, so the transverse diameter is to the* latus rectum [27].

Let there be the ellipse ABΓ whose minor axis AΓ. First let the maximal straight line come from the center, and be ΔB, then I say that ΔB is perpendicular to AΓ.

[Proof]. For let if that is not so, the perpendicular be ΔE. Then ΔE is the greatest straight line drawn from Δ, as is proved in Theorem 11 of this Book. But the greatest was ΔB, which is impossible. Therefore ΔB is perpendicular to AΓ.

Now let the maximal straight line come from another point namely Z, and let the straight line be ZH. Then I say that the angle ΓZH is acute, and that the perpendicular drawn from H to AΓ is such that as the length between its foot and Δ is to the length between its foot and Z, so the transverse diameter is to he *latus rectum*.

[Proof]. For let ZΓ be either greater than the half of the *latus rectum*, or smaller or equal to it. But if it were equal to it, it would be the maximal straight line, as we proved in Theorems 16 , 17, and 18 of this Book, and if it were greater than, then again ΖΓ would be the maximal, as is proved in Theorem 19 of this Book. Therefore ΖΓ is smaller than the half of the *latus rectum*.

Therefore if we make the ratio of a straight line adjoining ΖΔ to the sum of ΖΔ and that adjoining straight line equal to the ratio of the transverse diameter to the *latus rectum*, then that adjoining straight line is less than ΔΓ, let it be ΔΚ. Therefore as ΔΚ is to ΖΚ, so transverse diameter is to the latus rectum.

Then I say that straight line drawn from Κ perpendicular to ΑΓ meets Η.

[Proof]. For if it did not meet it, but fell like ΚΘ, then ΘΖ would be maximal, as is proved in Theorem 20 of this Book. But that is not so, therefore the perpendicular drawn from Η meets Κ, and as ΔΚ is to ΚΖ, so the transverse diameter is to the *latus rectum*.

## [Proposition] 23

*If there is drawn from a point on the minor of two axes of an ellipse one of the mentioned maximal straight lines, then that part of it intercepted between the section and the major axis is the smallest straight line that can be drawn [to the section] from the point of its meeting with the major axis* [28].

Let there be the ellipse ΑΒΓΔ whose major axis ΓΑ and minor axis ΔΒ. And let ΚΕ be the maximal straight line drawn from Κ.

I say that ΖΕ is the shortest of the straight lines from Z to meet the section.

[Proof]. For let from Ε a perpendicular ΕΗ to ΔΒ, and a perpendicular ΕΘ to ΑΓ, be drawn.

Then as ΔΒ is to the *latus rectum*, so the *latus rectum* is to ΑΓ, as is proved in Theorem 15 of Book I.

And as ΔΒ is to [its] *latus rectum*, so ΛΗ is to ΗΚ. Therefore as the *latus rectum* [of ΑΓ] is to ΑΓ, so ΛΗ is to ΗΚ, as is proved in Theorem 22 of this Book. But as ΛΗ is to ΗΚ, so ΘΖ is to ΘΛ. Therefore as ΛΘ is to ΘΖ, so ΓΑ is to *latus rectum* [of ΓΑ].

And ΘΕ is a perpendicular [to ΑΓ], and ΕΖ has been joined, and ΑΓ is the major axis. Therefore ΕΖ is the shortest straight line drawn from Ζ to the section, has is proved in Theorem 10 of this Book.

## [Preposition] 24

*If a point is taken on any conic section whatever, then only one of the minimal straight lines drawn from the axis meets it* [29] *.*

Let the section be, first, a parabola ΑΒ whose axis ΒΓ.

Let on thee section the point Α be taken.

I say that only one of the minimal straight lines can be drawn from the axis to Α.

[Proof]. For let if possible, two [minimal] straight lines ΑΓ and ΑΔ. Let from Α a perpendicular ΑΕ to ΒΓ, be drawn. Then ΕΔ is equal to the half of the *latus rectum*, as is proved in Theorem 13 of this Book. And similarly also ΕΓ is equal to the half of the *latus rectum*, but that is impossible. Therefore only one of the minimal straight lines can be drawn from the axis to Α.

[Proposition] 25

Furthermore let the section is the hyperbola or the ellipse ΑΒ whose the axis ΒΓ and the center Η, and let on the section an arbitrary point Α be taken.

I say that only one of the minimal straight lines can be drawn from the axis to Α[30] .

[Proof]. For if it is possible to draw more than one minimal straight line let two [minimal] straight lines ΑΕ and ΑΔ be drawn, and from Α, a perpendicular ΑΖ to ΒΓ , be drawn.

Then as ΖΗ is to ΖΕ, so the transverse diameter is to the *latus rectum*, as is proved in Theorems 14 and 15 of this Book.

Similarly also as ΖΗ is to ΖΔ , so the transverse diameter is to the *latus rectum*, but that is impossible. Therefore two minimal straight lines cannot be drawn from the axis to Α.

[Proposition] 26

*If a point is taken on an ellipse not on the minor axis, then only one of the maximal straight lines can be drawn from it to the minor axis*[31] .

Let there be the ellipse ΑΒΓ whose minor axis ΑΓ and a point Β on the section.

I say that only one maximal straight line can be drawn from Β to ΑΓ.

[Proof]. For let, if possible, two [maximal] straight lines ΒΔ and ΒΕ be drawn, and the perpendicular ΒΖ [to ΑΓ] be drawn, and let the center be Η.

Then ΒΕ is one of the maximal straight lines drawn from the axis, therefore as ΖΗ is to ΖΕ, so the transverse diameter is to the *latus rectum*, as is proved in Theorem 22 of this Book.

Similarly also it will be proved that as ZH is to ΔZ, so the transverse diameter is to the *latus rectum*, but that is impossible. Therefore only one maximal straight line can be drawn from B to the [minor] axis.

[Proposition] 27

*The straight line drawn from the end of one of the mentioned minimal straight lines tangent to the section is perpendicular to minimal of straight line* [32].

Let the section be, first, a parabola AB whose axis BΓ.

I say that the straight line drawn from the end of a minimal straight line tangent to the section AB is perpendicular to the minimal straight line.

[Proof]. If the minimal straight line is a part of BΓ, then what we said is evidently true].

But if minimal straight line is AΓ, we draw A a straight line tangent to the section AB, namely AΔ, that the angle ΔAΓ is right.

We draw the perpendicular AH. Then ΓH is equal to the half of the *latus rectum*, as is proved in Theorem 13 of this Book.

Furthermore AΔ is tangent to the parabola, and the perpendicular AH has been drawn from A [to the axis]. Therefore ΔB is equal to BH, as is proved in Theorem 35 of Book I.

Therefore as ΓH is to the *latus rectum*, so BH is to HΔ, therefore pl.ΓHΔ is equal to the rectangular plane under BH and the *latus rectum* which is equal to sq.AH, therefore sq.AH is equal to pl.ΓHΔ.

And the angle AHΔ is right, therefore the angle ΔAΓ [also] is right.

[Proposition] 28

Furthermore let the section be the hyperbola or the ellipse AB whose axis BΓ.

I say that the straight line drawn from the end of the minimal straight line tangent to the section is perpendicular to the minimal straight line [33].

[Proof]. If the minimal straight line is a part of BΓ, then it is evident that the straight line drawn from B tangent to the section is perpendicular to the minimal straight line because EZ is the axis.

But if it is not a part of BΓ, let the minimal straight line be AE, and let the tangent be AZ. Then I say that the angle ZAE is right.

Let the perpendicular AH [to the axis] be drawn, and let the center be Δ. Then since AE is the minimal straight line, and AH is a perpendicular, as ΔH is to

HE, so the transverse diameter is to the *latus rectum*, as is proved in Theorems 14 and 15 of this Book.

But as ΔH is to HE, so pl.ΔHZ is to pl.ZHE. Therefore as pl.ΔHZ is to pl.ZHE, so the transverse diameter is to the *latus rectum*. But as the transverse diameter is to the *latus rectum*, so pl.ΔHZ is to sq.AH, as is providing Theorem 37 of Book 1. Therefore pl.ZHE is equal to sq.AH.

And AH is a perpendicular [to the axis]. Therefore the angle ZAE is right.

[Proposition] 29

That may be proved in another way, that is as follows : let the conic section be AΓ and its axis be BΔ. Then I say that the straight line drawn from the end of the minimal straight line tangent to the section is perpendicular to the minimal straight line [34] .

Let the minimal straight line be AB and the tangent AΔ. Then I say that the angle ΔAB is right.

[Proof]. For if that is not so, we draw the perpendicular BE to AΔ.
Then AB is greater than BE.

Therefore how much the greater is it than BZ. [But] that is impossible for AB is minimal straight line, therefore the angle ΔAB is so right.

[Proposition] 30

*If a straight line is drawn from the end of one of the maximal straight lines drawn in the ellipse whichever one it may be, so as to be tangent to the section, then it is a perpendicular to the maximal straight line [35].*

Let the ellipse be ABΓ whose minor axis AΓ, and let there be drawn from a point on the axis to the section one of the maximal lines OB. Let from B a straight line ΔB tangent to the section be drawn.

I say that the angle ΔBO is right.

[Proof]. For let from the center of the section a perpendicular EK to the [minor axis], be drawn. Then EK is the half of the major axis, and AΓ is the minor axis. And since EK has cut one of the maximal straight lines, then the part of that straight line which fails between the section and the major axis is one of the minimal straight lines, as is proved in Theorem 23 of this Book.

Therefore BΛ is one of the minimal straight lines, and BΔ is tangent, therefore BΔ is a perpendicular to it, as is proved in three preceding Theorems.

[Proposition] 31

*If there is drawn from the end of a minimal straight line that is drawn in one of the [conic] sections a straight line at right angles [to the minimal straight line], and that end is one point on the section, then the drawn straight line is tangent to the section* [36].

Let there be the conic section AB with a minimal straight line ΓB.

I say that the straight line drawn from B such that it is a perpendicular to ΓB is tangent to the section.

[Proof]. For let, if it is possible for it not be tangent, let it cut it, as EBΘ. Let from a point Z outside the section, between it and BΘ, the straight line ZB be drawn, and from Γ a perpendicular ΓHZ to BZ, be drawn. Then the angle ΓBZ is acute and the angle ΓZB is right.

Therefore ΓZ is smaller than ΓB, and ΓH is much smaller than ΓB. But ΓB was minimal, that is impossible.

Therefore the straight line drawn from B perpendicular to BΓ is tangent to the section.

[Proposition] 32

*If there is a tangent to one of [conic] sections and a perpendicular is drawn to that straight line from the point of contact to meet the axis, then that drawn straight line is the minimal straight line that reaches that point [from the axis]* [37].

Let there be the conic section ABΓ, and let ΔE be a tangent to it. Let the point of contact a perpendicular BZ to ΔE, be drawn and continued until it reaches the axis AZH.

I say that BZ is one of the minimal straight lines.

[Proof]. For let, if that is not so, the minimal straight line which reaches B [from the axis] be BH. Then the angle ΔBH is right, as is proved in Theorems 27, 28, and 29 of this Book. But the angle ΔBZ also was right, that is impossible. Therefore BZ is one of the minimal straight lines.

[Proposition] 33

*If a perpendicular is drawn to one of the maximum straight lines, from that and of it, which is on the section, then it is tangent to the section* [38].

Let there be the conic section AB, and in it one of the maximal straight lines BΓ.

I say that the straight line drawn from B perpendicular to BΓ is a tangent to the section.

[Proof]. For let if that is not so, if cut it as ΕΒΔ. Let from Γ a straight line ΓΔΑ cutting ΒΔ, be drawn. Then ΔΓ is greater than ΓΒ, and ΑΓ is greater than ΔΓ.

Therefore is much greater than ΓΒ. But ΓΒ was one of the maximal straight lines, and that is impossible. Therefore the straight line drawn from Β perpendicular ΓΒ is tangent to the section.

[Proposition] 34

*If a point is taken outside a conic section on a continued maximal or minimal straight line, then the smallest length intercepted between that point and the section [on the straight lines drawn from that point on either side of the section but not continued to cut the section at more than one point] is the straight line which is the continued maximal or minimal straight line, and of the other straight lines those drawn closer to it are smaller than those drawn farther* [39].

Let there be a conic section ΑΒ with a maximal or minimal straight line ΒΓ in it. Let it be continued in a straight line, and let on it be taken, after it is continued [outside the section] an arbitrary point Δ. Let from Δ to the section ΔΑ, ΔΗ, and ΔΕ be drawn, let each of them cut the section in one point only.

I say that ΔΒ is the smallest of the straight lines drawn from Δ to the section, and that of the other straight lines those of them drawn closer to it are smaller than those drawn farther.

[Proof]. For let ΒΖ be drawn tangent to the section then the angle ΖΗΔ is right because of what was proved in Theorems 27, 28, 29, and 30 of this Book. Therefore ΔΖ is greater than ΔΒ and ΔΕ is much greater than ΔΒ.

Let ΗΒ and ΗΕ be joined. Then the angle ΔΕΗ is obtuse, and ΔΗ is greater than ΔΕ.

Similarly also it will be proved that ΔΑ is greater than ΔΗ.

And similarly it is possible for us to prove the same concerning the straight lines drawn to the other side of ΔΒ.

[Proposition] 35

*In every conic section, when minimal straight lines are drawn, the angle between a straight line drawn farther from the vertex of the section and the axis is greater than the angle between the straight line drawn closer [to the vertex] and the axis* [40].

Let the section be, first the parabola ΑΒΓ whose axis ΓΔ.

Let ΔA and BE be two of the minimal straight lines.

I say that the angle AΔΓ is greater than the angle BEΓ.

[Proof]. For let two perpendiculars AZ and BH [to the axis] be drawn. Then BE is one of the minimal straight lines and [hence] EH is equal to the half of the *latus rectum*, as is proved in Theorem 13 of this Book.

Similarly also it will proved that ZΔ is equal to the half of the *latus rectum*. Therefore EH is equal to ΔZ.

But the perpendicular AZ is greater than the perpendicular BH. Therefore the angle AΔZ is greater than the angle BEH.

[Proposition] 36

[Next] let the section [AB] be the hyperbola or the ellipse whose axis ΛE and center Δ. Let AE and BZ be two of the minimal straight lines.

Then I say that the angle AEΛ is greater than the angle BZΛ[41] .

[Proof]. For let two perpendiculars BΘ and AH [to the axis] be drawn , and the straight line ΔKB be joined.

Then as ΔH is to HE, so the transverse diameter is to the *latus rectum*, as is proved in Theorems 14 and 15 of this Book.

Similarly as ΔΘ is to ZΘ [so the transverse diameter is to the *latus rectum*]. Therefore as ΔH is to HE, so ΔΘ is to ΘZ. And *permutando* as ΔH is to ΔΘ, so EH is to ZΘ.

But as ΔH is to ΔΘ, so KH is to BΘ, therefore as HE is to ZΘ, so KH is to BΘ. And the angles AHE and BΘZ are right. Therefore the triangles KEH and BZΘ are similar. Therefore the angle AEH is greater than the angle BZΘ.

[Proposition] 37

*If there be a hyperbola, and one of the minimal straight lines is drawn in it so as to make an angle with the axis, then that angle is smaller than the angle between each of the asymptote to the section and the straight line drawn from the vertex of the section perpendicular to the axis* [42].

Let the hyperbola be AB whose axis ΓΔ. Let its asymptotes be ZΓ and ΓH, and let the minimal straight line be AΔ let through B pass the perpendicular ZBH to the axis.

I say that the angle AΔΓ is smaller than the angle ΓZH.

[Proof]. For let the half of the *latus rectum* be made BΘ, so that Θ falls between B and H or beyond them. Let ΓA be joined.

Then as ΓB is to BΘ, so the transverse diameter is to the *latus rectum*, and as ΓE is to EΔ, also so the transverse diameter is to the *latus rectum*, as was proved Theorem 14 of this Book. Therefore as ΓB is to BΘ, so ΓE is to EΔ.

And as KB is to BΓ, so AE is to ΓE. Therefore ex as KB is to BΘ, so AE is to EΔ. But the ratio KB to BΘ is smaller than the ratio ZB to BΘ, and as ZB is to BΘ, so ΓB is to BZ, as is proved in Theorem 3 of Book II . Therefore the ratio AE to EΔ is smaller than the ratio ΓB to BZ. And these
sides and close right angles. Therefore the angle ΓZB is greater than the angle AΔΓ.

[Proposition] 38

*If there are drawn in one of conic sections two minimal straight lines ending at the axis, then, when they are continued in a straight line, they will meet the other part of the section* [44].

Let there be the conic section AB whose axis ΓΔ, and let there be in the section two  of the minimal straight lines ΔA and EB.

I say that ΔA and EB, when continued towards the other side [of the axis] will meet each other [43].

[Proof]. The angle AΔΓ is greater than the angle BEΓ, as is proved in Theorems 35 and 36 of this Book. Therefore the sum of  the angles AΔE and BEΔ is greater than two right angles.

For that reason two angles adjoining them are less than two right angles.

Therefore two minimal straight lines AΔ and BE, when continued towards the other side of the section, will meet each other.

[Proposition] 39

*Maximal straight lines drawn in an ellipse to the minor axis cut each other in that part [of the ellipse]* [44].

Let there be the ellipse AΓB whose minor axis AΔ.

I say that the maximal straight lines drawn in the ellipse AΓB cut one another in the half of the section ABΔ.

[Proof]. For let if it is possible, they not cut one another, as the maximal straight lines BE and ΓZ. Let the perpendiculars BH and ΓΘ be drawn, and let the center be K. Then as KΘ is to ΘZ, so the transverse diameter is to the *latus rectum*, as is proved in Theorem 22 of this Book.

Similarly as KH is to HE also [so the transverse diameter is to the *latus rectum*. Therefore as KH is to HE, so KΘ is to ΘZ]. And *dividendo* as KH is to KE, so KΘ is to KZ, and *permutando* as KH is to KΘ, so KE is to KZ.

But KZ is smaller than KE. Therefore KΘ is smaller than KH also, but that is impossible. Therefore BE and ΓZ meet.

[Proposition] 40

*The point of meeting of the minimal straight lines drawn in an ellipse is within the angle between the half of the axis from which the minimal straight lines are drawn and the other axis* [45].

Let there be the ellipse AΔΓ whose major axis AΓ and minor axis BΔ. Let EΘ and ZH two of the minimal straight lines.

I say that EΘ and ZH will meet within the angle ΓBO.

[Proof]. For let these two straight lines be continued from H and Θ until they meet ΔB at K and Λ. Then EΘ and ZH are minimal straight lines, therefore EΛ is one of the maximal straight line, as is proved from the reverse of Theorem 23 of this Book.

Similarly also ZH when continued meets BO as ZK, and [hence] ZK is one of the maximal straight lines.

But EΘ and ZH, when continued, meet on the other side of the [major] axis, as is proved in Theorem 38 of this Book. And when EΛ and ZK are maximal straight lines, then they cut each other on the side [of the minor axis] on which they are, as is proved in Theorem 39 of this Book. Therefore, the place of meeting is within the angle between ΓB and BO.

[Proposition] 41

*The minimal straight lines drawn in a parabola or an ellipse to its axis, when continued, fall on the other side of the section* [46].

Now as to the fact that that is the case in the ellipse, that is evident.

Therefore let there be the parabola [ABΓ] whose axis BΔ, and minimal straight line AΔ.

I say that AΔ, when continued, meets the part BΓ of the section.

[Proof]. The section ABΓ is a parabola, and AΔ has been drawn from its diameter, therefore AΔ, when continued falls on the section BΓ, as is proved in Theorem 27 of Book 1.

[Proposition] 42

*If there is a hyperbola whose transverse diameter is not greater than the* latus rectum*, then none of the minimal straight lines drawn in it meet the other side of the section, but if the transverse diameter is greater than the* latus rectum*, then some of the minimal straight lines in the section will, when continued meet the section on the other side [of the axis] , and some of them will not meet it* [47].

Let there be the hyperbola ΑΒΓ whose axis ΔΕ and center Δ. Let the minimal straight line be ΑΕ.

[First] let the transverse diameter be not greater than the *latus rectum*. Then I say that ΑΕ will not meet the section when continued.

[Proof]. For let the asymptotes be ΔΖ and ΔΗ, and ΖΒ be a perpendicular to ΔΕ, and let the half of the *latus rectum* be ΒΘ. Then, since the transverse diameter is not greater than the *latus rectum* ΔΒ is not greater than ΒΘ.

And as ΔΒ is to ΒΘ, so sq.ΔΒ is to sq.ΒΖ, as is proved in Theorem 3 of Book II. Therefore ΔΒ is not greater than sq.ΒΖ, and ΔΒ is not greater than ΒΖ. Therefore the angle ΒΖΔ is not greater than the angle ΖΔΒ. But the angle ΒΖΔ is greater than the angle ΑΕΒ, as is proved in Theorem 37 of this Book.

Therefore the angle ΖΑΒ is greater than the angle ΑΕΒ. And the angle ΖΔΒ is equal to the angle ΒΔΗ. Therefore the angle ΒΔΗ is greater than the angle ΑΕΒ. And the angle adjacent to the angle ΑΕΒ is made common [to both sides], this angle together with the angle ΑΕΒ is equal to two right angles, and [hence] the angle ΕΔΗ together with the angle adjacent to the angle ΑΕΒ is greater than two right angles. Therefore ΑΕ and ΔΗ, when continued on the side ΕΗ, will not meet each other. Therefore ΑΕ will not meet side ΒΓ of the section for if it met it, ΑΕ would meet ΔΗ, as is proved in Theorem 8 of Book II .

[Proposition] 43

*Next let the transverse diameter be longer than the* latus rectum*, then I say that some of the minimal straight lines which occur in the section ΑΒΓ , when continued will meet the section on the other side [of the axis] and some of them will not meet it* [48].

[Poof]. For let the asymptotes ΖΔ and ΔΗ be drawn, and the transverse diameter be longer than the *latus rectum*. Then ΔΒ is greater than ΒΘ [equal to the half of the *latus rectum*, and [hence] as the ratio ΖΒ to ΒΘ is greater than ΖΒ to ΒΔ.

Therefore let as ΚΒ be to ΒΘ, so ΖΒ be to ΒΔ, and let ΔΚ be joined and continued, then it will meet the section, as is proved in Theorem 2 of Book II . Let it meet it at Α . Let from Α the perpendicular ΑΛ to ΔΕ be drawn, let as ΔΛ be to ΛΕ, so ΔΒ be to ΒΘ, and ΛΕ be joined . Then as ΔΒ is to ΒΘ , so ΔΛ is to ΛΕ, that is so the transverse diameter is to the *latus rectum*. And the perpendicular ΑΛ has been from Λ, and ΑΕ is joined. Therefore ΑΕ is one of the minimal straight lines, as is proved in Theorem 9 of this Book.

Furthermore as ΒΚ is to ΔΒ, so ΑΛ is to ΛΔ, and as ΔΒ is to ΒΘ, so ΔΛ is to ΛΕ. Therefore ex as ΑΛ is to ΛΕ, so ΒΚ is to ΒΘ.

But as ΒΚ is to ΒΘ, so ΖΒ is to ΒΔ. Therefore as ΑΛ is to ΛΕ, so ΒΖ is to ΒΔ. And the angles ΖΒΔ and ΑΛΕ are equal since they are right, therefore the triangles ΖΒΔ and ΑΛΕ are similar, therefore the angle ΖΔΒ is equal to the angle ΑΕΛ, and [the angle ΖΔΒ] is equal to the angle ΒΔΗ. Therefore the angle ΑΕΒ is equal to the angle ΒΔΗ. Therefore ΔΗ and ΛΕ are parallel, and, when continued, will not cut each other.

Therefore since they do not cut each other, ΑΕ will not meet the section anywhere but at Α, even if it is continued in a straight line for if it did meet it, it would meet ΔΗ and ΔΖ, as is proved in Theorem 8 of Book II .

But ΑΕ has been shown to be parallel to ΔΗ, which is impossible. Therefore ΑΕ does not meet the section ΑΒΓ at a point other than Α.

And as for the minimal straight lines drawn between Ε and Η, the angles which they form with ΒΕ are smaller than the angle ΑΕΒ, as is proved in Theorem 36 of this Book.

But the angle ΑΕΒ is equal to the angle ΒΔΗ. Therefore the angles which the minimal straight lines drawn between Β and Ε form [with the axis] are smaller than the angle ΒΔΗ, therefore when they are continued, they will not meet ΔΗ or the section ΒΓ [for the reason mentioned above].

As for the other minimal straight lines, since they form with the axis the angles greater than the angle ΑΕΒ, they will meet ΔΗ, and hence will meet the section ΒΓ.

[Proposition] 44

*If two of the minimal straight lines are drawn from the axis of one of the conic sections, and continued until they meet, and another straight line is drawn from their point of meeting cutting the axis and ending at the section, then the part of it falling between the section and the axis is not one of the minimal straight lines, and if the drawn straight line is not between two minimal straight* *lines, and a minimal straight line is drawn from the point at which it reaches the section, then [that minimal straight line] cuts off from the axis adjacent to the vertex of the section a segment greater than that cut off by the drawn straight line, but if the drawn straight line is between two minimal straight lines, then the minimal straight line drawn from the point at it reaches [the section] cuts off from the axis adjacent to the vertex of the section a segment smaller than the segment cut off [by the drawn straight line], and in the case of the ellipse the above said holds when two minimal straight lines and the drawn straight line all cut one and the same half of two halves of the major axis* [49].

First let the section be the parabola ΑΒΓ whose axis ΔΗ. Let two minimal straight lines that are in it be ΒΖ and ΓΕ, and let them meet at Ο.
Let there be drawn from Ο, first, a straight line ΟΛΚ outside ΟΓ and ΟΒ.

I say that ΛΚ is not one of minimal straight lines, and that the minimal straight line which is drawn Κ cuts of off from the axis next to the vertex of the section, which is Δ, a straight line longer than ΔΛ.

[Proof]. For let the perpendiculars ΟΗ, ΒΠ, ΓΝ, and ΚΜ be drawn. Let the half of the *latus rectum* be ΥΗ. Then ΒΖ is one of the minimal straight lines, and be ΒΠ is a perpendicular, therefore ΗΖ is equal to the half of the *latus rectum*, as is proved in Theorem 13 of this Book. Therefore ΠΖ is equal to ΘΗ, and ΠΘ is equal to ΖΗ, and as ΗΘ is to ΘΠ, so ΠΖ is to ΖΗ.

But as ΠΖ is to ΖΗ, so ΠΒ is to ΟΗ. Therefore pl.ΟΗΘ is equal to pl.ΒΠΘ.

And similarly also we will prove that pl.ΓΝΘ is equal to pl.ΟΗΘ. Therefore pl.ΒΠΘ is equal to pl.ΓΝΘ. And therefore as ΒΠ is to ΓΝ, so ΝΘ is to ΘΠ. So we join ΒΓ and continue it until it meets ΔΗ at Χ, and draw the perpendicular ΚΜ and continue it to [meet ΒΧ at] ο.

Then as ΒΠ is to ΓΝ, so ΠΧ is to ΧΝ, therefore as ΠΧ is to ΧΝ, so ΝΘ is to ΘΠ, and ΝΧ is to ΠΘ. Therefore ΧΜ is smaller than ΠΘ, and the ratio ΠΜ to ΜΧ is greater than the ratio ΠΜ to ΠΘ. And *componendo* the ratio ΠΧ to ΧΜ [equal to the ratio ΠΒ to Μο] is greater than the ratio ΜΘ to ΘΠ. Therefore pl.ΒΠΘ is greater than pl.οΜΘ.

Therefore pl.ΒΠΘ is much greater than pl.ΚΜΘ.

But we have [already] proved that pl.ΒΠΘ is equal to pl.ΟΗΘ. Therefore pl.ΟΗΘ is greater than pl.ΚΜΘ, therefore the ratio ΟΗ to ΚΜ [equal to the ratio ΗΛ to ΛΜ] is greater than the ratio ΜΘ to ΘΗ, and ΗΘ is greater than ΜΛ.

But ΗΘ is equal to the half of the *latus rectum*. Therefore ΜΛ is smaller than the half of the *latus rectum*, and [hence] the minimal straight line drawn from Κ cuts off from the axis adjacent to Μ a straight line greater than ΛΜ.

Therefore it cuts off from the axis adjacent to Δ a straight line greater than ΛΔ. So ΚΛ is not one of the minimal straight lines, as is proved in Theorem 24 of this Book.

Furthermore we draw on the other side if ΒΟ and ΓΟ the straight line ΟΑ [cutting ΗΔ at Φ], then I say that ΑΦ is not one of the minimal straight lines,

and that the minimal straight line drawn from Α cuts off from the axis a segment greater than ΔΦ.

[Proof]. For let ΑΡ be a perpendicular to ΔΗ. Now it has been proved that ΠΘ is equal to ΧΝ. Therefore ΧΡ is greater than ΠΘ, and the ratio ΡΠ to ΧΡ is smaller than the ratio ΡΠ to ΠΘ. And *divedendo* the ratio ΡΠ to ΠΧ is smaller than the ratio ΡΠ to ΡΘ. And *componendo* the ratio ΡΧ to ΧΠ is smaller than the ratio ΠΘ to ΘΡ, and the ratio ΡΨ to ΠΒ is smaller than the ratio ΠΘ to ΘΡ. Therefore pl.ΨΡΘ is smaller than pl.ΒΠΘ. Therefore pl.ΑΡΘ is much smaller than pl.ΒΠΘ.

But pl.ΒΠΘ is equal to pl.ΟΗΘ. Therefore pl.ΑΡΘ is smaller than pl.ΟΗΘ, and the ratio ΑΡ to ΟΗ [equal to the ratio ΡΦ to ΦΗ] is smaller than the ratio ΗΘ to ΘΡ. Therefore ΘΗ is greater than ΡΦ.

But ΘΗ is equal to the half of the *latus rectum*. Therefore ΡΦ is smaller than the half of the *latus rectum*, and the minimal straight line drawn from Α cuts off a segment greater than ΡΦ. Therefore the segment cut off [by the minimal straight line from Α] adjacent to Δ, which is the vertex of the section , is greater than ΔΦ, which is cut off by ΑΦ. Therefore ΑΦ is not one of the minimal straight lines, as is proved in Theorem 24 of this Book.

Furthermore let the drawn straight line ΟΣ fall between ΟΒ and ΟΓ. Then I say that ΣΥ is not one of the minimal straight lines, and that the minimal straight line drawn from Σ cuts off from the axis adjacent to Δ a straight line smaller than ΔΥ.

[Proof]. For let the perpendicular ΣΤ be drawn. Then it has been proved that ΠΘ is equal to ΧΝ. Therefore ΤΧ is greater than ΠΘ, and the ratio ΤΠ to ΤΧ is smaller than the ratio ΤΠ to ΠΘ. And *componendo* the ratio ΠΧ to ΧΤ is smaller than the ratio ΤΘ to ΘΠ.

But as ΠΧ is to ΧΤ, so ΒΠ is to ΤΞ. Therefore the ratio ΒΠ to ΤΞ is smaller than ratio ΤΘ to ΘΠ, and the ratio ΒΠ to ΠΘ is smaller than the ratio ΞΤ to ΤΘ. Therefore the ratio ΒΠ to ΠΘ is smaller than the ratio ΣΤ to ΤΘ.

But pl.ΟΗΘ is equal to pl.ΒΠΘ. Therefore pl.ΟΗΘ is smaller than pl.ΣΤΘ. Therefore the ratio ΟΗ to ΣΤ is smaller than the ratio ΤΘ to ΘΗ.

But as OH is to ΣT, so HY is to YT, and the ratio HY to YT is smaller than the ratio TΘ to ΘH. Therefore HΘ is smaller than YT. And HΘ is equal to the half of the *latus rectum*.

Therefore the minimal straight line drawn from Σ cuts off next to T a straight line smaller than TY, and therefore it cuts next to the vertex of the section [a segment] smaller than ΔY.

Therefore ΣY is not the minimal straight line, and the minimal straight line cuts off next to the vertex of the section a segment smaller than ΔY.

[Proposition] 45

Furthermore let the section be the hyperbola or the ellipse ΑΒΓΔ whose axis ΜΝΛ and center Ν, and let there be drawn in the section two minimal straight lines ΒΕ and ΓΖ, and let them meet at Θ, and let ΘΛΚ be drawn from Θ to the section. Then I say that ΚΛ, which is between the axis and the section, is not one of the minimal straight lines, but that the minimal straight line drawn from Κ cuts off the axis next to Δ a segment longer than ΔΑ [50] .

[Proof]. For let ΘΜ be the perpendicular from Θ to the axis, and there be a straight line through Ν parallel to ΜΘ, namely ΝΞ, and pass and through Θ a straight line parallel to ΜΝ, namely ΘΞ , and let ΝΞ be continued until it meets ΚΘ and ΒΘ, let it meets them at b and q [respectively]. Let each of the ratios ΞΠ to ΠΝ and ΝΟ to ΟΜ be equal to the ratio of the transverse diameter to the *latus rectum*.

Let ΟΣ, ΒΩ, ΓΗ, and ΚΦ are drawn as perpendiculars to the axis, and let ΒΓ be joined and continued in a straight line, and let through Π pass a straight line ΠΡ parallel to ΔΝ, and let it be continued to [meet the continued ΒΓ at] Υ.

Then since ΒΕ is one of minimal straight lines, and ΒΩ is a perpendicular, as ΝΩ is to ΩΕ, so the transverse diameter is to the *latus rectum*, as is proved in Theorems 9 and 10 of this Book. Therefore as ΝΟ is to ΟΜ, so ΝΩ is to ΩΕ. And *componendo* for the hyperbola and convertendo for the ellipse as ΟΝ is to ΝΜ, so ΩΝ is to ΝΕ.

And when subtract two lesser from two greater, we set as ΜΕ is to ΟΩ, so ΜΝ is to ΝΟ. But ΩΟ is to Τσ, therefore as ΕΜ is to Τσ so ΜΝ is to ΝΟ.

And since the ratio ΞΠ to ΠΝ also is equal to the ratio of the transverse diameter, as ΞΠ is to ΠΝ, so ΝΩ is to ΩΕ.

And *componendo* in the case of the hyperbola and *dividendo* in the case of the ellipse as ΞΝ is to ΝΠ, so ΝΕ is to ΕΩ.

But as NE is to EΩ, so NΘ is to BΩ because ΣE of the similarity of the triangles.

And adding in the case of the hyperbola and subtracting the lesser from the greater in the case of the ellipse as EΘ is to Bσ, so NE is to EΩ, that is the ratio ΞN to NΠ. Therefore as ΞΘ is to Bσ so ΞN is to NΠ.

Furthermore the ratio of the quadrangle NΘ to the quadrangle NT is compounded of [the ratios] ΞN to NΠ and MN to NO.

But we have [already] proved that as ΞN is to NΠ, so Ξθ is to Bσ, and we have [already] proved that as MN is to NO, so EM is to σT. Therefore the ratio of the quadrangle NΘ to the quadrangle NT is compounded of [the ratios] Ξθ to Bσ and EM to σT. But the quadrangle NΘ is equal to pl.ΞΘ,EM , because as Ξθ is to ΞΘ so ΘM is to ME. Therefore the quadrangle NT is equal to pl.BσT.

Similarly also it will be proved that the quadrangle NT is equal to pl.ΓδT. Therefore pl.BσT is equal to pl.ΓδT, and as Bσ is to Γδ, so δT is to Tσ. But as Bs is to Γδ, so σY is to Yδ, and as σY is to Yδ, so δT is to Tσ. And *dividendo* as σδ is to δY, so σδ is to σT. Therefore δY is equal to σT, and σT is greater than Yγ. Therefore the ratio γσ to γY is greater than the ratio γσ to σT, and *componendo* the ratio σY to Yγ is greater than the ratio γT to Tσ.

But as σY is to Yγ, so Bσ is to εγ. Therefore the ratio Bσ to εγ is greater than the ratio γT to Tσ, and pl.BσT is greater than pl.εγT. Therefore pl.BσT is much greater than pl.KγT.

But pl.BσT was equal to the quadrangle NT. Therefore the quadrangle NT is greater than pl.KγT. And the quadrangle NT is equal to the quadrangle PΣ because as NO is to OM, so ΘP is to PM. Therefore the quadrangle PΣ is greater than pl.KγT. But the quadrangle PΣ is equal to pl.ΘPT, therefore pl.ΘPT is greater than pl.KγT. Therefore the ratio ΘP to Kγ is greater than the ratio γT to PT. But as ΘP is to Kγ, so Pζ is to ζγ. Therefore the ratio Pζ to ζγ is greater than the ratio γT to PT. And *componendo* the ratio Pγ to γζis greater than the ratio γP to PT. Therefore PT is greater than γζ, and the ratio ΞΘ to PT is smaller than the ratio ΞΘ to γζ.

But as ΞΘ is to γζ, so Ξβ is to Kγ because of the similarity of the triangles. Therefore the ratio ΞΘ to PT is smaller than the ratio Ξβ to Kγ and ΞΘ is equal to MN, and PT is equal to MO. Therefore the ratio MN to MO is smaller than the ratio Ξβ to Kγ. But as MN is to MO, so ΞN is to NΠ because each of these two ratios NO to OM and ΞΠ to ΠN is equal to the ratio of the transverse diameter to the *latus rectum*. Therefore the ratio ΞN to NΠ is smaller than the ratio Ξβ to Kγ. And subtracting two lesser from two greater in the case of the hyperbola and adding in the case of the ellipse the ratio Nβ to $K\Phi$ is greater than the ratio $\Xi N$ to $N\Pi$ because $N\Pi$ is equal to $\Phi\gamma$ .

But as $N\beta$ is to $K\Phi$, so $N\Lambda$ is to $A\Phi$ because of the similarity of the triangles. Therefore the ratio $N\Lambda$ to $\Lambda\Phi$ is greater than the ratio $\Xi N$ to $N\Pi$.

And *dividendo* in the case of the hyperbola and *componendo* in the case of the ellipse the ratio $N\Phi$ to $\Phi\Lambda$ is greater than the ratio $\Xi\Pi$ to $\Pi N$.

But the ratio $\Xi\Pi$ to $\Pi N$ is equal to the ratio on the transverse diameter to the *latus rectum*. Therefore the ratio $N\Phi$ to $\Phi\Lambda$ is greater than the ratio of the transverse diameter to the *latus rectum*.

Therefore if we make the ratio of $N\Phi$ to another straight line equal to the ratio of the transverse diameter to the *latus rectum*, that other straight line will be longer than $\Phi\Lambda$.

Therefore the minimal straight line drawn from $K$ cuts off from the axis adjoining $\Delta$ a straight line longer than $\Delta\Lambda$, because of what is proved in Theorems 9 and 10 of this Book, and [hence] $K\Lambda$ is not one of minimal straight lines, because of what is proved in Theorem 25 of this Book.

Furthermore let $\Theta\eta A$ be drawn. Then I say that $A\eta$ is not one of minimal straight lines, and that the minimal straight line drawn from $A$ cuts off from the axis a segment longer than $\Delta\eta$.

[Proof]. For let to the axis the perpendicular $A$Ϙ be drawn and continued to [meet continued $\Gamma B$ at] $\Gamma$. Then since $Y\delta$ is equal to $\sigma T$, $Y\delta$ is greater than $TI$ , and the ratio $\delta I$ to $IT$ is greater the ratio $\delta I$ to $Y\delta$. And *componendo* the ratio $\delta T$ to $TI$ is greater than the ratio $IY$ to $Y\delta$. But as $IY$ is to $Y\delta$, so $\Gamma I$ is to $\Gamma\delta$. Therefore the ratio $\delta T$ to $TI$ is greater than the ratio $\Gamma I$ to $\Gamma\delta$. Therefore the ratio $\delta T$ to $TI$ is much greater than the ratio $AI$ to $\Gamma\delta$. Therefore pl.$\Gamma\delta T$ is greater than pl.$AIT$.

But we have shown that pl.$\Gamma\delta T$ is equal to the quadrangle $\Pi O$, therefore the quadrangle $\Pi O$ is greater than pl.$AIT$.

But the quadrangle $\Pi O$ is equal to the quadrangle $P\Sigma$ because the ratio $NO$ to $OM$ equal to the ratio $\Pi T$ to $TP$ is equal also to the ratio $\Xi\Pi$ to $\Pi N$ which is equal to the ratio $\Sigma T$ to $TO$. Therefore the quadrangle $P\Sigma$ is greater than pl.$AIT$ .

But the quadrangle $P\Sigma$ is pl.$\Theta PT$. Therefore pl.$\Theta PT$ is greater than pl.$AIT$, therefore the ratio $\Theta P$ to $AI$ is greater than the ratio $TI$ to $PT$. But as $\Theta P$ is to $AI$, so $P\kappa$ is to $\kappa I$. Therefore the ratio $P\kappa$ to $\kappa I$ is greater than the ratio $TI$ to $PT$.

And *componendo* the ratio $IP$ to $P\kappa$ is smaller than the ratio $IP$ to $IT$. Therefore $P\kappa$ is greater than $TI$ .

Let $T\kappa$ be common, then $PT$ is greater than $I\kappa$. Therefore the ratio $\Xi\Theta$ to $PT$ is smaller than the ratio $\Xi\Theta$ to $I\kappa$.

But as ΞΘ is to Iκ, so Ξα is to AI. Therefore the ratio Ξα to AI is greater than ΞΘ to PT.

But as for ΞΘ, that is equal to NM, and as for PT, that is equal to MO. Therefore the ratio Ξα to AI is greater than the ratio NM is to MO. But as NM is to MO, so ΞN is to NΠ, therefore the ratio Ξα to AI is greater than the ratio ΞN to NΠ.

So when we subtract two smaller from two greater in the case of the hyperbola, and add [them] in the case on the ellipse, the ratio αN to Ao is greater than the ratio ΞN to NΠ. But as αN is to AϘ, so Nη is to ηϘ. Therefore the ratio Nη to ηϘ is greater than the ratio ΞN to NΠ.

And *dividendo* in the case of the hyperbola and *componendo* in the case of the ellipse, the ratio No two oη is greater than the ratio ΞΠ to ΠN.

But as ΞΠ is to ΠN, so transverse diameter is to the *latus rectum*. And we make the ratio of No to another straight line equal to the ratio of the transverse diameter to the *latus rectum*, that straight line is greater than oη. Therefore the minimal straight line drawn from A cuts off from the axis a segment longer than Δη, because of what is proved in Theorems 9 and 10 of this Book. And Aη is not one of minimal straight line because of what is proved in Theorem 25 of this Book.

Furthermore let the straight line □ΨΘ between two minimal straight lines BE and ΓZ, then I say that □Ψ is not one of minimal straight lines, and that the minimal straight line drawn from □ cuts off from the axis a segment smaller than ΔΨ.

[Proof]. For let □μ as a perpendicular to the axis be drawn. Then since we have proved that Yδ is equal to σT, Yδ is smaller than ξT, and the ratio ξδ to δY is greater than the ratio δξ to ξT. And *componendo* the ratio ξY to Yδ is greater than the ratio δT to Tξ. But as ξY is to Yδ, so νξ is to Γδ. Therefore the ratio νξ to Γδ is greater than the ratio δT to Tξ, and pl.νξT is greater than pl.ΓδT.

But □ξ greater than νξ. Therefore pl.□ξT is much greater than pl.ΓδT.

And we have proved that pl.ΓδT is equal to the quadrangle NT, and that the quadrangle NT is equal to the quadrangle PΣ. Therefore pl.□ξT is greater than the quadrangle PΣ, therefore pl.□ξT is greater than the quadrangle PΣ. But the quadrangle PΣ is equal to pl.ΘPT, therefore pl.□ξT is greater than pl.ΘPT, and the ratio □ξ to ΘP is greaten than the ratio PT to Tξ.

But as □ξ is to ΘP, so ξι is to ιP, therefore the ratio ξι to ιP is greater than the ratio PT to Tξ. And *componendo* the ratio ξP to PT is greater than the ratio ξP to ξι ,therefore PT is smaller than ξι and the ratio ΞΘ to PT is greater than the ratio ΞΘ to ξι.

But as ΞΘ is to ξμ , so Ξο is to □ξ because of the similarity of the triangles. Therefore the ratio ΞΘ to PT is greater than the ratio Ξο to □ξ .

But ΞΘ is equal to NM and PT is equal to MO. Therefore the ratio NM to MO is greater than the ratio Ξο to □ξ . But as NM is to MO, so ΞN is to ΠN.

Therefore the ratio ΞN to ΠN is greater than the ratio Ξο to □ξ .

And when we subtract two lesser from two greater in the case of the hyperbola, and add [them] in the case of the ellipse, the ratio ΞN to ΠN is greater than the ratio οN to □μ .

But as οN is to □μ , so NΨ is to Ψμ because of the similarity of the triangles. Therefore the ratio ΞN to NΠ is greater than the ratio NΨ to Ψμ .

And *dividendo* in the case of the hyperbola and *componendo* in the case of the ellipse, the ratio ΞΠ to ΠN is greater than the ratio Nμ to μΨ .

But as ΞΠ is to ΠN, so the transverse diameter is to the *latus rectum*. Therefore the ratio of the transverse diameter to the *latus rectum* is greater than the ratio Nμ to μΨ .

And if we make the ratio of Nm to another straight line equal to the ratio of the transverse diameter to the *latus rectum*, that straight line is smaller than μΨ.

Therefore the minimal straight line drawn from □ cuts off from the axis a segment shorten than ΨΔ, as is proved in Theorems 9 and 10 of this Book. Therefore □Ψ is not one of minimal straight lines because of what is proved in Theorem 25 of this Book.

[Proposition] 46

*If there are drawn in one of quadrants of an ellipse two minimal straight lines to major axis, one of which passes through the center, and they are continued until they meet, then no [other] straight line can be drawn from the point where they meet to that quadrant of the section such that part of it intercepted between the axis and the section is one of minimal straight lines, and if straight lines are drawn from the point of meeting of two straight lines to the section, then the minimal straight lines drawn from the ends of those [straight lines] to the axis cut off from the axis adjacent to the vertex of the section a segment greater than the segment cut off by the straight lines themselves* [51]*.*

Let there be the ellipse ΑΒΓ whose major axis ΔΕ and center Z. Let from the center the perpendicular ZA to the axis be drawn and continued. Let BH be one of minimal straight lines, and let it meet ZA at K. Let [an arbitrary] straight line KΘΓ be drawn.

I say that ΓΘ is not one of minimal straight lines, and that the minimal straight line drawn from Γ to ΔE cuts off a segment greater than ΔΘ.

[Proof]. As for [the statement] that ΓΘ is not one of minimal straight lines, that is evident because BH is one of minimal straight lines, and the point of meeting of the minimal straight lines [falls] within the angle ΔZK, as is proved in Theorem 40 of this Book.

And BH meets ΓΘ only at K, therefore ΓΘ is not one of minimal straight lines.

As for [the statement] that the minimal straight line drawn from Γ meets ΔE and cuts off from it a segment greater than ΔΘ, that will be proved from the fact that the minimal straight line drawn from Γ meets BH [being a minimal straight line] within the angle HZK, as is proved in Theorem 40 of this Book.

Therefore it is evident that its cuts off from the axis a segment greater than ΔΘ.

## [Proposition] 47

*When minimal straight lines are drawn in a segment of an ellipse and are cut off by the major axis, no four of them meet at a single point* [52].

Let there be the ellipse ABΓΔ whose major axis ΔA.

I say that if there are drawn from the axis ΔA to the section ABΓΔ four minimal straight lines, they do not [all] meet at a single point.

[Proof]. For let, if possible there be drawn [minimal] straight lines KΓ, ΛE, MZ, and ΘB meeting at H. Then either one of these straight lines is perpendicular to AΔ or there is no perpendicular to AΔ among them.

First let one of them be perpendicular BΘ to it. Then since BΘ is one of the minimal straight lines and is perpendicular to AΔ, then Θ is the center, as is proved in Theorem 15 of this Book. And since one of minimal straight lines, BΘ has been drawn from the center, and KΓ is also one of minimal straight lines, and these two straight lines have met at H, and HE has been drawn from H, then EΛ is not one of minimal straight lines, has it proved in Theorem 46 of this Book. But it was a minimal straight line, which is impossible.

Therefore let none of BΘ, KΓ, ΛE, and MZ be a perpendicular to the axis AΔ, and let the center be N. Then if N is between BΘ and ΓK, then three minimal straight lines have been drawn from one of two halves of the axis, so as to meet at a single point, but it is impossible, because of what is proved in Theorem 45 of this Book. But if N is between ΓK and EΛ, then we draw from it a

perpendicular NP to AΔ, then the point of meeting of two straight lines EΛ and ZM occurs within the angle ΔNP, as is proved in Theorem
40 in this Book.

And similarly also two straight lines BΘ and HK must necessarily meet within the angle ANP. But the point of meeting of all [four] of them is H, which is impossible.

Therefore four drawn straight lines do not meet at a single point.

[Proposition] 48

*When maximal straight lines are drawn in one of the quadrants of an ellipse, no three of them meet at a single point* [53].

Let there be the ellipse ABΓ whose minor axis AΓ and major axis BΔ.

I say that no three of maximal straight lines drawn in the section ABΓ from one of quadrants meet at a single point.

[Proof]. For let, if it is possible, let there be drawn the [maximal] straight lines EΛ, ZK, and HΘ, and let them meet at a single point M.
Then since EΛ, ZK, and HΘ are maximal, and EN, ZH, and OH are minimal straight lines, as is proved in Theorem 23 of this Book.

So there have fallen in one of quadrants of this section three minimal straight lines so as to meet at a single point, that is impossible of what is proved in Theorems 45 and 46 of this Book. Therefore it is not the case that three maximal straight lines drawn from one of quadrants of the section ABΓ meet at a single point [54] .

[Proposition] 49

*If there is a conic section, and there is drawn from its axis a perpendicular to the axis such that that perpendicular cuts off from the axis on the side adjacent to the vertex of the section the segment no greater than the half of the* latus rectum [55] *, and a point is taken on that perpendicular and any straight line is drawn from it to the other part of the section between the perpendicular and the vertex of the section, then the minimal straight line drawn from the extremity of the straight line is not a part of that straight line, but it cuts off from the axis on the side of the vertex of the section a segment greater than that cut off by the drawn straight line.*

*In the case of the ellipse it is necessary that it be the major axis on which the perpendicular falls, and that the drawn straight line cut that the half of the axis on which the perpendicular falls* [56].

First let the section be the parabola AB whose axis BΓ, and the perpendicular ΔE. Let the segment cut from the axis by that perpendicular EB, be not greater than the half of the *latus rectum*. We take on ΔE an arbitrary point Δ, and draw from it the straight line ΔΘA.

I say that AΘ is not one of minimal straight lines.

[Proof]. For let the perpendicular AH be drawn. Now EB is not greater than the half of the *latus rectum*. Therefore EH is smaller than the half of the *latus rectum*. Let the segment equal to the half of the *latus rectum* be HΓ, and AΓ be joined. Then AΓ is a minimal straight line, as is proved in Theorem 8 of this Book.

And AΘ is not a minimal straight line, as is proved in Theorem 24 of this Book.

Rather the minimal straight line drawn from A cuts off from the axis a segment greater than BE and falls on the side [of the perpendicular ΔE] opposite to the vertex of the section.

[Proposition] 50

Furthermore let the section be the hyperbola or the ellipse AB [57] whose axis BΓ and center Γ, and let the perpendicular ΔE to the axis be drawn, and let BE be not greater than the half of the *latus rectum*, and let Δ be taken on ΔE and from it the straight line ΔZA [to meet the section at A] be drawn, then I say that AZ is not of minimal straight lines, and that the minimal straight line drawn from A cuts off from the axis a segment longer than BZ [57].

[Proof]. For let the perpendicular AH [to the axis] be drawn. Then BE is not greater of the half of the *latus rectum*, and ΓB is the half of the transverse diameter. Therefore the ratio of the transverse diameter to the *latus rectum* is not greater than the ratio ΓB to BE.

And the ratio ΓH to HE is greater than the ratio ΓB to BE. Therefore the ratio ΓH to HE is greater than the ratio of the transverse diameter to the *latus rectum*.

So we make the ratio ΓH to HΘ equal to the ratio of the transverse diameter to the *latus rectum*. Then AΘ is one of minimal straight lines, as is proved in Theorems 9 and 10 of this Book. Therefore AZ is not one of minimal straight lines, as is proved in Theorem 25 of this Book.

[Proposition] 51

*But if the mentioned perpendicular cuts off from the axis a segment greater than the half of the* latus rectum*, then I say that it is possible to generate a straight line such that when the drawn perpendicular is measured against it.*

*[1] if it is less than the perpendicular drawn to the axis then no straight line can be drawn from the end of the perpendicular to the section such that the part of it cut off [by the axis] is one of minimal straight lines,*
*but the minimal straight line drawn from it to the section cuts off from the axis adjacent to the vertex of the section a segment greater than that cut off by the straight line itself.*

*But [2] if the perpendicular is equal to the generated straight line, then it is possible to draw from its end only one straight line such that the part of it cut off [by the axis] is one of minimal straight lines, and the minimal straight line drawn from the ends of the others straight lines drawn from that point cut off from the axis adjacent to the vertex of the section straight lines greater than those cut off by the straight lines themselves.*

*[3] if the perpendicular is less than the generated straight line, then it is possible to draw from its end only two straight lines such that the part of each of them cut off [by the axis] is one of minimal straight lines, and the minimal straight line drawn from the ends of the other straight lines which fall between two straight lines from which two minimal straight lines are cut off from the axis adjacent to the vertex of the section segments less than those cut off by the straight lines themselves, but those drawn from the ends of the straight lines which are not between two minimal straight lines cut off from the axis straight lines greater than those cut off by the straight lines themselves.*

*However in the case of the ellipse our statement requires that the axis on which the perpendicular falls be the major axis* [58].

First we make the section the parabola ΑΒΓ whose axis ΓΖ. We draw the perpendicular ΕΖ to it, let the part cut off by it from the axis, namely ΓΖ, be greater than the half of the *latus rectum*.

I say that, if a certain straight line is cut off from ΕΖ, and [another] straight line is drawn from its end under the conditions stated above, what we stated in the enunciation will necessarily occur.

[Proof]. ΓΖ is greater than the half of the *latus rectum*. So let the half of the *latus rectum* be ΖΗ. We cut ΓΗ at Θ such that ΘΗ is double ΘΓ, and draw the perpendicular ΘΒ.

Let some straight line Κ be to ΘΒ as to ΘΗ be to ΗΖ [59] .

We take Ε on ΖΒ and, first, let ΖΕ be greater than Κ.

Then I say that no straight line can be drawn from E such that the axis cuts off from it a minimal straight line.

We join BE [meeting ΓZ at Λ]. [And I say that BΛ is not one of minimal straight lines].

Then as K is to ΘB, so ΘH is to HZ. And K is smaller than ZE. Therefore the ratio ZE to BΘ [equal to the ratio Zλ to ΛΘ] is greater than the ratio HΘ to HZ. And *componendo* the ratio ZΘ to ΘΛ is greater than the ratio ΘZ to ZH. Therefore ZH [equal to the half of the *latus rectum*] is greater than ΘΛ, and ΘΛ is smaller than the half of the *latus rectum*. Therefore the minimal straight line drawn from B [to the axis] falls on the side of Z [from Λ], as is proved from Theorem 8 of this Book. Therefore BΛ is not one of minimal straight lines, as is proved in Theorem 24 of this Book.

Furthermore we draw EIM [where I is between Λ and Γ], then I say that IM is not of minimal straight lines.

[Proof]. For let from B a straight line BO tangent to the section be drawn and the perpendicular MN be drawn and continued to [meet BO at] Ξ. Then since the section in a parabola, ΓO is equal to ΓΘ, as is proved in Theorem 35 of Book I . Therefore ΘO is equal to the double ΘΓ.

But ΘH had been [made equal to] the double ΘΓ. Therefore OΘ is equal to ΘH. And [thus] ΘH turns out to be greater than NO. Therefore the ratio ΘN to NO is greater than the ratio NΘ to ΘH. And *componendo* the ratio ΘO to ON [equal to the ratio ΘB to NΞ] is greater than the ratio NH to HΘ, and pl.BΘH is greater than pl.ΞNH.

Therefore pl.BΘH is much greater than pl.MNH. But pl.EZH is greater than pl.BΘH because the ratio EZ to BΘ is greater than the ratio ΘH to HZ, as we have proved above. Therefore pl.EZH is greater than pl.MNH, and the ratio ZE to MN [equal to the ratio ZI to IN] is greater than the ratio NH to ZH. And *componendo* the ratio ZN to NI is greater than the ratio NZ to ZH. Therefore ZH is greater than IN.

But ZH is equal to the half of the *latus rectum*. Therefore IN is smaller than the half of the *latus rectum*. Therefore MN is not one of minimal straight lines, but the minimal straight line drawn from M falls on the axis toward Z [from I], as is proved from Theorems 8 and 24 of this Book.

Furthermore we draw the straight line APE[where P is between Λ and Z], then I say that AP is not one of minimal straight lines.

For let the perpendicular AΣ be drawn and continued to [meet the tangent at] Π. Then ΘO is equal to ΘH, as we said above. And [therefore ΘO turns out to be greater than ΣH, therefore the ratio ΣΘ to ΘO is smaller than the ratio ΣΘ to ΣΗ. And *componendo* the ratio ΣΟ to ΟΘ is smaller than the ratio ΘΗ to ΣΗ. But as ΣΟ is to ΘΟ, so ΠΣ is to ΒΘ. Therefore the ratio ΠΣ to ΒΘ is smaller than the ratio ΘΗ to ΣΗ, and pl.ΠΣΗ is smaller than pl.ΒΘΗ.

Therefore pl.ΑΣΗ is much smaller than pl.ΒΘΗ.

But we have [already] proved that pl.ΕΖΗ is greater than pl.ΒΘΗ. Therefore pl.ΑΣΗ is smaller than ΕΖΗ, and the ratio ΑΣ to ΕΖ is smaller than the ratio ΖΗ to ΣΗ.

But as ΑΣ is to ΕΖ , so ΣΡ is to ΡΖ. Therefore the ratio ΣΡ to ΡΖ is smaller than the ratio ΖΗ to ΣΗ, and the ratio ΡΖ to ΣΡ is greater than the ratio SH to ΖΗ. And *componendo* the ratio ΣΖ to ΣΡ is greater than the ratio ΣΖ to ΖΗ. Therefore ΖΗ is greater than ΣΡ.

But ΖΗ is equal to the half of the *latus rectum*. Therefore ΣΡ is smaller than the half of the *latus rectum*. Therefore ΑΡ is not one of minimal straight lines, but the minimal straight line drawn from Α falls to the side of Ζ[from Ρ], as is proved from Theorems 8 and 24 of this Book.

Therefore when ΕΖ is grater than Κ, no straight line can be drawn from Ε to the section such that the axis cuts off from it a segment, which is one of minimal straight lines.

Furthermore [secondly] we make ΖΕ equal to Κ. Then I say that only one straight line can be drawn from Ε such that a minimal straight line is cut off from it [by the axis], and that other minimal straight lines drawn from the points where the straight lines from Ε meet the section fall on the farther side [of the original straight lines] from Γ.

[Proof]. As ΘΗ is to ΗΖ , so Κ [equal to ΕΖ] is to ΒΘ. But as ΕΖ is to ΒΘ, so ΖΛ is to ΛΘ. Therefore as ΘΗ is to ΗΖ, so ΖΛ is to ΛΘ, and ΖΗ is equal to ΛΘ.

But ΖΗ is equal to the half of the *latus rectum*. Therefore ΛΘ also is equal to the half of the *latus rectum*, and ΛΒ is one of minimal straight lines, as is proved in Theorem 8 of this Book.

Then I say that no other minimal straight line will be cut off [by the axis] from other straight lines drawn from Ε.

[Proof]. For let some straight line ΜΙΕ be drawn, and the perpendicular ΜΝ be drawn and continued to [meet the section at] Ξ. Let ΒΟ be a tangent to the section.

Then we will prove as we proved previously that pl.ΒΘΗ [equal to pl.ΕΖΗ] is greater than pl.ΜΝΗ.

And we will prove from that, as we proved above, that ΖΗ [equal to the half of the *latus rectum*] is greater than ΙΝ. Therefore ΜΙ is not one of minimal straight lines, but the minimal straight line drawn from M falls towards Z [from I].

But it is drawn like APE, then AP is not of the minimal straight lines, but the minimal straight line drawn from A falls towards Z.

[Proof]. For let the perpendicular AΣ be drawn and continued to [meet the section at] Π.

Similarly too [to the above] it will be proved that pl.AΣH is smaller pl.BΘH [equal to pl.EZH].

Hence we will prove, as we proved previously that PΣ is smaller than HZ. But PΣ is smaller than the half of the *latus rectum*. Therefore AP is not of minimal straight lines, but the minimal straight line drawn from A falls towards Z [from P].

Furthermore [thirdly] we make EZ smaller than K. Then I say that one can draw from E to the section ABΓ two straight lines such that two minimal straight lines can be cut off from them [by the axis] and that when minimal straight lines are drawn from the ends of other straight lines which fall between these two straight lines, they cut off from the axis segments smaller than the segments cut off by the drawn straight lines, and as for other straight lines, the minimal straight lines drawn from their ends cut of segments greater than those cut off by the straight lines themselves.

[Proof]. ZE is smaller than K. Therefore the ratio EZ to BΘ is smaller than the ratio K to BΘ [equal to the ratio ΘH to HZ], and pl.EZH is smaller than pl.BΘH.

Let pl.ΦΘH be equal to pl.EZH, and let TH be a perpendicular to HZ.

We pass through Φ the hyperbola [60] whose asymptotes TH and ΓH , as we showed in Problem 4 of Book II.

Then it cuts the parabola, let it cut it at A and M. We join EA and EM and draw the perpendiculars AΣ and MN then the section AΦM is a hyperbola and its asymptotes are TH and HΓ, and AΣ , MN, and ΦΘ have been drawn from the section at right angles [to an asymptote].

Therefore pl.MNH is equal to pl.ΦΘH, as is proved in Theorem 12 of Book II, and pl.ΦΘH is equal to pl.EZH. Therefore as MN is to EZ, so ZH is to NH. But as MN is to EZ, so NI is to IZ, therefore as ZH is to NH, so NI is to IZ. And *componendo* as NZ is to ZH, so ZN is to NI.

Therefore IN is equal to ZH, which is equal to the half of the *latus rectum*. Therefore MI is one of minimal straight lines as is proved in Theorem 8 of this Book.

Similarly also it will be proved that AP is one of minimal straight lines.

And since MI, and AP are minimal straight lines, and they meet at E, therefore of the straight lines drawn from E to the section for [any of] those falling between AE and EM, if a minimal straight line is drawn from the place where it reaches [the section] it falls towards the vertex of the section, and has for the other straight lines falling outside AE and EM [the minimal straight lines drawn from their ends] will fall on the side [of the straight lines] farther from the vertex of the section, as was proved in Theorem 44 of this Book [61-63] .

[Proposition] 52

Furthermore we make the section the hyperbola or the ellipse ΑΒΓ whose axis ΕΓΔ and center Δ, and draw from the axis perpendicular ZE, and let ΕΓ be greater than the half of the *latus rectum*.

Then I say that in this case [too] the same property necessarily results as in the parabola [64].

[Proof]. ΔΓ is the half on the transverse diameter, and ΓE is greater than of the half of the *latus rectum*. Therefore the ratio ΔΓ to ΓE is smaller than the ratio of the transverse diameter to the *latus rectum*.

Therefore if we make the ratio ΔH to HE equal to the ratio of the transverse diameter to the *latus rectum*, the point H falls between Γ and E.

We take two straight lines ΘΔ and ΔK in continuous proportion between HΔ and ΔΓ.

Let KB be a perpendicular to the axis, and let the ratio of some straight line Λ, to KB be equal to the ratio compounded of the ratios ΔE to EH and HK to KΔ [65-66] .

In the first instance we make EZ greater than Λ.

Then I say that it is not possible to draw from Z to the section any straight line such that what is cut off from it [by the axis] is one of minimal straight lines, and that the minimal straight lines drawn from the ends of the straight lines drawn from Z to the section cut off from the axis adjacent to the vertex of the section segments greater than those cut off by the straight lines [from Z] themselves.

[Proof]. For let the straight line ZMB be joined then I say that BM is not one of minimal straight lines for we make the ratio ZN to NE equal to the ratio of the transverse diameter to the *latus rectum*, and draw the straight lines ZoO and ΝΩΞ parallel to ΕΓΔ, and draw Hωo and ΔO parallel to EZ. Then since EZ is greater than Λ, the ratio EZ to BK is greater than the ratio Λ to BK.

But the ratio EZ to BK is compounded of the ratios ZE to EN and KX to KB because KX is equal to EN.

And as for the ratio Λ to KB we had made it equal to the ratio compounded of the ratios ΔE to EH and HK to KΔ, then the ratio compounded of the ratios ZE to EN and KX to KB is greater than the ratio compounded of the ratios ΔE to EH and HK to KΔ.

But as ZE is to EN, so ΔE is to EH, because both of the ratios ZN to NE and ΔH to HE are equal to the ratio of the transverse diameter to the *latus rectum*. Therefore the remaining ratio KX to KB is greater than the ratio HK to KΔ. Therefore pl.XKΔ is greater than pl.BKH.

But pl.XKΔ is the quadrangle ΔX. Therefore pl.KBH is smaller than the quadrangle ΔX.

We make the quadrangle HX that is pl.KXΩ common [to both sides] then pl.BXΩ is smaller than the quadrangle ΔΩ. But the quadrangle ΔΩ is equal to the quadrangle ϘN because as ZN to NE, so ΔH is to HE. Therefore pl.BXΩ is smaller the quadrangle ϘN.

And we had proved in the proof of Theorem 45 of this Book that, when that is the case, then BM is not one of minimal straight lines, and that the minimal straight line drawn from B cuts off from the axis adjacent to the vertex of the section a segment longer than ΓM.

Furthermore we draw ZςP to a point other than B, then I say that Pς is not one of minimal straight lines, and that the minimal straight line drawn from P cuts off from the axis adjacent to the vertex of the section a segment longer than Γς.

[Proof]. We draw from B a tangent BΞ to the section, and draw to the axis the perpendicular PΠ and continue it to [meet the tangent at] Σ. Then, since the ratio XK to KB is greater than the ratio HK to KΔ, we make the ratio YK to KB equal to the ratio HK to KΔ, and draw through Y a straight line TYΦ parallel to EΓΔ. Then since BσT is tangent to the section, and BK is perpendicular to the axis, pl.KΔσ is equal to sq.ΔΓ, as is proved in Theorem 37 of Book I . Therefore as KΔ is to ΔΓ, so ΔΓ is to Δσ.

Therefore the third proportional to KΔ and ΔΓ is Δσ. And the third proportional to HΔ and ΔΘ was KΔ. And as KΔ is to ΔΓ, so HΔ is to ΔΘ. Therefore, as HΔ is to ΔK, so ΔK is to Δσ.

And when we subtract two lesser from two greater, the ratio of the remainders HK to Kσ is equal to the ratio HΔ to ΔK.

But as HΔ is to ΔK, so YB is to BK because the ratio HK to KΔ was made equal to the ratio YK to KB. Therefore as HK is to Kσ, so BY is to BK.

But as BY is to BK, so YT is to Kσ. Therefore as HK is to Kσ, so YT is to Kσ, and HK is equal to YT.

But HK is equal to YΦ. Therefore YΦ is equal to YT, and Tβ is smaller than YΦ, and the ratio Yβ to Tβ is greater than the ratio Yβ to YΦ.
And *componendo* the ratio YT to Tβ is greater than the ratio βΦ to YΦ. But as YT is to Tβ, so YB is to Σβ, and the ratio YB to Σβ is greater than the ratio βΦ to ΦY. Therefore pl.BYΦ is greater than pl.ΣβΦ.

Therefore pl.BYΦ is much greater than pl.PβΦ.

Furthermore as HK is to KΔ, so YK is to KB. Therefore pl.BKH is equal to pl.ΔKY.

We make pl.YKH common [to both sides].

Then pl.BYΦ is equal to pl.ΔH,YK because YΦ is equal to HK. And pl.ΔH,YK is the quadrangle ΔΦ. Therefore pl.BYΦ is equal to the quadrangle ΔΦ.

But pl.BYΦ was [shown to be] greater than pl.PβΦ, therefore the quadrangle ΔΦ is greater than pl.PβΦ.

In the case of the hyperbola we make pl.βγΩ. Then pl.βγΩ is smaller than the sum of the quadrangles ΔΦ and βΩ.

In the case of the ellipse when we subtract pl.βγΩ [from both sides] the quadrangle ΔΦ without the quadrangle βΩ is greater than pl.PγΩ.

Thus pl.PγΩ is much smaller than the quadrangle ΔΩ [in both cases].

But the quadrangle ΔΩ is equal to the quadrangle ϘN because as ZN is to NE, so ΔH is to HE. Therefore pl.PγΩ is smaller than the quadrangle οN.

But we showed in the proof of Theorem 45 of this Book that in that case PΓ is not one of minimal straight lines, and that minimal straight line drawn from P cuts off from the axis adjacent to the vertex of the section longer than ΓΓ.

Furthermore we draw ZεA [on the other side of ZMB], then I say that Aε is not one of minimal straight lines, and that the minimal straight line drawn from A cuts off from the axis adjacent to the vertex of the section a segment longer than Γε.

[Proof]. For let the perpendicular Aζθ be drawn and continued to [meet the tangent at] δ. We have already proved that ΦY is equal to YT. Therefore Φζ is smaller than YT. Therefore the ratio ζY to Φζ is greater than the ratio ζY to YT. And *componendo* the ratio YΦ to Φζ is grater than the ratio ζT to TY.

But as ζT is to TY, so ζδ is to BY. Therefore the ratio YΦ to Φζ is greater than the ratio δζ to BY, and pl.BYΦ is greater than pl.δζΦ.

And we will prove by the method that we followed previously that pl.AθΩ is smaller than the quadrangle ΩZ.

And it will be proved from that as was shown in the proof of Theorem 45 of this Book, that Aε is not one of minimal straight lines, and that the minimal straight line drawn from A cuts off from the axis adjacent to the vertex of the section a segment longer than Γε.

Furthermore [secondly] we make ZE equal to Λ, then I say that only one straight line can be drawn from Z such that the part of it cut off [by the axis] is one of minimal straight lines, and that the minimal straight lines drawn from the ends of the remaining straight lines cut off from the axis adjacent to the vertex of the section segments longer than those cut off by the straight lines themselves.

[Proof]. We proceed as we did in the first case for the construction of the perpendicular BK, and join ZB. Then the ratio ZE to BK, is equal to the ratio Λ to BK. Now ZE to BK is compounded of the ratios ZE to EN and KX to KB for KX is equal to EN, and the ratio Λ to BK is compounded of the ratios ΔE to EH and HK to KΔ according to our previous construction the ratio compounded of the ratios ZE to EN and KX to KB is equal to the ratio compounded of the ratios ΔE to EH and HK to KΔ.

But as ZE is to EN, so ΔE is to EH. Therefore the remaining ratio KX to KB is equal to the ratio HK to KΔ.

Therefore pl.XKΔ [which is the quadrangle ΔX] is equal to pl.BKH.

We make pl.XKB common [to both sides], by adding in the case of the hyperbola and subtracting in the case of the ellipse, then pl.BXΩ is equal to the quadrangle ΔΩ. But the quadrangle ΔΩ is equal to the quadrangle ΩZ.

Therefore the quadrangle ΩZ is equal to pl.BXΩ.

And we had shown in the proof of Theorem 45 of this Book that, when that is the case, BM is one of minimal straight lines.

I say that no other straight line can be drawn from Z such that the part of it cut off [by the axis] in one of minimal straight lines.

[Proof] For let ZςP and the perpendicular PΠ be drawn. Then we will prove by the same method as before that XΩ is equal to XΞ. Therefore Ξγ is smaller than XΩ, and the ratio Xγ to γΞ is greater than the ratio Xγ to XΩ.
And *componendo* the ratio XΞ to Ξγ is greater than the ratio γΩ to ΩX.

But as XΞ is to Ξγ, so BX is to Σγ. Therefore the ratio BX to Σγ is greater than the ratio γΩ to ΩX, and pl.BXΩ is greater than pl.ΣγΩ.

Therefore pl.BXΩ is much greater than pl.PγΩ.

And we had proved that pl.BXΩ is equal to the quadrangle ΩZ. Therefore pl.PγΩ is smaller than the quadrangle ΩZ.

But we showed in the proof of Theorem 45 of this Book that, when that is the case, Pς is not one of minimal straight lines, and that the minimal straight line drawn from P cuts off from the axis adjacent to the vertex of the section a segment greater than Γς.

Similarly too it can be proved that Aε is not one of two minimal straight lines, and that the minimal straight line drawn from A cuts off from the axis adjacent to the vertex of the section a segment longer than Γε.

Furthermore [thirdly] we make ZE smaller than Λ. Then I say that only two straight lines can be drawn from E such that the part of [each of] these two cut off [by the axis] is one of minimal straight lines, and that the minimal straight lines drawn from the ends of the straight lines drawn between these two minimal straight lines cut off from the axis adjacent to the vertex of the section segments smaller than those cut off by the straight lines themselves, and that the minimal straight lines drawn from the ends of the remaining straight lines cut off from the axis adjacent to the vertex to the sections segments greater than those cut of by the straight lines themselves.

[Proof]. The ratio ZE to BK is smaller than the ratio ΛBK. And hence it will be proved by a method similar to the preceding that the ratio KX to KB is smaller than the ratio HK to KΔ, and that the quadrangle ΩZ is smaller than the ratio HK to KΔ. Therefore we make pl.IXΩ equal to the quadrangle ΩZ, and draw a hyperbola [67] passing through I with asymptotes ΞΩ and ΩH, then it is constructed as we showed Problem 4 of Book II, that is the section AIP.

We draw the perpendiculars Aθ and Pγ. Then each of pl.AθΩ and pl.PγΩ is equal to pl.IXΩ because of what is proved in Theorem 12 of Book II .

And pl.IXΩ was made equal to the quadrangle ΩZ. Therefore pl.AθΩ is equal to pl.PγΩ, which is equal to the quadrangle ΩZ.

And when that is the case, then it will be proved as we showed in the preceding part of this Theorem, that each of two straight lines Aε and Pς is one of minimal straight lines.

And they have been drawn, so as to meet at Z, and we have shown in Theorem 45 of this Book, that when that is the case no other straight line can be drawn from Z such that the part of it cut off [by the axis] is one of minimal straight lines, and that for the straight lines drawn from Z between Aε and Pς, when minimal straight lines are drawn from their ends to the axis, they cut off from the axis adjacent to the vertex of the section segments smaller than the segments cut off by the straight lines themselves, and that the minimal straight lines drawn from the ends of the remaining straight lines are in the opposite case, that is they cut of segments greater [than those cut of by the straight lines themselves].

In the case of the ellipse this enunciation depends on the axis, which is used the major axis [68-73] .

[Proposition] 53

*If a point is taken outside of one of two halves of an ellipse into which the major axis divides it, such that the perpendicular drawn from it to the axis falls on the center of the section, and [such that] the ratio of that perpendicular together with the half of the minor axis to the half of the minor axis is not smaller than the ratio on the transverse diameter to the latus rectum, then no straight line can be drawn from that point to the section such that the part of it falling between the axis and the section is one of straight lines, rather the minimal straight line drawn from its extremity falls on that side of the drawn straight line which is farther from the vertex of the section* [74].

Let there be the half of the ellipse ΒΑΓ with major axis ΒΓ. We take a point outside of it [such that] when a perpendicular [to the major axis] is drawn from it, it falls on the center, that [taken point] is Δ. We draw from Δ a perpendicular ΔΕ to ΓΒ. Let Ε on which the perpendicular falls be the center of the section, and let the ratio ΔΑ to ΑΕ be not smaller than the ratio of the transverse diameter to the *latus rectum*.

Then I say that no straight line can be drawn from Δ such that the part of it cut off between the section and ΒΓ is one of minimal straight lines, and that, if a straight line is drawn from it, such as ΔΚ, then the minimal straight line drawn from Κ falls on the side [of ΔΚ] towards Ε.

[Proof]. For let two perpendiculars ΚΗ and ΚΖ be drawn. Then the ratio ΑΔ to ΑΕ is not smaller than the ratio of the transverse diameter to the *latus rectum*.

But the ratio ΔΑ to ΑΕ is smaller than the ratio ΔΖ to ΖΕ. Therefore the ratio ΔΖ to ΖΕ [equal to the ratio ΕΗ to ΗΘ] is greater than the ratio of the transverse diameter to the *latus rectum*.

So let the ratio ΕΗ to ΗΛ be equal to the ratio of the transverse diameter to the *latus rectum*. Then ΚΛ is one of minimal straight lines, as is proved in Theorem 10 of this Book, therefore ΚΘ is not one of minimal straight lines, as is proved in Theorem 25 of this Book, and the minimal straight line drawn from Κ falls on the side of Ε from ΚΛ.

[Proposition] 54

*If a point is taken outside of one of two halves of an ellipse into which the major axis divides it, and a perpendicular is drawn from it to [the major axis] such that it ends at the center, and the ratio of that perpendicular together with the half of the minor axis to the half of the minor axis is smaller than the ratio of the transverse diameter to the* latus rectum*, then amongst the straight lines drawn from that point to the section in each of two quadrants [into which the minor axis divides the half of the ellipse] there is only one straight line such that the part of it cut of between the section and the major axis is minimal straight line, and for other straight lines drawn on that side no minimal straight line is cut off from them [between the axis and the section, but for those of them drawn closer to the vertex of the section than the straight line from which a minimal straight line is cut off, the minimal straight lines drawn from their ends are farther [from the vertex]. And for those of them that are farther [from the vertex of the section than is the minimal straight line], the minimal straight lines drawn from their ends are drawn closer [to the vertex]*.

Let there be the ellipse ΒΑΓ whose major axis ΒΓ, and the let us take outside of it a point such that when a perpendicular is drawn from it, it falls on the center, that is Δ. We draw from it a perpendicular ΔΕ to ΓΒ let it fall on the center, and let the ratio ΔΑ to ΑΕ be smaller than the ratio of the transverse diameter to the *latus rectum*.

I say that of straight lines drawn from Δ in one of two quadrants only one is such that the part of it cut off between ΒΑΓ and ΒΓ is a minimal straight line and that for those of the remaining straight lines drawn closer to Β the minimal straight line drawn from the end [of each] of them is farther [from Β] and for those of them drawn farther from Β the minimal straight line drawn from the end [of each] of them is closer [to Β].

[Proof]. The ratio ΔΑ to ΑΕ is smaller than the ratio of the transverse diameter to the *latus rectum*. We make the ratio ΔΗ to ΗΕ equal to the ratio of the transverse diameter to the *latus rectum*, and draw ΗΘ and ΘΚ parallel to ΑΕ and ΒΓ, and join ΘΔ [cutting ΒΓ at Λ].

Then I say that ΛΘ, which is a part of ΔΘ, is a minimal straight line because the ratio ΔΗ to ΗΕ [equal to the ratio ΕΚ to ΚΛ] is equal to the ratio of the transverse diameter to the *latus rectum*, and Ε is the center of the section. Therefore ΘΛ is one of minimal straight lines as is proved in Theorem 10 of this Book.

And ΑΕ is also one of minimal straight lines, as is proved in Theorem 11 of this Book.

And both these straight lines meet at Δ.

So for those of straight lines drawn from Δ whose distance from B is greater than the distance of ΦΘ [from B], the minimal straight line drawn from the end of [each of] them is closer to B than it, and for those of them whose distance from B is smaller [than that of ΔΘ], the minimal straight line drawn from the end of [each of] them is farther from B than it, as is proved in Theorem 46 of this Book [75].

[Proposition] 55

*If a point is taken outside of one of two halves of an ellipse into which the major of its two axes divides it and a perpendicular is drawn from it to the axis, so as not to fall on the center, then there can be drawn from that point to the section a straight line such that the part of it cut off between the section and the major axis is one of minimal straight lines, and it cuts the other of two halves of the major axis on which the perpendicular does not fall, and no other straight line can be drawn from that point cutting that half [of the axis] such that the part of it cut off is a minimal straight line*[76].

Let there be the ellipse ΑΒΓ whose major axis ΑΓ and center Δ, and let the taken point be Ε, and the perpendicular drawn from it to the axis ΑΓ be the perpendicular ΕΖ, where the center is not Ζ.

I say that there can be drawn from Ε a straight line cutting ΔΓ such that the part of it falling between ΑΒΓ and ΔΓ is one of minimal straight lines.

For let the ratio ΕΗ to ΗΖ be made equal to the ratio of the transverse diameter to the *latus rectum*, and likewise be made the ratio ΔΘ to ΘΖ.

We draw through Η a straight line ΚΛ parallel to ΑΓ, and draw through Θ a straight line ΜΘΛ parallel to ΕΗ.

We construct a hyperbola passing through Ε with asymptotes ΜΛ and ΛΚ, as is shown in Problem 4 of Book II . Let that section be ΕΝ, and let it cut the ellipse at Ν.

Then I say that, when we join ΝΕ this straight line is one of minimal straight lines.

[Proof]. For let ΕΝ be continued to meet ΛΜ and ΛΚ. Let it meet them at Μ and Κ.

We draw two perpendiculars ΝΟ and ΚΠ to ΑΓ. Then ΜΕ is equal to ΚΝ, as is proved in Theorem 8 of Book II . Therefore ΖΘ is equal to ΠΟ, and the ratio ΕΗ to ΗΖ is equal to the ratio of the transverse diameter to the *latus rectum*, and is equal to the ratio ΖΠ to ΠΞ. Therefore the ratio ΖΠ to ΠΞ is equal to the ratio of the transverse diameter to the *latus rectum*.

But the ratio ΔΘ to ΘZ was also [made] equal to the ratio of the transverse diameter to the *latus rectum*. Therefore the ratio ZΠ to ΠΞ is equal to the ratio ΔΘ to ΘZ.

But ΘZ is equal to ΠO, and [hence] ΔΘ is equal to the sum of ΠO and ΔZ. So, when we subtract ZΘ and ΠO from ZΠ, and ΠO from ΠΞ, the ratio of the remainder ΔO to the remainder OΞ is equal to the ratio of the whole ZΠ, to the whole ΠΞ, which is equal to the ratio of the transverse diameter to the *latus rectum*.

Therefore the ratio ΔO to OΞ is equal to the ratio of the transverse diameter to the *latus rectum*. And NO is a perpendicular [to the axis] and Δ is the center. Therefore NΞ is one of minimal straight lines, as is proved in Theorem 10 of this Book.

[Proposition] 56

And what we said in the preceding theorem concerning the fact that the hyperbola will meet the ellipse will be proved by us drawing from Γ a tangent Go to the ellipse. Then the ratio ΔΘ to ΘZ is equal to the ratio of the transverse diameter to the *latus rectum*.

But the ratio ΔΘ to ΘZ is smaller than the ratio ΓΘ to ΘZ. Therefore the ratio ΓΘ to ΘZ is greater than the ratio of the transverse diameter to the *latus rectum*, which is equal to the ratio EH to HZ. Therefore the ratio ΓΘ to ΘZ is greater than the ratio EH to HZ, and pl.ΓΘ,HZ is greater than pl.ΘZ,EH
But HZ is equal to ΓϘ, and ZΘ is equal to HΛ, therefore pl.Θγ Ϙ is greater than pl.EHΛ.

So the hyperbola passing through E with asymptotes MΛ and ΛϘ cuts ΓϘ, as is proved from the converse of Theorem 12 of Book II . And ΓϘ is tangent to the section ABΓ [at Γ]. Therefore the mentioned hyperbola cuts the section ABΓ.

[Proposition] 57

Furthermore now we make the ellipse ABΓ whose major axis ΓA, and take the point Δ below the axis, and draw from it the perpendicular ΔZ, and let the center be E, and draw from Δ the straight line ΔHB from which one of minimal straight lines is cut off [between the axis and the section], let the minimal straight line be BH, and let it cut ΓHE, and draw ΔK and ΔΘ [on either side of ΔHB, meeting ΓE at Π and Ξ] and from the center E draw EN parallel to ΔZ, now

ΒΗ is one of minimal straight lines, so it meets the minimal straight line drawn from the center inside the angle ΓΖΔ, let it meet it at Ν. Then the straight line joining Ν and Θ cannot have a minimal straight line cut off from it between the section and its [major axis], but the minimal straight line drawn from Θ is closer to Γ [than ΝΘ],as is proved in Theorem 46 of this Book.

Therefore ΘΞ is not one of minimal straight lines, as is proved in Theorem 25 of this Book.

Similarly too it will be proved that ΚΠ is not one of minimal straight lines, and that the minimal straight line drawn from Κ falls on the side of Α [from ΓΙ].

[Proposition] 58

*For every point taken outside one of conic sections provided that it is not of the axis wherever the axis is continued in a straight line, it is possible for us to draw from it some straight line such that the part of it which falls between the section and its axis is one of minimal straight lines* [77].

Let the section first be the parabola ΑΒ whose its continued axis ΓΖ. We take outside of the section the point Δ, not on the axis.

I say that there can be drawn from Δ a straight line such that the part of it which falls between ΑΒ and ΒΓ is one of minimal straight lines.

[Proof]. For let the perpendicular ΔΕ to ΓΖ wherever it falls on it be drawn let ΕΖ be equal to the half of the *latus rectum*, and let ΖΗ be a perpendicular to ΖΓ.

We construct the hyperbola ΔΑΘ passing through Δ with asymptotes ΗΖ and ΖΓ, as is shown in Problem 4 of Book II .

Then it will cut the parabola, let it cut it at Α. We join ΔΑ and continue it [on both sides] to Η and Γ, and drop a perpendicular ΑΚ from Α onto ΓΖ. Ν

Then ΔΗ is equal to ΑΓ, as is proved in Theorem 8 of Book II ,therefore ΖΕ is equal to ΚΓ.

But ΖΕ is equal to the half of the *latus rectum*. Therefore ΚΓ is equal to the half of the *latus rectum*. And ΚΑ is a perpendicular [from the axis to the section]. Therefore ΑΓ is one of minimal straight lines, as is proved in Theorem 8 of this Book.

[Proposition] 59

Furthermore we make the section the hyperbola or the ellipse ΑΒ whose axis ΒΔ and center Γ, and take outside of the section the point Ε not on the continuation of the axis, and draw from it the perpendicular ΕΖ to ΒΔ, and first let that perpendicular not fall on the center.

I say that it is possible for us to draw from Ε a straight line such that the part of it falling between ΑΒ and ΒΔ is a minimal straight line.

[Proof]. For let the ratio ΓΗ to ΓΖ be equal to the ratio of the transverse diameter to the *latus rectum*. We draw ΗΜ at right angles [to ΓΖ], and make the ratio ΕΘ to ΘΖ equal to the ratio of the transverse diameter to the *latus rectum*, and pass through Θ a straight line ΚΛ parallel to ΒΔ. We construct the hyperbola passing through E with the asymptotes ΜΚ and ΚΛ, as is shown in Problem 4 of Book II . Then it will meet the section ΑΒ. Let that hyperbola be ΕΑΞ, and let it meet the section ΑΒ at Α. We join ΕΑ and continue it a straight line [on both sides] to Μ and Λ and draw the perpendicular ΑΝ [to ΒΔ]. Then ΜΕ is equal to ΑΛ, as is proved in Theorem 8 of Book II, therefore ΚΘ is equal to ΟΛ, and [hence] ΟΚ is equal to ΘΛ, and ΝΗ is equal to ΘΛ.

And the ratio ΖΔ to ΘΛ is equal to the ratio ΖΕ to ΕΘ, which is equal to the ratio ΓΖ to ΓΗ because both ratios ΓΗ to ΗΖ and ΕΘ to ΘΖ are equal to the ratio of the transverse diameter to the *latus rectum*. Therefore the ratio ΖΔ to ΝΗ is equal to the ratio ΓΖ to ΓΗ.

And when we add the ratios in the case of the hyperbola and separate them in the case of the ellipse, the ratio ΔΓ to ΓΝ is equal to the ratio ΖΓ to ΓΗ.

And convertendo in the case of the ellipse and *dividendo* in the case of the hyperbola the ratio ΓΗ to ΗΖ [equal to the ratio of the transverse diameter to the *latus rectum*] is equal to the ratio ΓΝ to ΝΔ, and ΝΑ is a perpendicular to ΒΔ. So ΑΔ is one of minimal straight lines, as is proved in Theorems 9 and 10 of this Book.

The proof is similar if the perpendicular falls outside of Β.

[Proposition] 60

Furthermore we make the perpendicular which is drawn from the point taken outside of the hyperbola fall on the center as the perpendicular ΓΔ, and make the ratio ΓΕ to ΕΔ equal to the ratio of the transverse diameter to the *latus rectum* and draw ΕΑ parallel to ΔΖ [to meet the section at Α], and join ΓΑ and continued it to [meet the axis at] Ζ, then I say that ΑΖ is one of minimal straight lines[79].

[Proof]. For let from Α the perpendicular ΑΗ to ΔΖ be drawn. Then the ratio ΓΕ to ΕΔ is equal to the ratio of the transverse diameter to the *latus rectum*, and is equal to the ratio ΓΑ to ΑΖ.

But the ratio ΓA to AZ is equal to the ratio ΔH to HZ. Therefore the ratio ΔH to HZ is equal to the ratio of the transverse diameter to the *latus rectum*. And AH is a perpendicular [from the section to the axis]. Therefore AZ is one of minimal straight lines, as is proved in Theorem 9 of this Book.

[Proposition] 61

Furthermore [in the case of the hyperbola] we make the perpendicular falling from the taken point be on the other side of the center as the perpendicular ΓΔ, and let the center be E, and the section AB, and make the ratio EZ to ZΔ equal to the ratio of the transverse diameter to the *latus rectum*, and also make the ratio ΓH to HΔ equal to the ratio of the transverse diameter to the *latus rectum*, and draw HΘ parallel to ΔB, and ZK and EM parallel to ΓΔ, and construct the hyperbola passing through E with the asymptotes ΘK and KZ, then [that hyperbola] will cut the section AB, let it cut it at A, and let the hyperbola be AE.

We join ΓA and continue it to [meet ΔB at] Λ.

I say that AΛ is one of minimal straight lines [80] .

[Proof]. For let ΘAO perpendicular to ΔO be drawn . Then the ratio ΓH to HΔ is equal to the ratio EZ to ZΔ. Therefore pl.ΓHK [equal to pl.ΓH,ZΔ] is equal to pl.KME [equal to pl.ZE,ΔH].

But pl.KME is equal to pl.KΘA because of the asymptotes, as is proved in Theorem 12 of Book II.

Therefore pl.ΓHK is equal to pl.KΘA, and the ratio AΘ to ΓH is equal to the ratio HK to KΘ. But the ratio AΘ to ΓH is equal to the ratio ΘN to NH. Therefore the ratio HK to KΘ is equal to the ratio NΘ to NH, and KΘ [equal to ZO] is equal to NH. Therefore the ratio AΔ to NH is equal to the ratio AΔ to ZO, and [also] is equal to the ratio AΓ to ΓN. Therefore the ratio ΛΔ to ZO is equal to the ratio ΛΓ to ΓN.

But the ratio ΛΓ to ΓN is equal to the ratio ΔΓ to ΓH. Therefore the ratio ΛΔ to ZO is equal to the ratio ΔΓ to ΓH. But the ratio ΔΓ to ΓH is equal to the ratio ΔE to EZ, and the ratio ΛΔ to ZO is equal to the ratio ΔE to EZ.

Therefore the ratio of the remainder [of ΛΔ without ΔE, namely ΛE], to the remainder [ of ZO without EZ, namely EO], is equal to the ratio ΔE to EZ.

And *dividendo* the ratio EO to OΛ is equal to the ratio EZ to ZΔ, which is equal to the ratio of the transverse diameter to the *latus rectum*.

Therefore the ratio EO to OΛ is equal to the ratio of the transverse diameter to the *latus rectum*. Therefore ΛA is one of minimal straight lines, as is proved in Theorem 9 of this Book.

[Proposition] 62

It is possible for us to draw one of minimal straight lines through any point, which is between one of conic sections and its axis [81].

Let the section first be the parabola ΑΒ whose axis ΒΗ. We take in the mentioned place the point Γ.

Then I say that it is possible for us to draw through Γ one of minimal straight lines.

[Proof]. For let from Γ the perpendicular ΓΔ [to the axis] be drawn. Let the half of the *latus rectum* be ΔΕ.

We draw from Ε the perpendicular ΕΘ to ΔΗ, and construct a hyperbola passing through Γ with asymptotes ΘΕ and ΕΗ, then this hyperbola will cut the parabola. So [let it cut it at Α, and] let the hyperbola be ΑΓ. We join the straight line ΑΓ and continue it to [meet ΕΔ at] Η [and to meet ΕΘ at Θ].

Then I say that ΑΗ is one of minimal straight lines.

[Proof]. For let The perpendicular ΑΖ be drawn. Then ΓΗ is equal to ΘΑ, as is proved in Theorem 8 of Book II. Therefore ΔΗ is equal to ΕΖ.

But ΕΔ is the half of the *latus rectum*. Therefore ΖΗ is the half of the *latus rectum*. So ΑΗ is one of minimal straight lines, as is proved in Theorem 8 of this Book.

[Proposition] 63

Furthermore we make the section the hyperbola or the ellipse ΑΒ whose axis ΒΛ and center Γ, and take in the mentioned place the point Δ.

I say that it is possible for us to draw through Δ one of minimal straight lines [82].

[Proof]. For let the perpendicular ΔΕ [to the axis] be drawn, and make the ratio ΓΘ to ΘΕ equal to the ratio of the transverse diameter to the *latus rectum*, and likewise [make] the ratio ΔΖ to ΕΖ [equal to the ratio of the transverse diameter to the *latus rectum*].

We draw ΚΗ [through Ζ] parallel to ΒΓ, and ΘΞ parallel to ΔΕ, and construct a hyperbola passing through Δ with asymptotes ΞΗ and ΗΚ. Then this section will cut the hyperbola and the ellipse, so [let it cut it at Α, and let the section be ΑΔ. We join the straight line ΑΔ and continue it [on both sides] to Ξ and Κ, and drop the perpendicular ΑΜ.

Then I say that ΑΛ is one of minimal straight lines.

[Proof]. ΞA is equal to ΔK, as is proved in Theorem 8 of Book II . Therefore HN is equal to ZK, and the ratio of KZ to the difference between KZ and EΛ is equal to the ratio ΔZ to ZE.

But KZ is equal to NH, and NH is equal to ΘM. Therefore the ratio of ΘM to the difference between ΘM and EΛ is equal to the ratio ΔZ to ZE.

But the ratio ΔZ to ZE is equal to the ratio ΓΘ to ΘE. Therefore the ratio of ΘM to the difference between ΘM and EΛ is equal to the ratio ΓΘ to ΘE, and *dividendo* in the case of the ellipse and *componendo* in the case of the hyperbola the ratio ΓM to MΛ is equal to MΛ the ratio ΓΘ to ΓE.

But the ratio ΓΘ to ΘE is equal to the ratio of the transverse diameter to the *latus rectum*, and MA is a perpendicular to ΓB. Therefore AΛ is one of minimal straight lines.

## [Proposition] 64

*If a point is taken below the axis of a parabola or a hyperbola, such that the straight line drawn from it to the vertex of the section forms with the axis an acute angle, and [such that] it is not possible to draw from that point to the section a straight line such that the part of it falling between the section and the axis is one of the minimal straight lines, or if only one of straight lines drawn from that point to one side [of the axis], which is different from the side where the point is, can have cut off from it [by the axis and the section] a minimal straight line, then the straight line drawn from that point to the vertex of the section is the shortest of the straight lines drawn from that point to that side of the section, and of the remaining straight lines those drawn closer to it are shorter than those drawn farther* [83].

Let the section first be the parabola ABΓ whose axis AE, and let there be the point Z below the axis AE and let there be the point Z below the axis, and let the angle ZAE which is formed by the straight line ZA drawn from Z to vertex of the section and the axis AE be an acute angle, and first let it not be possible to draw from Z to the section any straight line such that the part of it cut off between the section and the axis is one of minimal straight lines

Then I say that the shortest of straight lines drawn from Z to the section AΓ is AZ, and that of the remaining straight lines [drawn from Z to the section] those drawn closer to it are shorter than those drawn farther .

That will be proved after we prove that when straight lines drawn from Z ending at points of the section, in the case where not one of these straight lines can have a minimal cut off from it [between the axis and the section], then the minimal straight lines drawn from the points on the section and falling on the axis fall on that side of the straight lines drawn from Z which is farther from A. We prove that as follows.

We draw from Z the perpendicular ZE, then AE is either equal to the half of the *latus rectum*, or greater [than it], or smaller than it.

First let it be equal to it or smaller than it. Then for straight lines from drawn from Z to the section the part of them cut off between the section and the axis is not one of minimal straight lines, but the minimal straight lines drawn to the axis from the points to which [the straight lines drawn from Z] reach fall on that side of drawn straight lines which is farther from A, as is proved in Theorem 49 of this Book.

Furthermore we make AE greater than the half of the *latus rectum*, and let the half of the *latus rectum* be EΘ, and let ΘH be the double HA, and draw from H the perpendicular HB to AE, and [let Λ be such that] the ratio Λ to HB is equal to the ratio ΘH to ΘE, then ZE is either equal to Λ, or smaller than it, or greater than it.

Now that ZE is not equal Λ is evident for it was proved in Theorem 51 of this Book that when Λ is equal to EZ, then one straight line can be drawn from Z such that the part of it cut off between the section and the axis is a minimal straight line, but we have stated that no straight line can be drawn from Z such that the part of it cut off between the section and the axis is a minimal straight line. Therefore Λ is not equal to EZ.

Similarly too it will be proved that EZ cannot be smaller than Λ for it was proved in Theorem 51 of this Book that, when EZ is smaller than Λ, then there can be drawn from Z two straight lines such that the part which the axis cuts off from each of them is a minimal straight line, but we had made Z a point such that it is not possible to draw from it a straight line such that a minimal straight line is cut off from it between the axis and the section.

Therefore ZE is not smaller than Λ. And it was proved that is not equal to it.

And it was also proved in Theorem 51 of this Book that, when ZE is greater than Λ, then no straight line can be drawn from Z such that the part of it falling between the section and its axis is a minimal straight line, and the for the straight lines drawn from Z to the section, when minimal straight lines are drawn from their ends to the axis, they fall on the axis [removed] from those straight lines on the side which farther from A.

Therefore it has been proved that if AE is equal to for smaller than the half of the *latus rectum*, then it must be that for the straight lines drawn from Z to the section, when minimal straight lines are drawn from the points of their ends, they fall on the side which is farther from A [than the original straight lines], and [it has also been proved that] if AE is greater than the half of the *latus rectum*, then ZE is greater than Λ, as we proved, and in that case it must also be that for the straight lines drawn from Z to the section, when minimal straight lines are drawn from the points of their ends, they fall on the side which is farther from A.

Therefore since that has been proved, then I say that ZA is the shortest of the straight lines drawn from Z to the section ABΓ, and that of the remaining straight lines [drawn to ABΓ from Z], those drawn closer to it are shorter than those drawn farther.

[Proof]. For let ZB and ZΓ be drawn. Then, if possible, first let AZ be equal to BZ. We draw from A the straight line AK tangent to the section. Then AK is perpendicular to the axis AE, as is proved in Theorem 17 of Book I because it is parallel to the ordinates dropped on the axis. Therefore the angle ZAK is obtuse. Therefore we draw from A the perpendicular AN to AZ, then it falls in side of the section because it is not possible for any other straight line to fall between the tangent and section, as is proved in Theorem 32 of Book I .

We draw from B the tangent BΞ to the section. Then the minimal straight line drawn between B and the axis falls on the side of BZ farther from A, as we proved above. And [that minimal straight line] forms a right angle with BΞ, as is proved in Theorem 27 of this Book. Therefore the angle ZBΞ is acute.

So if we make Z center, and with radios BZ draw a circle, then [that circle] will cut BΞ. And NA will be outside of it for the angle ZBΞ is acute, and the angle NAZ is right.

Therefore let the circle be the circle BΞOA .Then it cuts the section AB, let it cut it at O.

We join OZ and draw OΔ tangent to the section. Then OΔ falls outside of the circle, and the minimal straight line drawn between O and the axis is farther from A than OZ, as we proved [above].

And it forms a right angle with OΔ, as is proved in Theorem 27 of this Book. Therefore the angle ΔOZ is acute, and OΔ cuts the circle. But it [also] fell outside of it, which is impossible. Therefore AZ is not equal to ZB.

So, if possible, let AZ be greater than ZB. Then, when we make Z center, and with the radius BZ draw a circle, the circle will cut AZ. And a part of BΞ will be inside of the circle, as we proved. And the circle will cut the section because it cuts AZ. Let [it cut the section at X, and let] the circle be BPX Ϙ.

We join ZX, and draw from X a tangent XΣ to the section. Then it falls inside the circle for the minimal straight line drawn between the axis and X falls on the side of XZ farther from A, and [hence] the angle ZXΣ is acute. Therefore ΣX cuts the circle.

But we had proved that it falls outside of it, which is impossible. Therefore AZ is not greater than BZ, and we had proved that it is not equal to it. Therefore it is smaller than it.

Then I say that of the remaining straight lines [drawn from Z to the section] those drawn closer to AZ are smaller than those drawn farther.

[Proof]. For let the tangent ΞB be continued to Y. Then the angle ZBΞ is acute [hence] the angle YBZ is obtuse. So we draw from B the perpendicular BM to BZ, then BM falls inside of the section. We draw from Γ the tangent ΓΩ to the section.

First let BZ, if that is possible, be equal to XZ. Then if we describe a circle on the center Z with the radius ZΓ, it will fall outside of ΓΩ because the angle ZΓΩ is acute. But it falls inside of BM because BM is perpendicular to BZ. Therefore it cuts the section.

And when we joined the point at which it cuts it and Z with a straight line, the absurdity of that is proved as is was in the case of the equality of AZ and ZB.

Similarly too if ZB is greater than ZΓ the impossibility is proved as it was proved in the case of AZ and ZB, where AZ was made greater than ZB. Therefore ZA is the smallest of the straight lines drawn from Z to the section ABΓ, and of the remaining straight lines those drawn closer to it are shorter than those drawn farther.

Therefore it has been proved that, if Z is in the situation that there cannot be drawn from it to the section any straight line such that the part of it cut off [between the axis and the section] is one of minimal straight lines, and the angle ZAE is acute, then the smallest of straight lines drawn from Z to the section is AZ, and that those [of the other straight lines] drawn closer to ZA are shorter than those drawn farther.

But if a minimal straight line can be cut off from only one of straight lines drawn from Z to the section, and the angle ZAE is again acute , then it will be proved, in Theorem 67 of this Book, that AZ is again the smallest of straight lines drawn from Z to the section, and that of the remaining straight lines those drawn closer to it are smaller than those drawn farther.

[Proposition] 65

Furthermore if we make the section the hyperbola ΑΒΓ with axis ΑΕ and center Δ, and take some point Ζ below the axis such that, when we join ΖΑ, the angle ΖΑΕ is acute and [such] that for none of straight lines drawn from Ζ to the section is the part of it cut off between the section and the axis one of minimal straight lines, then I say that ΖΑ is the shortest of straight lines drawn from Ζ to the section ΑΒΓ, and that of the remaining straight lines those drawn closer to it are shorter than those drawn farther [84].

[Proof]. All of minimal straight lines drawn from each of the points on the section ΑΒΓ to the axis ΑΕ fall on the side farther from Α than the straight line joining that point to Ζ for we draw from Ζ the perpendicular ΖΕ to the axis then ΑΕ is either equal to or greater than or smaller than the half of the *latus rectum*.

Now if it is equal to it or smaller than it, then for straight lines drawn from Ζ to the section ΑΒΓ, when minimal straight lines are drawn from their ends to the axis, they are farther from Α than those [straight lines], as is proved in Theorem 50 of this Book.

But if ΑΕ is greater than the half of the *latus rectum*, then we make the ratio ΔΘ to ΘΕ equal to the ratio of the transverse diameter to the *latus rectum*, and we imagine two straight lines ΗΔ and ΔΚ in continuous proportion between ΘΔ and ΔΑ, and draw from Κ the perpendicular ΚΒ to ΑΕ, And construct[the straight line Λ such that] the ratio Λ to ΚΒ is equal to the ratio pl.ΔΕ,ΘΚ to pl.ΔΚ,ΘΕ.

Then I say that ΖΕ is greater than Λ.

[Proof]. For let, if it is possible, for it not to be greater than it, then first let it be equal to it. Then it was proved Theorem 52 of this Book that in this case one can draw from Ζ a [single] straight line such that the part of it cut off [between the axis and the section] is one of minimal straight lines. But that is not so, therefore ΕΖ is not equal to Λ.

Similarly too it will be shown that ΖΕ is not smaller than Λ for if it were smaller than it, then it would be possible to draw from Ζ two straight lines such that the part of [each of] them cut off [between the axis and the section] is one of minimal straight lines, therefore ΖΕ is greater than Λ.

And it was proved in Theorem 52 of this Book that, when ΖΕ is greater than Λ, no straight line can be drawn from Ζ such that the pare of it cut off between the section and the axis is one of minimal straight lines, and that the minimal straight lines drawn from the ends of those straight lines are farther from Α than the straight lines themselves.

Therefore it has been proved that for all of straight lines drawn from Ζ to the section, when minimal straight lines are drawn from their ends two the axis, then these [minimal straight lines] are farter from Α than other straight lines.

And that will be proved by the method similar to that by which it was proved in the case of the parabola in the preceding theorem, that ΑΖ is smaller than all [other] straight lines drawn from Ζ to the section ΑΒΓ, and that of the remaining straight lines those drawn closer to it are smaller than those drawn farther.

[Proposition] 66

Furthermore we make the section the ellipse ΑΒΓ whose major axis ΑΓ and center Δ, with the point Δ below the major axis, and let the angle ΖΑΓ be acute, and draw from center Δ the perpendicular ΔϘ to the axis, and let Ζ be a point such that it is not possible to draw from it to [the quadrant] ΑϘ a straight line such that the part of it cut off between the section and the axis is one of minimal straight lines, then I say that ΑΖ is the shortest of straight lines drawn from Ζ to [the quadrant] ΑϘ, and that of the remaining straight lines those drawn closer to it are shorter than those drawn farther [85].

[Proof]. For the perpendicular drawn from Ζ to the axis falls between Α and Δ, for if it were possible for it to fall between Δ and Γ, then it would be possible to draw from Ζ to the section a straight line such that the part of it cut off between the section and the axis is one of minimal straight lines, as is proved in Theorem 55 of this Book, but that is not so, therefore the perpendicular does not fall between Δ and Γ.

Furthermore it does not fall on the center Δ for if it fell on the center Δ, when it is continued in a straight line, the part of it falling between the section and the axis would be one of minimal straight lines, as is proved in Theorem 11 of this Book. Therefore it falls between Α and Δ, as the perpendicular ΖΕ.

Now ΑΕ is either equal to the half of the *latus rectum*, or smaller than it, or greater than it.

But if it is smaller than it or equal to it, then for the straight lines drawn from Ζ to the section ΑϘ, no minimal straight line can be cut off from them [between the axis and the section], and when minimal straight lines are drawn from their ends to the axis, they fall on the side which is farther from Α than the straight lines themselves, as is proved in Theorem 50 of this Book.

And if ΑΕ is greater than the half of the *latus rectum*, we make the ratio ΔΘ to ΘΕ equal to the ratio of the transverse diameter to the *latus rectum*, and take two straight lines ΗΔ and ΔΚ in continuous proportion between ΑΔ and ΔΘ, and draw ΗΒ at right angles [to the axis], and construct [a straight line Λ such that] as Λ is to ΗΒ, so pl.ΔΕ,ΘΗ is to pl.ΔΗ,ΘΕ. Then ΖΕ is either equal to Λ or greater than it or smaller than it.

Now if ΕΖ is equal to Λ, then a [single] straight line can be drawn from Ζ to ΑϘ such that the part of it cut off [between the axis and the section] is one of minimal straight lines, as is proved in Theorem 52 of this Book. But that is not so.

And if ΕΖ were smaller than Λ, then there could be drawn [from Ζ to ΑϘ] two straight lines such that the parts of them cut off [between the axis and the section] are both minimal straight lines, and if ΕΖ is greater than Λ, then no straight line can be drawn from Ζ to ΑϘ such that the part of it cut off [between the axis and the section] is one of minimal straight lines, and when a straight line is drawn from Ζ to the section ΑϘ, the minimal straight line drawn from its and end to the axis is farther from Α than the straight line itself, as is proved in Theorem 52 of this Book.

Thus it has been proved in every case that the minimal straight lines drawn from every point of the section ΑϘ to the axis are farther from Α than the straight lines joining those points to Ζ.

Next we can prove, as we did in the case of the parabola that ΑΖ is shorter than all [other] straight lines drawn from Ζ to the section ΑϘ, and that of the remaining straight lines those drawn closer to it are shorter than those drawn farther.

And the proof for that is the same for all three sections, now that we have proved for each of the sections that the minimal straight lines drawn from the section to the axis fall on the side which is farther from Α than the straight lines themselves.

## [Proposition] 67

Furthermore we make the section the parabola or the hyperbola ΑΒΓ whose axis ΔΕ, and let there be some point Ζ below the axis, and let the angle ΖΑΕ be acute, and let there be just one straight line among those drawn from Ζ to the section such that the part of it cut off [between the axis and the section] is one of minimal straight lines, then I say again that ΖΑ is the shortest of straight lines drawn from Z to the section ΑΒΓ, and that of the remaining straight lines those drawn closer to it are shorter than those drawn farther [86] .

[Proof] . For let from Z to the axis perpendicular ZE be drawn. Then I say that for all straight lines drawn from Z to the section ΑΒΓ, then minimal straight lines are drawn from their ends to the axis, these straight lines are farther from A than the straight lines themselves, except for one single straight line.

For EA in the cases of the parabola and the hyperbola is greater than the half to the *latus rectum*, for if it were not greater than it, then it would not be possible to draw from Z a straight line such that the part of it cut off [between the axis and the section] is one of minimal straight lines, as is proved in Theorems 49 and 50 of this Book. Therefore AE is greater than the half to the *latus rectum*.

Then if the section is a parabola we cut off from AE next to E a straight line equal to the half of the *latus rectum*, and do the other construction as we did in Theorem 64 of this Book, until we find the constructed the straight line against which we measured EZ. Then EZ is equal to it for if it were smaller than it, then it would be possible to draw from Z two straight lines such that the part cut off from [each of] them [between the axis and the section] is one of minimal straight lines, as is proved in Theorem 51 of this Book. But that is not so.

Therefore ZE is equal to the constructed straight line. And it was proved in that theorem that when that is so, then only one straight line can be drawn from Z [to the section] such that the part of it cut off is one of minimal straight lines, and that the minimal strait lines drawn from the ends of other straight lines [between Z and the section] are farther from A than the straight lines themselves.

That will also be shown in the same way in this section if it is a hyperbola for we make the center Δ and divide ΔE into two parts such that the ratio of one to other is equal to the ratio of the transverse diameter to the *latus rectum*, and carry out the rest of the construction as we did in Theorem 65 of this Book until we find the constructed straight line against which we measured ZE.

Then in this case too, as in the case on the parabola, ZE is equal to the found straight line. Therefore only one straight line can be drawn from Z [to the section] such that the part of it cut off [ between the axis and the section] is one of minimal straight lines, and for other straight lines drawn from Z to the section, when minimal straight lines are drawn from their ends to the axis, these [minimal] straight lines are farther from A than the straight lines themselves, as is proved in Theorem 52 of this Book.

And a similar was shown too in the case of the parabola. Then let the straight line drawn from Z to the section ABΓ such that the part of it cut off by the axis is one of minimal straight lines ZB.

We draw from Z to the section between A and B two straight lines ZO and ZΠ. Then we prove as we proved in Theorem 64 of this Book that AZ is the shortest of straight lines drawn from Z and ending at the section between A and B, and that of the remaining straight lines such as ZO and ZΠ between those two points, those drawn closer to it are shorter than those drawn farther.

Then I say that ZΠ is shorter than ZB. For if it is not shorter than it, first, let it be equal to it. We draw ZK [to the section between ZΠ and ZB], then ZK is greater than ZΠ as we proved previously. Therefore it is greater than ZB.

So we cut off from ZK a straight line ZΦ greater than ZB but shorter than ZK, and make Z center and draw a circle with the radius ZΦ. Then it will cut the straight line KB and the arc KB of the section. So let it cut them as the circle ΦN [where N is on the section]. We join ZN, then ZK is closer than ZN to AZ. Therefore ZK is smaller than ZN. But KN is equal to ZΦ. ZK is smaller than ZΦ. But it was [constructed as] greater than it, that impossible. Therefore ZΠ and ZB are not equal.

Again we make, if possible, ZΠ greater than ZB, and cut off from ZΠ the straight line ZY greater than ZB but smaller than ZΠ. We make Z center and draw a circle with the radius ZY, then that circle will cut the straight line ZΠ and will cut the arc ΠB of the section. So let it cut them as the arc YIΛ, we join ZI. Then ZΠ is smaller than ZI because it is closer to AZ.

But ZI is equal to ZY. Therefore ZΠ is smaller than ZY, but that is impossible. Therefore ZΠ is not greater than ZB.

And we had [already] proved that it is not equal to it. Therefore it is smaller than it.

Thus it has been proved that all straight lines drawn from Z to [the arc] AB are shorter than ZB.

Again we draw Zι and ZΩ in the remaining arc BΓ of the section, on the other side of ZB. Then I say that ZB is smaller than Zι , and that Zι is smaller than ZΩ.

[Proof]. For let the tangents ιΨ and ΩX to the section be drawn. Then the angles ZιΨ and ZΩX are obtuse because the minimal straight lines drawn from ι and Ω to the axis are farther from A than straight lines drawn from their vertices to Z, each [being farther from A] than its corresponding [straight line].

Therefore we draw from ι the perpendicular ιΣ to Zι then it falls inside of the section. Then from that we can prove, as we proved in Theorem 64 of this Book that ιZ is shorter than ZΩ.

Similarly of the straight lines drawn from Z on the other side of ZB all of those drawn closer to A are smaller than those drawn farther.

And I say that ZB is the shortest of them.

[Proof]. The axis cuts off from ZB a minimal straight line. Therefore the angle between the tangent drawn from B and ZB is right.

First we make, if possible, ZB equal to Zι , and draw ZP between them. Then ZP is smaller than Zι because it is closer to AZ. Therefore ZP is smaller than ZB.

We make ZΞ [on ZB] smaller than ZB but greater than ZP, and make Z center, and draw a circle with the radius ZΞ, then it will cut BP between B and P. Let the circle be MTΞ, and let it cut it at T. We join ZT. Then ZT is smaller ZP because it is closer to AZ.

But ZT is equal to ZM. Therefore ZM is smaller than ZP. But it is [also ] greater than it, which is impossible. Therefore Zι is not equal to ZB.

Therefore, if possible, let it be smaller than it. We make ZϘ [on ZB] greater than Zι but smaller than ZB. Therefore when we make Z center and draw a circle with the radius ZϘ, it will cut the arc BI of the section let it cut it at Γ, and let it be the circle ϘςΘ. We join ςZ. Then ςZ is smaller than Zι because it is closer to AZ.

But Zς is equal to ZΘ. Therefore ZΘ is smaller than Zι . But it is [also] greater than it, which is impossible. Therefore Zι is not smaller than ZB. And we had [already] proved that it is not equal to it. Therefore it is greater than it . Therefore BZ is the shortest of straight lines drawn from Z to the arc BΓ of the section.

Thus it has been proved from what we said, that AZ is shorter than all straight lines drawn from Z to ABΓ, and that of the remaining straight lines those drawn closer to it are shorter than those drawn farther.

[Proposition] 68

If AB is the parabola whose axis BΓ, and AΔ and ΔE are the tangents to the section [where E is closer to the vertex B than A], then EΔ is smaller than ΔA [87] .

[Proof]. For let AE be joined and from Δ the straight line ΔH [meeting

AE at H] parallel to BΓ be drawn. Then AH is equal to EH, as is proved in Theorem 30 of Book II . We draw from A the perpendicular AΓ to the axis. Then the angle AΘΔ is right, therefore the angle AHΔ is obtuse. And ΔH is common to the triangles AΔH and EΔH. Therefore the sides AH and HΔ are [respectively] equal to the sides EH and HΔ. And the angle EHΔ is smaller than the angle AHΔ. Therefore the base EΔ is smaller than the base AΔ.

[Proposition] 69

If there is the hyperbola AB whose axis ΔE and center E, and two tangents to it ZH and HA [where Z is closer to the vertex B], ZH is smaller than HA [88].

[Proof]. For let BH is joined and continued in a straight line two [meet AZ at] Γ, and AΓZ be joined. Then AΓ is equal to ΓZ, as is proved in Theorem 30 of Book II . Therefore we draw the perpendicular AΘΔ, and continue EΓ to [meet it at] Θ. Then the angle AΔE is right, and the angle AΘE is greater than the angle AΔE therefore the angle AΘE is obtuse, and the angle HΓA is obtuse. Therefore the angle HΓZ is smaller than the angle HΓA. And AΓ is equal to ΓZ, and ΓH is common to the triangles AΓH and ZΓH. Therefore the base ZH is smaller than the base HA.

[Proposition] 70

If there is the ellipse ABΓΔ whose major axis AΓ and minor [axis] BΔ, and there are drawn between B and Γ on one of the quadrants of the section, and two tangents PH and ΘH to the section, then the closer of these two to the minor axis is greater than the farther [89].

[Proof]. For let ΘP be joined, and HZ be drawn from H to the center Z [cutting ΘP at E]. Then PE is equal to EΘ, as is proved in Theorem 30 of Book II. And EP is closer to ZB, the half of the minor axis, than ZΘ , and ZΘ is closer to ZΓ, the half of the major axis. Therefore ZΘ is greater than ZP.

And EΘ and EZ are [respectively] equal to PE and EZ. Therefore the angle ΘEZ is greater than the angle PEZ, and the angle PEH is greater than the ΘEH. And PE and EH are [respectively] equal to ΘE and EH. Therefore the base PH is greater than the base ΘH.

[Proposition] 71

If ABΓ is the ellipse whose major axis AΓ and minor axis BH [and center Δ], and XE and ΘΦ are perpendiculars to the major axis, XE being greater than

ΦΘ, and XY and ΘY are tangent to the section, and it is evident that they will meet each because of that we said in Theorem 27 of Book II, then XY is greater than ΘY [90].

[Proof]. For let ΘKX and ΔKY be joined, and let XE be continued to [meet the section at] Λ, and let ΛΔ be joined and continued to [meet the section at] O. Then ΛΔ is equal to ΔO, as is proved in Theorem 30 of Book I .

And ΛE is equal to EX, and ΔE is a perpendicular to ΛX. Therefore ΛΔ is equal to ΔX.

But ΛΔ was [shown to be] equal to ΔO. Therefore ΔX is equal to ΔO.

We join OX, then it is parallel to EΦ. And when we draw the perpendicular OΠ [to the major axis], it is also parallel to XE, therefore it is equal to it.

But XE was [assumed] greater than ΘΦ. Therefore OΠ is greater than ΘΦ. Therefore ΔΘ is closer to [the half of the major axis] ΓΔ than ΔO. Therefore ΔΘ is greater than ΔO, as is proved in Theorem 11 of this Book.

And we had proved that ΔO is equal to ΔX. Therefore ΔΘ is greater than ΔX.

But ΘK is equal to KX as is proved in Theorem 30 of Book II. Therefore the angle ΔKΘ is greater than the angle ΔKX, and the angle YKX is greater than the angle YKΘ. And the sides XK and KY are [respectively] equal to the sides ΘK and KY. Therefore the base XY is greater than the base ΘY.

[Proposition] 72

*If a point is taken below the axis of a parabola or a hyperbola, and it is possible to draw from it two straight lines such that the part which the axis cuts off from each of them is one of minimal straight lines, then the closer of those two straight lines to the vertex of the section is greater than all [other] straight lines drawn from that point to the arc of the section from the vertex of the section to the other, second, straight line, and of the remaining straight lines drawn to that arc on both sides those drawn closer to it are greater than those drawn farther, and second straight line is smaller than all straight lines drawn from the point to the remaining [part] on that side of the section, that is the complement of the first arc on that side, and of the remaining straight lines drawn to that other [complementary] arc those drawn closer to it are smaller than those drawn farther* [91].

Let the section be ABΓ with the axis ΓE, and the point Δ below it, and two straight lines ΔA and ΔB drawn from it to the section such that the parts that cuts off them are two minimal straight lines.

I say that ΔB is greater than all [other straight lines drawn from Δ to the arc] ΓBA, and that those [straight lines] on both sides, which are closer to ΔB are greater than those drawn farther, and that ΔA is smaller than all straight lines drawn from Δ to AP [where P is an arbitrary point on the other side of A from B], and that of those straight lines those drawn closer to ΔA are smaller than those drawn farther.

[Proof]. For let from Δ the perpendicular ΔE to ΓE be drawn. We construct against the straight line which we measure ΔE as we constructed it in Theorems 64 and 65 of this Book. Then ΔE is smaller than that straight line for if it were greater than it, it would not be possible to draw from Δ a straight line such that the part of it cut off [between the axis and the section] is one of minimal straight lines, and if it were equal to it, then it would be possible to draw only one straight line [of that kind], as is proved in Theorems 51 and 52 of this Book.

Therefore since ΔE is smaller than the constructed straight line, then only two straight lines can be drawn from it such that the part of [each of] them cut off is one of minimal straight lines, and the minimal straight lines drawn from the ends of the straight lines between ΔA and ΔB are closer to A than the straight lines themselves, but as for minimal straight lines drawn from the ends of the remaining straight lines, they are farther [from the vertex], as is proved in Theorems 51 and 52 of this Book.

Now as to [the statement] that ΔB is greater than all [other] straight lines drawn from Δ to [the arc] ΓB, which will be proved as we proved it in Theorem 64 of this Book.

And similarly it will be proved that of those straight lines which are on the side of Γ [from B] those drawn closer to ΔB are greater than those drawn farther.

But as to [the statement] that ΔB is the greatest of the straight lines drawn [from Δ] to [the arc] AB, and that of those straight lines drawn closer to it are greater than those drawn farther, that will be proved as follows. We draw ΔM and ΔN [between ΔB and ΔA] and draw from B and M tangents BΞ and XMΘ to the section. Then BΠ is one of minimal straight lines, and BΞ is tangent to the section, so the angle ΞBΠ is right, as is proved in Theorems 27 and 28 of this Book, and the angle ΞMΔ is obtuse because the minimal straight line drawn from M to [the axis] ΓE is closer to Γ than MΔ, as is proved in Theorems 51 and 52 of this Book. And [thus] the angle ΞBΔ is right, and the angle ΞMΔ is obtuse. Therefore the sum of sq.ΞB and sq.BΔ is greater than the sum of sq.ΞM and sq.MΔ.

But ΞB is smaller than ΞM, as is proved in Theorems 68 and 69 of this Book. Therefore BΔ is greater than ΔM.

Similarly too it will be proved that MΔ is greater than ΔN because the angle ΘMΔ is acute, and, when we make NΘ tangent the angle ΘNΔ is obtuse.

Similarly also it will be proved that NΔ is greater than ΔΛ.

Therefore ΔB is greater than all [other] straight lines drawn from Δ to the arc AΓ of the section, and of those straight lines drawn closer to it are greater than those drawn farther.

Now as to [the statement] that ΔA is smaller than all straight lines drawn from Δ to [the arc] AP, which will be proved by a method like we followed in Theorem 64 of this Book.

And similarly too it will be proved that of straight lines drawn [from Δ] to AP those [straight lines] drawn closer to AΔ are smaller than those drawn farther.

[Proposition] 73

*If a point is taken below the major or two axes of an ellipse not on the continuation of the minor axis, and of straight lines drawn from that point to the section only one can have cut off from it [between the major axis and the section] one of minimal straight lines, then only that [minimal] straight line is greater than all other straight lines [drawn from that point to the section], and of the remaining straight lines those drawn closer to it are greaten than those drawn farther, and the shortest on straight lines drawn from that point to that half of the section to which the greatest straight line is drawn is the straight line joining that point and the vertex of the section adjacent to that point* [92].

Let there be the ellipse ABΓ whose [major] axis AΓ and center Δ. We draw through Δ the perpendicular BΔE to the axis, and take below the axis the point Z, let Z be a point such that only one straight line can be drawn from it to ABΓ such that the part of it which the axis AΓ cuts off is one of minimal straight lines.

Now concerning this straight line from which a minimal straight line is cut off, since no other straight line can be drawn from that point to the section such that the axis cuts from it one of minimal straight lines, but it is [always] possible for us to draw from Z [just one] straight line such that the part of it cut off by the axis is one of minimal straight lines, provided that it cuts the other one of two halves of the axis, that is to say the half on which the perpendicular drawn from Z [to the axis] does not fall, as is proved in Theorem 55 of this Book, Therefore the straight line drawn from Z to ABΓ such that the part cut off from it is one of minimal straight lines cuts ΓΔ.

So let that straight line be ZHΘ, we join ZA.

Then I say that ZΘ is the greatest of straight lines drawn from Z to ABΓ, and that of straight lines on either side of it those drawn closer to it are greater than those drawn farther, and that the shortest of all them is ZA.

[Proof]. The section ABΓ is the ellipse, and Z has been taken below its major axis, being a point such that only one straight line can be drawn from it to the section such that a minimal straight line can be cut off from it.

Now it has been proved in Theorem 57 of this Book that, when that is the case, the remaining minimal straight lines drawn from a point on the section to the axis, whatever point that may be, are farther from A or from Γ, than the straight lines joining that point to Z, and that can be proved for any of straight lines whether they are farther from A, or from Γ. So we draw some straight lines ZK, ZΛ, and ZM from Z to the section [where K and Λ are on AB, and M is on BΘ], and draw from A a tangent AΞ to the section, then the angle ZAΞ is obtuse. So we draw from A the perpendicular AO to AZ, then it falls in side of the section, as is proved in Theorem 32 of Book I.

We draw from K the tangent ΠKP to the section. Then the minimal straight line drawn from K to the axis is farther from A than KZ, as is proved in Theorem 57 of this Book. Therefore the angle ΠKZ is acute. But the angle OAZ was [made] right . So we can prove as we proved in Theorem 64 of this Book by drawing the perpendicular [to ZK] from K, that AZ is not greater than ZK, and not equal to it. Therefore AZ is smaller than ZK.

Furthermore ΠKP is tangent to the section, and the angle PKZ is obtuse, so we draw from K the perpendicular KϘ to KZ. Then it falls in side of the section, since no straight line can fall between the tangent and the section, as is proved in Theorem 32 of Book I.

We also draw through Λ the tangent TΛY to the section. Then the minimal straight line drawn from Λ is farther from A than ΛZ, as is proved in Theorem 57 of this Book. Therefore the angle TΛZ is acute. So again it can be proved as it was proved in Theorem 64 of this Book that ZK is smaller than ZΛ.

Furthermore we join ZB and draw through B the tangent XBΨ to the section, then the angle XBΔ is right, and the angle XBZ is acute. And therefore ΛZ is smaller than ZB, as is proved in Theorem 64 of this Book.

I also say that ZB is shorter than ZM for we draw through M the tangent ΨMΩ to the section. Then since ABΓ is an ellipse, and the perpendicular BΔE to its axis, has been drawn through its center, and BΨ and ΨM are tangents, then

BΨ is greater than ΨM, as is proved in Theorem 70 of this Book. But the sum of sq.ΨB and sq.BZ is smaller than the sum of sq.ZM and sq.MΨ because the angle ΨBZ is obtuse, and the angle ΨMZ is acute. Therefore ZB is smaller than ZM.

Similarly too it will be proved that ZM is smaller than ZN by drawing [the tangent] ΩNI.

So it has been proved that of these straight lines those drawn closer to Θ are greater than those drawn farther.

Now I say that ΘZ is greater than ZN. We draw ΘI tangent to the section. Then the angle IΘZ is right, as is proved in Theorem 28 of this Book, and the angle INZ is obtuse, and NI is greater than IΘ , as is proved in Theorem 71 of this Book. Therefore ΘZ is greater than ZN. Therefore ΘZ is the greatest of straight lines drawn from Z to [the arc] AΘ, and of these straight lines those drawn closer to it are greater than those drawn farther,
and ΛZ is the shortest of them.

So we draw Zι ,Zζ and ZΓ to [the arc] ΘΓ, and draw from Γ the tangent Go to the section, and Γ□ perpendicular to ΓZ. Then it falls in side of the section, as is proved in Theorem 32 of Book I .

So we draw from ζ the tangent ζΦ to the section. Then the minimal straight line drawn from ζ to the axis is farther from Γ than ζZ, therefore the angle ΦζZ is acute. Hence it will be proved that ZΓ is smaller than Zζ , and we will prove as we proved in Theorem 64 of this Book that of straight lines drawn from Z to the section between ZΓ and ZΘ those drawn closer to ZΓ are shorter than those drawn farter. Therefore Zζ is smaller than Zι.

Then I say that Zι is smaller than ZΘ.

[Proof]. If it is not smaller than it, then it is equal to it or greater than it.

So it possible let it be greater than it. We make ZΣ greater than ZΘ and smaller than Zι .Then when we make Z center, and draw a circle with the radius ZΣ, then it will cut the arc Θι of the section, let it cut it at the point α, as the circle Σαβ . We join Zα, then Zα is farther from ZΓ than Zι . Therefore Zα is greater than Zι.

But Zα is equal to Zβ, therefore Zβ is greater than Zαι. But it is [also] smaller than it, that is impossible. So Zι is not greater than ZΘ.

So, if possible, let it be equal to it. We draw between these two straight lines Zγ. Then Zγ is greater than Zι, therefore Zγ is greater than ZΘ. So we make Zδ greater than ZΘ and smaller than Zγ. Then when we make Z center, and draw a circle δεσ, with the radius Zδ it will cut the arc Θγ of the section, let it cut it at ε. We join Zε. Then Zε is greater than because it is farther from ZΓ.

But $Z\varepsilon$ is equal to $Z\delta$, therefore $Z\delta$ is greater than $Z\gamma$. But it is [also] smaller than it, which is impossible. Therefore $Z\iota$ is greater than $Z\Theta$.

So $Z\Theta$ is the greatest of straight lines drawn from $Z$ to the section $AB\Gamma$, and those [straight lines] drawn closer to it are greater than those drawn farter, and $Z\Gamma$ is the shortest of straight lines drawn from $Z$ to [the arc] $\Gamma\Theta$. But $Z\Gamma$ is greater than $ZA$.

Therefore $ZA$ is the shortest of straight lines drawn from $Z$ to the section $AB\Gamma$, and the greatest of them is $Z\Theta$, and those [straight lines] drawn closer to it are greater than those drawn farther.

[Proposition] 74

*If a point is taken below the major of the axes of an ellipse, and it is possible for us to draw from that point to the arc of the section opposite to it just two straight lines such that the parts cut off from them [by the axis] are minimal straight lines, then the greatest of straight lines drawn from that point to that side of the section is that one of two straight lines from each of which a minimal straight line can be cut off which meets the minor axis, and of straight lines on either side of it those drawn closer to it are greater than those drawn farther, and the shortest of those straight lines is the straight line drawn from that point to that one of two vertices of the section which is closer to it* [93].

Let the ellipse be $AB\Gamma$ whose major axis $A\Gamma$, and let there be a point $Z$ below the major axis, and let the center of the section be $\Delta$.

We draw through $\Delta$ the perpendicular $B\Delta E$ to the axis. Let it be possible for us to draw from $Z$ just two straight lines such that the parts of them cut off between $AB\Gamma$ and the axis of the section are minimal straight lines, let those two straight lines which we stated to be drawn from $Z$ be $ZH$ and $Z\Theta$, and let there be no other straight line apart from them which can be drawn from it so that the part of it cut off [by the axis] is one of minimal straight lines.

Then I say that $Z\Theta$ which cuts the minor axis is the greatest of all straight lines drawn from $Z$ to the section $AB\Gamma$, and that [for straight lines] on both sides of it those drawn closer two $Z\Theta$ are greater than those drawn farther, and that $ZA$ is the shortest of mentioned those straight lines.

[Proof]. For let from $Z$ the perpendicular $ZN$ to the axis be drawn. Then it is evident that $ZN$ does not fall on the center for if it were to fall on the center, then it would be impossible to draw from $Z$ a straight line such that the part of it which the axis cuts off is one of minimal straight lines except for perpendicular $ZN$ alone [when continued to meet the section], or [else] would be possible to draw two straight lines besides it such that the part of each of them cut off [by the axis] is one of minimal straight lines, as is proved in Theorems 53 and 54 of this Book. But that is not the case [here by hypothesis].

So let the perpendicular ZN fall between A and Δ. Then AN is greater than the half of the *latus rectum* for, if it were not greater than it, then it would not be possible to draw from Z a straight line between A and B such that the part of it cut off [by the axis] is one of minimal straight lines, as is proved in Theorem 50 of this Book. Therefore AN, as we said, is greater than the half of the *latus rectum*.

So we make the ratio ΔK to KN equal to the ratio of the transverse diameter to the *latus rectum*, and take two mean proportionals between AΔ and ΔK, and construct the perpendicular as we constructed it in Theorem 64 of this Book, and do the rest of what we did so as to generate the straight line against which we measure ZN.

Then ZN is equal to that generated straight line for if it were greater than it, then it would not be possible to draw from Z to AB a straight line such that the part of it cut off [by the axis] is one of minimal straight lines, and if it were smaller than it, then it would be possible to draw to [the quadrant] AB two straight lines such that the part of them cut off [by the axis] is one of minimal straight lines, as is proved in Theorem 52 of this Book, and it would also be possible to draw from Z another, third, straight line to [the quadrant] BΓ, as is proved in Theorem 55 of this Book. Therefore, ZN is equal to the generated straight line.

And it was proved in Theorem 52 of this Book that, when that is the case, then only one straight line can be drawn from Z to [the quadrant] AB such that the part of it cut off [by the axis] is one of minimal straight lines, and that the minimal straight lines drawn from the ends of the remaining straight lines drawn two AB are farther from A than the straight lines themselves.

So we draw from Z to the section the straight lines ZA, ZO, and ZΠ. Then it will be proved, as we proved in Theorems 72 and 73 [of this Book] that ZA is smaller than ZO, and ZO is smaller than ZΠ.

Then I say that ZΠ is smaller than ZH for if it is not smaller than it, let it be greater than it or equal to it, and, first it be equal to it. We draw between them ZY, where ZY is greater than ZΠ, and ZΠ is equal to ZH. Therefore ZY is greater than ZH. So we cut off from ZY the straight line ZI shorter than ZY, but greater than ZH, make Z center and draw the circle IΛM with the radius ZI, then it cuts the arc YH [of the section], Let it cut it at Λ. We join ZΛ. Then ZΛ is greater than ZY because it is farther from ZA.

And ZΛ is equal to ZI, therefore ZI is greater than ZY. But it is [also] smaller than it, which is impossible.

In a similar way it will be proved that ZH is not smaller than ZΠ. Therefore it is greater than it. So ZH is greatest of straight lines drawn from Z to [the arc] AH, and of these straight lines those drawn closer to it are greater than those drawn farther,  and the shortest of them is ZA.

Similarly too it will be proved that ZB is the greatest of straight lines drawn between H and B, and that of these straight lines those drawn closer to it are greater than those drawn farther, just as we proved the matter of straight lines drawn to [the arc] AH.

Then I also say that ZH is the smallest of straight lines drawn to [the arc] HB.

[Proof]. For let ZΣ be drawn [to HB]. Then, if it is possible, for ZΣ not to be greater than ZH, it is equal to it or smaller than it.

First, let it be equal to it. We draw ZΞ between ZH and ZΣ. Then ZΞ is smaller than ZΣ, therefore ZΞ is smaller ZH. We make ZϘ greater than ZΞ but smaller than ZH and make Z center, and draw the circle ϘPT with the radius ZϘ . Then it will cut the arc ΞH [of the section], let it cut it at P. We join ZP.  Then ZP is smaller than ZΞ because it is farther from ZB, and ZP is equal to ZT. Therefore ZT is smaller ZΞ . But it is [also] greater than it, which is  impossible. So ZΣ is not equal to ZH.

Similarly too it will be proved that it is not greater than it.

Therefore ZB is greater than all [other] straight lines drawn from Z to [the quadrant] BA, and of these straight lines those drawn closer to it are greater than those drawn farther.

Now ABΓ is the ellipse whose major axis AΓ and minor axis BΔE,  with Z inside of the angle AΔE, from which ZΘ has been drawn to the arc BΓ of the section. So it will be proved as we proved in the preceding theorem that ZΘ is the greatest of straight lines drawn from Z to BΓ, and that of these straight lines those drawn closer to it are greater than those drawn farther.

And it has [already] been proved that ZB is the greatest  if straight lines drawn to [the arc] AB, and that of these straight lines those drawn closer to it are greater than those drawn farther.

So ZΘ is the greatest of straight lines drawn from Z to the section ABΓ, and of the remaining straight lines those drawn closer to it are greater than those drawn farther, and ZA is the smallest of them.

## [Proposition] 75

*If a point is taken below the major of two axes an ellipse, and it is possible to draw from it to the section three straight lines such that the parts of them which the axis cuts off are minimal straight lines, two of these straight lines being on that one of two sides of the minor axis on which is the point, and one straight line being on the opposite side, then of straight lines drawn from that point to the arc of the section between the midmost of three straight lines and that vertex of the section which is farther from the point, the greatest is that one of three straight lines which is drawn on the side opposite to that in which is the point, and those of these straight lines drawn closer to it are greater than those drawn farther, but as for straight lines drawn from that point to the section which is between the midmost of three straight lines and that vertex of the section which is next to the point, the greatest of them is the straight line next to that vertex of the section which is on the side on which is the point, and those of these straight lines which are closer to it are greater than those which are farther, and the greatest of these straight lines and [also] of other straight lines mentioned previously is that one of three straight lines which is drawn to the side opposite to the side on which is the point* [94].

Let there be the ellipse ΑΒΓ whose major axis ΑΓ and center Ξ. Let the perpendicular passing through the center be ΒΞ, and the point below the axis be Ε. And let there be drawn from it three straight lines ΕΗ, ΕΖ, and ΕΔ such that the parts cut off from them [by the axis] are minimal straight lines , two of these straight lines ΕΖ and ΕΔ are on the side [of the minor axis] on which is Ζ, and one straight line ΕΗ is on other side.

Then I say that ΕΗ is the greatest of straight lines drawn from Ε to the section ΑΒΓ, and that of straight lines between Δ and Α those drawn closer to it on both sides are greater than those drawn farter, and that ΖΕ is the greatest of straight lines drawn between Γ and Δ, and that those of these straight lines that are closer to it are greater than those drawn farther.

[Proof]. ΔΑ and ΖΘ are minim al straight lines. So we will prove as we proved in the case of the parabola in Theorem 72 of this Book that ΕΖ is the greatest of straight lines drawn from Ε to [the arc] ΓΒ, and that of these straight lines those drawn closer to it are greater than those drawn farther.

Furthermore ΔΛ is one of minimal straight lines, and ΗΚ is also one of minimal straight lines. So it will be proved then, as is was proved in the preceding theorem that ΕΗ is the greatest of straight lines drawn from Ε to[the arc] ΑΔ.

And I also say that ΕΗ is greater than ΕΖ. For let from Ζ, Η, and Ε the perpendiculars ΖΜ, ΗΝ, and ΕΟ be drawn. Then the ratio ΜΞ to ΜΘ is equal to the ratio of the transverse diameter to the *latus rectum* as is proved in Theorem 15 of this Book.

And likewise too the ratio ΞN to NK is equal to the ratio of the transverse diameter to the *latus rectum*, as is proved in Theorem 15 of this Book. Therefore the ratio ΞM to MΘ is equal to the ratio ΞN to NK.

But the ratio OM to MΘ is smaller than τηε ratio ΞM to MΘ. Therefore the ratio OM to MΘ is smaller than the ratio ΞN to NK. Therefore the ratio OM to MΘ is much smaller than the ratio ON to NK. And *dividendo* the ratio OΘ to ΘM is smaller than the ratio OK to KN.

Now as for the ratio OΘ to ΘM, it is equal to the ratio EO to ZM, and as for the ratio OK to KN, it is equal to the ratio EO to HN. Therefore the ratio EO to ZM is smaller than the ratio EO to HN. Therefore ZM is greater than HN.

Therefore the straight line drawn from Z parallel to AΓ is farter from A than H, let that straight line be ZΠ [which cuts ΞB at Σ].

We continue the perpendicular EO to [meet ZΠ at] P. Then ZΣ is equal to ΣΠ. Therefore PΠ is greater than ZP.

And EP is common to the triangles EPZ and EPΠ, and is a perpendicular to ZΠ. Therefore EΠ is greater than EZ. But EH is greater than EΠ. Therefore EH is greater than EZ. So EH is the greatest of straight lines drawn from E to the section ABΓ.

And the situation with to straight lines drawn closer to and farter from it is as we declared in the enunciation.

[Proposition] 76

*If a perpendicular is drawn some point to the major axis of an ellipse, so as to fall on its center, and no other straight line can be drawn from that point to one of quadrants of the section which are on the opposite side of the section to the side in which is the point, such that the part of it cut off [by the axis] is one of minimal straight lines, then the greatest of straight lines drawn from that point to the section is that perpendicular, when continued [to meet the section], and of the remaining straight lines [drawn from that point], those drawn closer to it are greater than those drawn farther* [95].

Let the ellipse be ABΓ whose major axis AΓ, and the taken point be E, and the perpendicular drawn from it to the center be EΔ, which has been continued to [meet the section at] B. And let it not be possible to draw from E to [the quadrant] BΓ any straight line except BΔ such that the part of it cut off [by the major axis] is one of minimal straight lines.

Then I say that EB is the greatest of straight lines drawn from E to [the quadrant] BΓ.

[Proof]. No straight line can be drawn from E to the section between B and Γ such that the part of it cut off is one of minimal straight lines.

And [so] the minimal straight lines drawn from the ends of those straight lines are farther from Γ than the straight lines themselves, as is proved in Theorem 53 of this Book. Hence it will be proved by means of the tangents, as it was proved in Theorem 72 of this Book, that EB is the greatest of straight lines drawn from E to the quadrant AB.

And similarly it will be proved that it is the greatest of straight lines drawn [from E] to the other quadrant. Therefore it is the greatest of straight lines drawn from E to the section.

And [it will be proved] that those of these straight lines that are closer to it are greater than those drawn farther.

## [Proposition] 77

*If a perpendicular is drawn from some point to the major of two axes on an ellipse, so that it falls on the center, and it is possible to draw from that point to a quadrant of the section [one] straight line such that the part of it cut off by the axis is one of minimal straight lines, then that straight line is greatest of straight lines drawn from that point to that quadrant, and of these straight lines those drawn closer to it are greater than those drawn farther*[96].

Let the ellipse be ABΓ whose major axis AΓ and center Δ, and the point taken below is E from which the perpendicular EΔ has been drawn to AΓ, and let it be possible to draw from it to ΓB only one straight line such that the part of it cut off [by the axis] is one of minimal straight lines, let that straight line be EHZ.

Then I say that EZ is the greatest of straight lines drawn from E to [the quadrant] BΓ, and that those [straight lines] drawn closer to it on both sides are greater than those drawn farter.

[Proof]. For let BΔ and ZH are two minimal straight lines which have been continued to meet at E. So the minimal straight lines drawn from [any] point on the section between Γ and Z are farter from Γ than the straight lines joining that point and E, as is proved in Theorem 46 of this Book. And the minimal straight lines drawn from [any] point on the section between B and Z are closer to Γ than the straight lines joining that point and E, as is proved in Theorem 46 of this Book. And when that is the case, then it can be proved, as it was proved in Theorem 72 of this Book by means of the tangents, that EZ is the greatest of the straight lines drawn from Ε to ΒΓ, and that of these straight lines those drawn closer to it are greater than those drawn farther.

# BOOK SIX

## Preface

Apollonius greets Attalus

I have sent you the sixth Book of the Conics. My aim in it is to report on conic sections which are equal[1] to each other and those unequal to each other, and those unequal to each other, and on those similar to each other and dissimilar to each other, and on segments of conic sections. In this we have enunciated more than what was composed by others among our predecessors. In this Book there is also how to find a section in a given right cone equal to a given section, and 257or to find a right cone, containing a given conics section, similar [2] to a given cone. What we have stated on this [subject] is fuller and clearer than the statements of our predecessors. Farewell.

## Definitions

1. Conic sections which are called equal are those which can be fit one on another, so that the one does not exceed the other[3] Those which are said to be unequal are those for which that is not so.

2. And similar [conic section] are such that, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates are drawn in them to the lengths they cut off from the vertex of the section are equal to one another, while the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios [4]. Sections that are dissimilar are those in which what we stated above does not occur.

3. The line that subtends a segment of the circumference of a circle or of a conic section is called the base of that segment [5] .

4. The line that bisects all the lines drawn in that segment parallel to the base is called the diameter to that segment [6] .

5. And the point on the section from which the diameter is drawn is called the vertex of the segment [7].

6. Segments that are called equal from their bases up are those that can be applied, one to another, so that one does not exceed the other. And seg-

ment that are called unequal are those for which what we stated is not the case.

7.And segments that are called similar are those in which the angles formed between their bases and their diameters are equal, and for which, an equal number of lines having been drawn in each of them parallel to their base, the ratios of these lines, and also the ratio of each base, to the ratios of these lines, and also the ratio of each base to the lengths which they cut off from the diameter from the vertex of the section are equal for every segment similarly the ratio of the part cut off from the diameter of one to the part cut off from the diameter of the other.

8.A conic section is said to the be placed in a cone, or a cone is said to contain a conic section, when the whole of the section is in the surface bounding the cone between its vertex and its base, or in that surface after it has been produced beyond the base, so that the whole of the section is in the surface below the base, or else some of the section is in this surface and some in the other surface.

9. Right cones that are said to be similar are those for which the ratios of their axes to the diameters of their bases are equal.

10.The *eidos* that I call the *eidos* of the section corresponding to the axis or to the diameter is that [*eidos*] under the axis or diameter together with the *latus rectum* [8].

[Proposition] 1

*Parabolas in which the latera recta which are perpendiculars to the axes are equal, them selves equal, and if parabolas are equal, their latera recta are equal* [9].

Let there be two parabolas whose axes AΔ and ZΘ and equal *latera recta* AE and ZM.

I say that these sections are equal.

[Proof]. When we apply the axis AΔ to the axis ZΘ, then the section will coincide with the section so as to fit on it for if it does not fit on it, let there be a part of the section AB that does not fit on the section ZH. We take the point B on the part of it that does not coincide with ZH, and draw from it [to the axis] the perpendicular BK, and complete the rectangular plane KE. We make ZΛ equal to AK, and draw from Λ the perpendicular ΛH to the axis [meeting the section at H], and complete the rectangular plane ΛM. Then KA and AE are equal to ΛZ and ZM each to its correspondent.

Therefore the quadrangle KE is equal to the quadrangle ΛM. And KB is equal in square to the quadrangle EK, as is proved in Theorem 11 of Book I.

And similarly too ΛH is equal in square to the quadrangle ΛM. Therefore KB is equal to ΛH.

Therefore when the axis [of one section] is applied to the axis [of the other], AK will coincide with ZΛ, and KB will coincide with ΛH, and B will coincide with H. But it was supposed not to fall on the section ZH, which is impossible. Therefore it is impossible for the section [AB] not to be equal to the section [ZH]

Furthermore we make the section [AB] equal to the section [ZH], and make AK equal to ZΛ, and draw the perpendiculars [to the axis] from K and Λ , and complete rectangular planes EK and MΛ, then the section AB will coincide with the section ZH, and therefore the axis AK will coincide with the axis ZΛ for if it does not coincide with it, the parabola ZH has two axes which is impossible.

Therefore let it coincide with it. Then K will coincide with L because AK is equal to ZΛ, and B will coincide with H. Therefore BK is equal to ΛH, the quadrangle EK is equal to the quadrangle ΛM, AK is equal to ZΛ, and AE is equal to ZM.

## [Proposition] 2

*If the* eidoi *corresponding to the transverse axes of hyperbolas of ellipses are equal and similar*[10], *then the sections will be equal, and if the sections are equal, then the* eidoi *corresponding to their transverse axes are equal and similar, and their situation is similar*[11] .

Let there be two hyperbolas or ellipses AB and ΓH whose axes AK and ΓΘ. Let the *eidoi* corresponding to their transverse axes be equal and similar, these are ΔE and NΛ.

I say that the sections AB and ΓH are equal.

[Proof]. We apply the axis AK to the axis ΓΘ, then the section [AB] will coincide with the section [ΓH] for if that it no so, let a part of the section AB not coincide with the section ΓH we take the point B on that part, and draw from it the perpendicular BK to the axis, and complete the rectangular plane ΔZ We cut off from ΓΘ a segment ΓΘ equal to AK, and draw from Θ the perpendicular ΘH to ΓΘ, and complete the rectangular plane NM. Then AE and AK are [respectively] equal to ΛΓ and ΓΘ. Therefore the quadrangle EK is equal to the quadrangle ΛΘ.

Furthermore the rectangular planes ΛM and EZ are similar and similarly situated because they are similar to the rectangular planes ΔE and NΛ [respectively], and AK is equal to ΓΘ. Therefore the quadrangle EZ is equal to the quadrangle ΛM. And the rectangular planes KE and ΘΛ were [already proved] equal. Therefore the quadrangle AZ is equal to the quadrangle ΓM, and the straight lines equal to them in square are [respectively] BK and ΘH, as is proved in Theorems 12 and 13 of Book I.

Therefore when the axis is applied to the axis, BK will coincide with ΘH, and B will coincide with H. But it was supposed to fall on the section ΓH, which is impossible. Therefore the whole section AB will fit on the section ΓH.

Furthermore we make two sections equal, and make AK and ΓΘ equal, and draw from them the perpendiculars KB and ΘH, and complete [the rectangular planes] ΔE, ΔZ, NΛ, and NM, then the section AB will fit on the section ΓH, and the axis AK will coincide with the axis ΓΘ for if it did not coincide with it, then the hyperbola would have two axes and the ellipse three axes, which is impossible. Therefore AK coincides with ΓΘ, and it is equal to it. So K will coincide with Θ, and KB will coincide with ΘH, and [hence] B will coincide with H, and KB will fit on HΘ, therefore KB is equal to HΘ.

For that reason the quadrangle AZ is equal to the quadrangle ΓM.

But AK is equal to ΓΘ, therefore KZ is equal to ΘM.

Furthermore we make AΞ equal to ΓΠ, then it will be proved, as we proved above, that ΞT is equal to ΠX. Therefore ΣZ is equal to MY, and ΣT is equal to YX. Therefore the rectangular planes ZT and MX are equal and similar.

Therefore the quadrangle ΔE is similar to the quadrangle NΛ, and also the quadrangle ΔZ is similar to the quadrangle NM. But KZ is equal to ΘM. Therefore ΔK is equal to NΘ. But it was [assumed] that AK is equal to ΓΘ. Therefore ΔA is equal to NΓ and the quadrangle ΔE is similar to the quadrangle NΛ. Therefore AE is equal to ΓΛ, and the quadrangle ΔE is equal to the quadrangle NΛ. And these are the *eidoi* corresponding to the axes.

## Porisms

If there are [a number of] parabolas, and ordinates falling on one of their diameters meet the diameters at equal angles, and their *latera recta* are equal, then the sections are equal, and if there are [a number of] hyperbolas or ellipses, and the ordinates falling on one of their diameters meet the diameter at equal angles, and *eidoi* corresponding to those diameters are equal and similar, then the sections are equal [12].

That is proved as it was proved for the axes.

[Proposition] 3

*As for the ellipse it is evident that it cannot be equal to any of other sections because it is bounded, but they are unbounded.*

Then I also say that no parabola can be equal to a hyperbola [13] .

[Proof]. For let there be the parabola ΑΒΓ and the hyperbola ΗΙΚΝ. Then, if possible, let it be equal to it, and let the axes of the sections be ΒΖ and ΚΜ, and let the transverse axis of the hyperbola be ΚΘ, and let ΒΕ and ΒΖ be equal to ΚΛ and ΚΜ [respectively]. We draw from the axes the perpendiculars ΑΕ, ΔΖ, ΙΛ, and ΗΜ. Now the section fits on the section because it is equal to it, and [hence] Ε, Ζ, Α, and Δ coincide with Λ, Μ, Ι, and Η [respectively], and as ΖΒ is to ΕΒ, so ΔΖ is to ΑΕ, as is proved in Theorem 20 of Book I. Therefore as ΜΚ is to ΚΛ, so ΜΗ is to ΛΙ. But that is impossible because as sq.ΜΗ is to sq.ΙΛ, so pl.ΘΜΚ is to pl.ΘΛΚ, as is proved in Theorem 21 of Book I.

Therefore the parabola is not equal the hyperbola.

[Proposition] 4

*If there is an ellipse and a straight line passes through its center such that its extremities end at the section, then it cuts the boundary of the section into two equal parts. And the surface is also bisected [by it]* [14] .

Let there be the ellipse ΑΓΒ whose center Θ, and let the straight line AB pass through its center. And first let ΑΒ be one of the axes of the section.

Then I say that the line ΑΓΒ fits on the line ΑΕΒ, when it is applied to it, and the surface ΑΓΒ coincides with the surface ΑΕΒ.

[Proof]. For let, if possible, the line ΑΓΒ not coincide wholly with the line ΑΕΒ. We take Γ on the part of it that does not coincide with it, and draw from it the perpendicular ΓΔ to ΑΒ, and continue it to [meet the section again at] Ε. Then ΓΔ coincides with ΔΕ because the angles at Δ are right, and ΓΔ is equal to ΔΕ. Therefore Γ coincides with Ε.

But it had been assumed not to coincide with it, which is impossible. Therefore the line ΑΓΒ coincides with the line ΑΕΒ so as to fit to it, and the surface ΑΓΒ will coincide with the surface ΑΕΒ. Hence the line ΑΓΒ is equal to the line ΑΕΒ, and the surface ΑΓΒ to the surface ΑΕΒ.

[Proposition] 5

Furthermore we do not make AB one of the axes [15]. And let the axes be ΓΔ and KΛ, and we draw two perpendiculars AE and BH [to the axis], then the line ΓAΔ fits on the line ΓZΔ, as was proved in the preceding theorem, and Z coincides with A, and the surface AΓE coincides with the surface ΓZE. Furthermore [the line] KΓΛ coincides with [the line] KΔΛ, and EΘ coincides with ΘH, and EZ with BH because EΘ is equal to ΘH, and EZ to BH, and the surface ΓEZ coincides with the surface ΔHB. Therefore the surface AΓE coincides with the surface BΔH. So it is equal to it, and [hence] the line AΓ is equal to the line ΔB.

Furthermore [Δ]AEΘ is equal to [Δ]ΘBH. Therefore [the surface] AΓΘ is equal to [the surface] ΘBΔ, hence the remainder [line] AK is equal to the remainder [line] BΛ. And [hence] the line AKΔ is equal to the line ΓΛB. Therefore the whole surface AKΔB is equal to the whole surface AΓΛB, and the line AKΔB is equal to the line AΓΛB.

[Proposition] 6

*If there is a conic section, and a part of it coincides with another part of another section so as to fit on it, then the [first] section is equal to the[second] section* [16].

Let the arc AB of the section AB, when applied to the arc ΓΔ of the section ΓΔE fit on it. I say that the section AB is equal to the section ΓΔE.

[Proof]. For let, if that is not so, then the part AB coincide with the part ΓΔ, and let the remainder of the section not coincide with the other section, but let them be as the sections ΔΓM and ΔΓN. We take the point Θ on ΓM, and join it to Δ, and draw in the section ΓΔE the diameter KΛ bisecting ΔΘ. Then the tangent to the section ΓΔE at K is parallel to ΔΘ, and the diameter KΛ bisects the straight lines parallel to ΔΘ. Therefore we draw from Γ the straight line ΓZ parallel to ΔΘ. Then KΛ bisects it, and it is parallel to the tangent to the section ΔΓM at K. And that [tangent] is also the tangent to the section ΔΓN. Therefore KΛ is a diameter to the section AΓN, as is proved in Theorem 7 of Book II. Therefore it bisects the diameter ΔN at L. But ΔΘ was [assumed to be] bisected at [the same point] Λ, which is impossible. Therefore the whole section AB coincides with the section ΓΔE so as to fit on it, therefore it is equal to it.

[Proposition] 7

*The perpendiculars drawn from a parabola or a hyperbola to its axis, and continued to the other side, cut off from the section on both sides of the axis the segments which, when one is applied to an other fit so as not to exceed or fall short of it, but do not fit on any other part of the section if placed on it*[17] .

Let there be the parabola or the hyperbola ΓΒΑ whose axis ΓΗ. We take on the section two points Β and Α, and draw from them two perpendiculars to ΓΗ, and continue them to the other side of the section, these are ΒΖΔ and ΑΗΕ. Let them cut off from the section two segments ΒΓΔ and ΑΓΕ. I say that the line ΒΓ fits on the line ΓΔ, and the line ΒΑ on the line ΔΕ and the surface ΑΓΗ on the surface ΗΓΕ, and the arc ΑΒΓ of the section on the arc ΓΔΕ.

[Proof]. The proof of that is like the preceding proofs for all perpendiculars drawn from the arc ΑΒΓ to the axis ΓΗ are equal in square to figures that are equal to those figures to which the perpendiculars drawn from the arc ΓΔΕ to the axis ΓΗ, being continuous with those perpendiculars, are equal in square. Therefore ΒΖ is equal to ΖΔ, and ΑΗ is equal to ΕΗ, and the angles at Ζ and Η are right.

Therefore the arc ΓΒ, when applied to the arc ΓΔ, will fit on it, and the arc ΑΒ will coincide with the arc ΔΕ, and the [corresponding] surfaces will coincide with the surfaces.

Therefore let the arc ΘΚ be another arc which is not cut off by these two perpendiculars. Then I say that the arc ΔΕ, if applied to it, will not fit on it.

[Proof]. For let if that it not so, and if possible, it fit. Then, when ΔΕ is applied to ΚΘ so as to fit on it, the line ΓΔ will coincide with the arc, which is adjacent to the arc ΘΚ, as is proved in the preceding theorem. And the point Γ of the arc ΓΔΕ will fall on a place different from its position on the arc ΚΘΓ because the arc ΚΘΓ is not equal to the arc ΓΔΕ, and the axis ΓΗ will fall on a place different from the position it has [now]. Therefore the parabola or the hyperbola has two axes, which is impossible. So the arc ΔΕ does not coincide with the arc ΔΚ.

[Proposition] 8

*In every ellipse perpendiculars which are drawn to the axis and continued in a straight line to the other side of it cut off from the section on either side of the axis arcs which fit when one is applied to another, and if they are applied to the arcs cut off by the perpendiculars whose distance from the center towards other side is equal to the distance of the perpendiculars drawn [above], they will fit on them, but will not fit on [any] other arc of the section* [18] .

Let there be the ellipse ΑΓΔΒ whose axis ΑΒ and ΚΛ. Let there be drawn in it two perpendiculars to ΑΒ, and let them be continued in a straight line to both sides [of the section], let them be ΓΕ and ΔΖ. And let them cut off from it two arcs ΓΔ and ΕΖ. And let there also be drawn in the section two other perpendiculars of this kind whose distance from the center is [respectively] equal to the distance of those two perpendiculars, these are ΜΞ and ΝΘ .

Now as to [the statement] that when one of ΓΔ and ΕΖ is applied to the other, it will fit on it, which will be proved as it was proved in the preceding theorem.

And similarly it will be proved that ΜΝ will fit on ΞΘ. And because the surface ΚΑΛ, when applied to the surface ΚΒΛ, lies on it, as is proved in Theorem 4 of this Book, ΓΕ will coincide with ΝΘ because the distance of each from the center is one and the same.

And ΔΖ will coincide with ΜΞ, and [hence] the arc ΓΔ will coincide with the arc ΜΝ.Therefore it will fit on the arc ΞΘ because one of them fits on other.

And likewise too the arc ΕΖ [will fit on ΞΘ and ΜΝ].

Therefore let there be another arc ΠΡ of the section, apart from these four. Then I say that none of these arc will fit on it.

[Proof]. For let if possible the arc ΜΝ fit on it. Then it will necessarily follow, as it did in the preceding theorems, that the ellipse would have more than two axes, which is impossible. Therefore ΜΝ will not fit on ΠΡ.

[Proposition] 9

*In equal sections those parts of them at equal distances from their vertices will fit one on another, and those [parts] not at equal distances from their vertices will not fit one on another* [19] .

Let there be two equal sections with axes ΓΔ and ΚΛ. Let the distance of the arc ΑΒ from Γ be equal to the distance of the arc ΕΗ from Κ.

Then I say that ΑΒ will fit on ΕΗ.

[Proof]. Then the section ΓΑ is applied to the section ΚΕ, the point Β will coincide with H because the distance of each from the vertices of two sections is equal. And A will coincide with Ε, and [hence] the section ΑΒ will coincide with the section ΕΗ. Then I say that it will not coincide with any other arc so as to fit on it.

[Proof]. For let, if possible, it coincide with the arc ΖΘ. Now we have proved that it fits on ΕΗ. Therefore the arc ΖΘ will fit on the arc ΕΗ. But the arcs ZΘ and EH are not the arcs cut off by two perpendiculars, and their distances from the vertices are not equal. That is impossible as is proved in two preceding theorems two.

## [Proposition] 10

*In the sections that are unequal no part of one of them will fit on a part of another*[20] .

Let there be two unequal sections ΑΒΓ and ΔΕΖ.

That no part of one of them will fit on a part of another.

[Proof]. For let, if possible, the part ΑΒ fit on a part ΔΕ. Then the whole ΟΛΕ section ΑΒΓ will fit on the section ΔΕΖ, as is proved in Theorem 6 of this Book. Therefore the section ΑΒΓ is equal to the section ΔΕΖ, which is impossible. So no part of ΑΒΓ fits on a part of ΔΕΖ.

## [Proposition ] 11

*Every parabola is similar to every parabola* [21] .

Let there be two parabolas ΑΒ and ΓΔ whose axes ΑΚ and ΓΟ.

I say that two sections are similar.

[Proof]. For let their *latera recta* ΑΠ and ΓΡ, and let as ΑΚ be to ΑΠ, so ΓΟ be to ΓΡ. We cut ΑΚ at two arbitrary points Ζ and Θ, and cut ΓΟ into the same number of arcs with the same ratio at the points Μ and Ξ. We draw from the axes ΑΚ and ΓΟ the perpendiculars ΖΕ, ΘΗ, ΚΒ, ΜΛ, ΝΞ, and ΔΟ [and continue them to meet the sections again at Ι, Σ, Τ, Υ, Φ, and Χ]. Then as ΠΑ is to ΑΚ, so ΓΡ is to ΓΟ, and ΚΒ is the mean proportional between ΑΠ and ΑΚ, and ΟΔ is the mean proportional between ΓΡ and ΡΟ, because of what is proved in Theorem 11 of Book I.

As ΚΒ is to ΚΑ, so ΔΟ is to ΟΓ. And ΒΤ is equal to the double ΒΚ, and ΔΧ is equal to the double ΔΟ. Therefore as ΒΤ is to ΑΚ, so ΔΞ is to ΓΟ.

Furthermore as ΠΑ is to ΑΚ, so ΓΡ is to ΓΟ. And as ΑΚ is to ΑΘ, so ΟΓ is to ΓΞ, and as ΑΠ is to ΑΘ, so ΓΡ is to ΓΞ.

Hence it will be proved, as we proved above, that as ΗΣ is to ΑΘ, so ΝΦ is to ΓΞ.

And similarly too it will be proved that as ΕΙ is to ΖΑ, so ΛΥ is to ΜΓ.

Therefore the ratio of [each of] ΒΤ, ΗΣ, and ΕΙ ,which are perpendiculars to the axis, to the amounts ΑΚ, ΑΘ, and ΑΖ which they cut off from the axis is equal to the ratio of ΔX, NΦ, and ΛY ,which are perpendiculars to the axis, to the amounts OΓ, ΞΓ, and MΓ which they cut off from the axis.

And the ratios of the segments cut of from one of the axes to the segments cut off from the other are equal. Therefore the section AB is similar to the section ΓΔ.

[Proposition] 12

*Hyperbolas and ellipses in which the* eidoi *corresponding to their axes are similar are also [themselves] similar, and if the sections are similar, then the* eidoi *corresponding to their axes are similar* [22].

Let there be two hyperbolas or ellipses AB and ΓΔ whose *eidoi* corresponding to their axes AK and ΓO are similar, the transverse diameters of these conic are AΠ and PΓ. We cut off from the axes the segments AΓ and ΓO and let as AK be to AΠ, so ΓO be to ΓP.

We cut AK arbitrarily at Z and Θ, and cut ΓO into the same number of segments as AK, and in the same ratios at M and Ξ we draw from Z, Θ, K, M, Ξ, and O the BK, ΘH, ZE, OΔ, ΞN, and MΛ to the axes, [and continue them to meet the sections again at T, Σ, I, X, Φ, and Y].

Then because the *eidoi* of the sections are similar as sq.BK is to pl. ΠKA, so sq.ΔO is to pl.POΓ, as may be proved from Theorem 21 of Book I.

But as pl.ΠKA is to sq.KA, so pl.POΓ is to sq.OΓ. Therefore as sq.BK is to sq.KA, so sq.ΔO is to sq.OΓ , and as BK is to KA , so ΔO is to OΓ, and as BT is to KA, so ΔX is to OΓ.

Furthermore as ΠA is to AK, so PΓ is to ΓO, and as KA is to AΘ, so OΓ is to ΓΞ. Therefore as AΠ is to AΘ, so PΓ is to ΓΞ. Hence it will proved, as we proved above, that as HΣ is to ΘA, so NΦ is to ΞΓ, and that as EI is to ZA, so ΛY is to MΓ.

Therefore the ratios of the perpendiculars BT, HΣ and EI to the amounts AK, AΘ, and AZ they cut of from the axis are [respectively] equal to the ratios of the perpendiculars ΔX, NΦ, and ΛY to the amounts OΓ, HΓ, and MΓ they cut off from the axis.

And the ratios of the parts of AK that the perpendiculars cut of to the parts of ΓO which the perpendiculars cut off are equal. Therefore the section AB is similar to the section ΓΔ.

Furthermore we make the section AB similar to the section ΓΔ. Then since two sections are similar we draw in the section AB some perpendiculars BT, AΣ, and EI to the axis, and in the section ΓΔ the perpendiculars ΔX, NΦ, and ΛY, and let the ratios of these perpendiculars to the amounts they cut off from the axes be equal [respectively], and likewise the ratios of the parts they cut off from one of the axes to the parts they cut off from other axis, then as ΒΚ is to ΑΚ, so ΔΟ is to ΟΓ, and as ΚΑ is to ΑΘ, so ΟΓ is to ΓΞ, and as ΑΘ is to ΘΗ, so ΓΞ is to ΝΞ. Therefore as ΒΚ is to ΘΗ, so ΔΟ is to ΝΞ.

And as sq.ΒΚ is to sq.ΗΘ, so sq.ΔΟ is to sq.ΝΞ. Therefore as pl.ΠΚΑ is to pl.ΠΘΑ, so pl.ΡΟΓ is to pl.ΡΞΓ because of what was proved in Theorem 21 of Book I. and because as ΚΑ is to ΑΘ, so ΟΓ is to ΓΞ, [and as ΚΑ is to ΑΠ, so ΟΓ is to ΓΡ], as ΚΠ is to ΠΘ, so ΡΟ is to ΡΞ, and [hence] as ΠΘ is to ΚΘ, so ΡΞ is to ΟΞ. But as ΚΘ is to ΑΘ, so ΟΞ is to ΞΓ. Therefore as ΠΘ is to ΘΑ, so ΡΞ is to ΞΓ And [hence] as pl.ΠΘΑ is to sq.ΘΑ, so pl.ΡΞΓ is to sq.ΞΓ.

But as sq.ΑΘ is to sq.ΘΗ, so sq.ΓΞ is to sq.ΝΞ. Therefore as pl.ΠΘΑ is to sq.ΘΗ, so pl.ΡΞΓ is to sq.ΞΝ.

But the ratio pl.ΠΘΑ to sq.ΘΗ is equal to the ratio of ΠΑ to the *latus rectum* [of ΑΒ], as is proved in Theorem 21 of Book I. Therefore the *eidoi* corresponding to ΠΑ and ΡΓ are equal [23-24] .

[Proposition] 13

Let there be two hyperbolas or ellipses whose centers Ζ and I, and diameters ΓΛ and ΕΜ. Let the angles that those diameters form with their ordinates be equal, and let the *eidoi* corresponding to ΓL and ΕΜ be similar.

If those *eidoi* of hyperbolas or ellipses that are corresponding to diameters other than the axes are similar, and the ordinates falling on those diameters form equal angles with the diameters, then the sections are similar[25].

I say that the sections are similar.

[Proof]. For let from Γ and Ε the tangents ΓΘ and ΕΟ to the sections be drawn. Then these tangents are parallel to the ordinates fallen. We draw through Α and Δ the straight lines ΤΑΥ and ΦΔΧ parallel to the tangents.
Now the *eidoi* corresponding to ΓΛ and ΕΜ are similar *latus rectum* proved in Theorem 37 of Book I. And likewise [the ratio pl.ΙΞΟ to sq.ΕΞ] is equal to the ratio of the [transverse] diameters to [its] the *latus rectum*. Therefore the ratios of the transverse diameter ΚΔ to [its] *latus rectum*. Therefore two ratios of the [transverse] axes ΑΒ and ΚΔ to their *latera recta* are equal. And the *eidoi* corresponding to the axes of these sections are similar. Therefore two sections are similar as is proved in the preceding theorem .

And it is evident too that in the case on two ellipses this requires that the axes ΒΑ and ΚΔ both be the major axes or the both be the minor axes because the ratio of BA to its *latus rectum* in both cases is equal to the ratio of KΔ to its *latus rectum*. And the rule is one and the same for major and minor [axes].

[Proposition] 14

*A parabola is not similar to a hyperbola and to an ellipse* [27].

Let there be the parabola AB whose axis AH, and the hyperbola or the ellipse ΓΔ similar to it. And let the axis of ΓΔ be the straight line ΓΔ, and let the side of the *eidos* of the section, the transverse axis, be ΓM.

Let there be the perpendiculars BI and ZN in the sections [in the parabola], and ΔΞ and KO [in the hyperbola on the ellipse], and let the ratios of these

[perpendiculars] to the segments they cut off from the axes in one of the sections be equal to [their] ratios to the segments they cut off from the axis of other section, and let the ratios of the segments cut off from one of the axes to the segments cut off from the other axis be equal. Then as ZH is to HA, so KΛ is to ΛΓ, and as HA is to AE, so ΛΓ is to ΓΘ.

But as AE is to EB, so ΓΘ is to ΘΔ. Therefore as ZH is to EB, so KΛ is to ΔΘ, and as sq.ZH is to sq.BE, so sq.KΛ is to sq.ΔΘ.

But as sq.ZH is to sq.BE, so HA is to AE, as is proved in Theorem 20 of Book I. And as HA is to AE, so ΛΓ is to ΓΘ. Therefore as sq.KΛ is to sq.ΔΘ, so ΛΓ is to ΓΘ, but as KΛ is to sq.ΔΘ, so pl.MΛΓ is to pl.MΘΓ, as is proved in Theorem 21 of Book I. Therefore as ΛΓ is to ΓΘ, so pl.MΛΓ is to pl.MΘΓ. Therefore MΘ is equal to MΛ, but that is impossible. Therefore the parabola is not equal to any other section

[Proposition] 15

*A hyperbola is not similar to an ellipse* [28].

Let there be the hyperbola AB and the ellipse ΓΔ. Let their axes be [respectively] AK and ΓM, and let their transverse diameters be AE and ΓZ.

Then, if these two sections are similar, then there are in the sections some perpendiculars, for instance BN, ΘΞ, ΔO, and ΛH, such that the ratios of these [perpendiculars] to the segments they cut off from the axes in both sections are [respectively] equal. Then we will prove as we proved in the preceding theorem that as sq.ΘK is to sq.BH, so sq.ΛM is to sq. ΔI, and pl.EKA is to pl.EHA, and pl.ZMI is to pl.ZIΓ. Therefore as pl.EKA is to pl.EHA, so pl.ZMΓ is to pl.ZIΓ. And when what is so and as KΛ is to AH, so MΓ is to ΓI, and [hence] as

KE is to EH, so ZM is to ZE, that is impossible, therefore the section AB is not similar to the section ΓΔ.

[Proposition] 16

*Opposite hyperbolas are similar and equal* [29].

Let there be two opposite hyperbola A and B whose axis AB.

I say that the hyperbolas A and B are similar and equal.

[Proof]. The *latera recta* of the hyperbolas A and B are equal, as is proved in the proof of Theorem 14 of Book I.

And the straight line AB is a side common to their *eidoi*. Therefore the *eidoi* corresponding to the axis of the hyperbolas A and B are similar and equal. Therefore the hyperbola A is similar to the hyperbola B and is equal to it, as is proved in Theorem 12 of this Book.

[Proposition] 17

*If there are similar sections, and tangents are drawn to them ending at their axes and forming equal angles with the axes, and diameters are drawn to the sections from the points of contact, and a point is taken on each of those diameter, and the ratios of the segments between the taken points and the vertices of those diameter to the tangents are equal and straight lines are drawn through [each] taken point parallel to the tangents so that they cut off segments from the sections then those segments are similar, and their position is similar, and if segments are similar and their position is similar, then the ratios of their diameters to the [corresponding] tangents are equal, and the angles which the tangents form with the axes are equal* [30].

First let the similar sections be two parabolas AB and KΛ, let their axis be AZ and KO, and the tangents to them are ΓZ and MO. Let the angles AZΓ and MOK be equal. We draw through Γ and M the diameters ΓE and MΞ to the sections. Let as EΓ is to ΓZ, so MΞ be to MO. We draw through E and Ξ the straight lines ΔB and NΛ parallel to ΓZ and MO.

I say that the segments BΓΔ and ΛMN are similar and similarly situated.

[Proof]. We draw from A and K the perpendiculars AH and KP to the axes [cutting ZΓ and OM at Θ and Π] , and continue the diameters EΓ and ΞM until they meet them at H and P.

We make the ratio ΣΓ to the double ΓZ equal to the ratio ΘΓ to ΓH, and the ratio TM to the double MO equal to the ratio ΠM to MP. Then ΣΓ and TM are *latera recta* corresponding to the diameters ΓΕ and ΜΞ [respectively]. Therefore sq.ΔΕ is equal to pl.ΣΓΕ, as is proved in Theorem 49 of Book I. And likewise sq.ΝΞ is equal to pl.ΤΜΞ. And the angle ΚΟΜ is equal to the angle ΑΖΓ, the angle ΚΟΜ is equal to the angle ΡΜΟ, and the angle ΑΖΓ is equal to the angle ΗΓΖ because ΞΡ and ΕΗ are parallel to ΟΚ and ΖΑ [respectively], as is proved from Theorem 46 of Book I. Therefore the angle ΡΜΟ is equal to the angle ΗΓΖ,and the angles at Η and Ρ are equal, therefore the triangle ΘΓΗ is similar to the triangle ΡΜΠ, and [hence] as ΘΓ is to ΓΗ, so ΠΜ is to ΜΡ. Therefore as ΣΓ is to ΓΖ, so ΤΜ is to ΜΟ.

But the ratio ΓΖ to ΓΕ had been made equal to the ratio ΜΟ to ΜΞ therefore as ΣΓ is to ΓΕ, so ΤΜ is to ΜΞ.

Hence it will be proved, as we proved in Theorem 11 of this Book that, if the straight lines are drawn to ΓΕ parallel to ΔΒ and the straight lines are drawn to ΜΞ parallel to ΛΝ, and the ratio of these straight lines which are parallel to [the segment] bases ΔΒ and ΛΝ to the segments they cut off from the [corresponding] diameters adjacent to Γ and M are equal, and the ratios of the segments cut off from one of the diameters to those cut off from other diameter are also equal, and the angles formed by the coordinates to parallel to these bases and the diameters in both sections are equal [because the angles at Γ and Μ are equal], then the segment ΒΓΔ is similar to the segment ΛΜΝ, and its position is similar to its position.

Furthermore we make the segment ΔΓΒ of one section similar to the segment ΛΜΝ of other section, and let their diameters be ΓΕ and ΜΞ, and their bases be ΒΔ and ΛΝ, and the points of their vertices be Γ and Μ and let ΓΖ and MO be tangents to the sections at these points. Then I say that the angle ΑΖΓ is equal to the angle ΚΟΜ, and that as ΕΓ is to ΓΖ, so ΜΞ to ΜΟ.

We draw the straight lines that we drew previously. Then since the sections are similar, two angles formed by ΔΒ and ΓΕ are equal to two angles formed by ΛΝ and ΜΞ. And ΖΓ and ΟΜ are parallel to ΒΔ and ΛΝ [respectively]. Therefore the angles at Γ, Ε, Μ, and Ξ are equal.

Therefore, since that is so, and [since] the angles ΖΓΕ and ΟΜΞ are obtuse, the angle ΖΓΕ is equal to the angle ΟΜΞ. Therefore the angle at Ζ is equal to the angle at Ο.

Furthermore as ΔΒ is to ΕΓ, so ΝΛ is to ΞΜ because of the similarity of the segments of the sections, and [hence] as ΔΕ is to ΓΕ, so ΝΞ is to ΞΜ, and as ΣΓ is to ΔΕ, so ΔΕ is to ΕΓ, and as ΤΜ is to ΞΝ, so ΝΞ is to ΞΜ. Therefore as ΣΓ is to ΓΕ, so ΤΜ is to ΜΞ. And as ΖΓ is to ΓΣ, so ΜΟ is to ΜΤ because that the triangle ΓΘΗ is similar to the triangle ΠΜΡ. Therefore as ΓΖ is to ΓΕ, so ΟΜ is to ΜΞ. And we had [already] proved that the angles at Ζ and Ο are equal .

[Proposition] 18

Furthermore we make the mentioned sections hyperbolas or ellipses, and let every thing else be as we stated in the preceding theorem [31] , and let the diameters ΓΕ and ΜΞ end at the centers Ι and Φ of the sections, and let the ratio of [abscissa] ΓΕ to the tangent ΓΖ be equal to the ratio of [abscissa] ΞΜ to [the tangent] ΜΟ, and let the angles ΑΖΓ and ΚΟΜ be equal, then I say that the segments ΔΓΒ and ΛΜΝ are similar, and let the ratio ΣΓ to the double ΓΖ be equal to the ratio ΘΓ to ΓΗ, and let the ratio ΤΜ to the double ΜΟ be equal to the ratio ΠΜ to ΜΡ. Then ΓΣ and ΤΜ are *latera recta* corresponding to the diameters ΓΕ and ΜΞ [respectively], as is proved in Theorem 50 of Book I.

Therefore we draw from Α, Κ, Γ, and Μ the perpendiculars ΑΗ, ΚΡ, ΓΥ, and ΜΧ to the axes. Then, since two sections are similar, the *eidoi* corresponding to their axes are also similar, as is proved in Theorem 12 of this Book, and since the *eidoi* of these two sections corresponding to their axes are similar, as pl.ΙΥΖ is to sq.ΓΥ, so pl.ΦΧΟ is to sq.ΜΧ because of what is proved in Theorem 37 of Book I.

And we had constructed the angles at Ζ and Ο as equal, and the angles at Υ and Χ are equal because they are right. Therefore the triangle ΓΥΖ is similar to the triangle ΜΧΟ.

And we had [already] proved that as pl.ΙΥΖ is to sq.ΓΥ, so pl.ΦΧΟ is to sq.ΜΧ. Therefore the triangle ΓΥΙ is similar to the triangle ΜΦΧ[32] .

And [hence] the angle at I is equal to the angle at Φ, and the angle ΖΓΙ is equal to the angle ΦΜΟ. And the angles at Ε and Ξ are equal because the tangent is parallel to the ordinates. And the angles at Α and Κ are right, and the angles at Φ and I have [already] been proved equal. Therefore the remaining angles [in the triangles ΙΗΑ and ΦΡΚ] at Η and Ρ are equal. And it has [already] been proved that the angle ΖΓΙ is equal to the angle ΟΜΦ. Therefore the triangle ΘΓΗ is similar the triangle ΠΜΡ, and [hence] as ΘΓ is to ΓΗ, so ΠΜ is to ΜΡ. But we had made the ratio ΓΣ to the double ΓΖ equal to the ratio ΓΘ to ΓΗ, and the ratio ΤΜ to the double ΜΟ equal to the ratio ΠΜ to ΜΡ. Therefore as ΓΣ is to ΓΖ, so ΜΤ is to ΜΟ.

But as ΓΖ is to ΓΙ, so ΟΜ is to ΜΦ. Therefore as ΓΣ is to ΓΙ, so ΜΤ is to ΜΦ, and as ΓΣ is to ΓΨ, so ΜΤ is to ΜϘ. Therefore the *eidoi* of which one is pl.ΣΓΨ and the other is pl.ΤΜϘ are similar.

Furthermore as ΓΣ is to ZΓ, so MT is to MϘ, and we had made the ratio ΓZ to ΓE equal to the ratio MO to MΞ.Therefore as ΓΣ is to ΓE, so MT is to MΞ.

And since that is so, and since the *eidos* pl.ΣΓΨ is similar to the *eidos* pl.TMO, then, when we divide ΓE into partitions and draw through the points of partition straight lines parallel to ΔB which is the base of the segment [ΔAB], and divide MΞ in the same ratios as the partitions of ΓE, and again draw through the points of partition straight lines parallel to ΛN which is the base of the segment [ΛMN], then it will be proved, as we proved in Theorem 12 of this Book, that the ratios of the parallel straight lines cutting ΓE to the portions they cut off from it adjacent to Γ are equal to the ratios of the parallel straight lines cutting MΞ to the portions they cut off from it adjacent to M. And the angles formed by the base ΔB with ΓE are equal to the angles formed by the base ΛN with MΞ, because these angles are equal to the angles at Γ and M continued by the tangent and the diameter.

Therefore two segments ΔΓB and NMΛ are similar, and their position is similar.

Furthermore we make the segment ΔΓB similar to the segment NMΛ, then I say that the angle ΓZA is equal to the angle MOK, and that as ΓE is to ΓZ, so ΞM is to MO.

[Proof]. For, since two segments are similar, there can be drawn in them some straight lines parallel to ΔB and NΛ equal, to number, cutting ΓE and MΞ at equal angles, and [then] the ratios between them and [also] the ratios of the bases ΔB and ΛN to the portions they cut off from the diameters are equal, and also the ratios of the partitions of ΓE [continued by these straight lines] to the partitions of MΞ are equal to each other, and the straight lines drawn to ΓE in the segment ΔΓB parallel to ΔB are equal in square to the rectangular planes applied to ΓΣ and greater than it [in the case of the hyperbola] or smaller than it [in the case of the ellipse] by are rectangular plane similar to pl.ΣΓΨ, as is proved in Theorem 50 of Book I, and likewise too the straight lines drawn to MΞ in the segment NMΛ parallel to ΛN are equal in square to the rectangular planes applied to TM and greater and smaller than it by a plane similar to pl.TMϘ.

Therefore, since that is so, then it will be proved, as we proved in Theorem of this Book, that as ΓΣ is to ΨΓ, so MT is to MϘ.

And when that is so, and the ordinate meet two diameters at equal angles, and [for that reason] as pl.IYZ is to sq.ΓY, so pl.ΦXO is to sq.MX, and the angles at Y and X are right, and the angle ZΓI is equal to the angle OMΦ, then the triangle IΓZ is similar to the triangle ΦMO .

And that will be proved in the case of the hyperbola by a proof that is universally applicable, but in the case of the ellipse it will be proved [only] by the axes ΑΙ and ΚΦ being either both major or both minor axes.

Then, since as ΓΣ is to ΓΨ, so MT is to MϘ, as pl.ΓΕΨ is to sq.ΔΕ, so pl.Μ ΞϘ
is to sq.ΝΞ, as is proved in Theorem 21 of Book I. And as sq.ΔΕ is to sq.ΓΕ, so sq.ΝΞ is to sq.ΜΞ. Therefore as pl.ΨΕΓ is to sq.ΕΓ, so pl.ϘΞΜ is to sq.ΞΜ, and as ΨΕ is to ΕΓ, so ϘΞ is to ΞΜ.

But as ΙΓ is to ΓΖ, so ΦΜ is to ΜΟ because of the similarity of the triangles ΙΓΖ and ΦΜΟ. And ΓΨ is equal to the double ΓΙ, and MϘ is equal to the double ΜΦ. Therefore as ΓΖ is to ΓΕ, so ΜΟ is to ΜΞ. And the angles at Ζ and Ο are equal.

[Proposition] 19

*When straight lines are drawn in a parabola or a hyperbola as perpendiculars to the axis, then two segments cut off by each pair of perpendiculars on either side [of the axis] are similar and similarly situated, but as for other segments [in that section], they are dissimilar to them* [34].

Let there be the parabola or the hyperbola whose axis ΑΛ, and let a pair of straight lines be drawn in the section as perpendiculars ΒΘ and ΓΚ to the axes, and let them cut off from the section the segments ΒΓ and ΘΚ, and let the segments ΔΕ and ΘΚ be two segments not cut off by the same [pair of] perpendiculars. Then I say that the segments ΒΓ and ΘΚ are similar, and that the segments ΔΕ and ΘΚ are dissimilar.

[Proof]. As for [the statement] that the segments ΒΓ and ΘΚ are similar, that is evident because each of them will fit on other, as is proved in Theorem7 of this Book. But as for [the statement] the segments ΔΕ and ΘΚ are dissimilar, that will be proved as follows. Let, if possible, the segments ΔΕ and ΘΚ be similar. We join ΔΕ and ΓΒ, and continue them to [meet the continued axis at] Ζ and Η. Now the segments ΔΕ and ΘΚ are similar, therefore the segment ΘΚ will fit on the segment ΒΓ, as is proved in Theorem 7 of this Book. Therefore the section ΔΕ is similar to the section ΒΓ. Therefore when the straight lines ΒΓ and ΔΕ are continued in a straight line, they will meet the axis at equal angles because of what was proved in two preceding theorems. We draw ΜΞ bisecting ΓΒ and ΔΕ, draw from Μ [lying on the section] ΜΙ parallel to ΔΕΖ. Then ΜΞ is the diameter to the section because of what is proved in Theorem 28 of Book II. And ΜΙ is parallel to the ordinates falling on it, therefore it is tangent to the section. And the segments ΓΒ and ΔΕ are similar, therefore as ΜΙ is to ΜΞ, so

MI is to MN, as is proved in two preceding theorems. But that is impossible. Therefore the segment ΔME is dissimilar to the segment ΘK.

[Proposition] 20

*When straight lines are drawn in an ellipse as perpendiculars to its axis, then every pair of these perpendiculars cuts off on either side [of the axis] two segments similar to each other and similar to two segments cut off by the pair of perpendiculars whose distance from the center is equal to the distance of that pair of perpendiculars, and the position of these four segments is similar, and no other segment [in that ellipse] is similar [to these]*[34].

Let there be the ellipse whose axis AΛ, and let there be in it the pair of straight lines BΘ and ΓK cutting the axis at right angles. And let there be the other pair of straight lines ZI and HO cutting the axis at right angles, the distance of which from the center is equal to the distance of those [straight lines]. Then I say that the segments BΓ, ΘK, ZH, and IO are similar, and that none of other segments is similar to them.

[Proof]. As for [the statement] that the segments BΓ, ΘK, ZH, and OI are similar and similarly situated, that is evident because these segments will fit one on another as is proved in Theorem 8 of this Book.

But as for [the statement] that no other segment is similar to them; this will be proved as follows. Let, if possible the segment ΔE be similar to those segments. We join ΔE and ΓB. Then, when they continued, if one of them meets the axis, the other will meet it at the same angle as the first, as is proved in Theorem 18 of this Book. Therefore ΔE and ΓB are parallel. Therefore we bisect them and draw through two points of bisection MNΞ. Then MNΞ is a diameter to two segments, as is proved in Theorem 28 of Book II. Therefore since the segments ΔE and ΓB are similar, as ΓB is to ΞM, so ΔE is to MN. That is impossible for when we join MB and MΓ and continue them, they will not pass through Δ and E. Therefore the segment ΔE is dissimilar to the segment ΓB.

[Proposition] 21

*When straight lines are drawn in parabolas so as to be perpendiculars to the axes and to cut off from the axes in the directions of the vertices of the sections the segments whose ratios to the latera recta in all sections are equal, then the segments that those perpendiculars cut off from one on the sections are similar to the segments that the other perpendiculars cut off from the other* *section, and their situation is similar, but they are not similar to any of other segments that are taken from those sections* [35].

Let there be two parabolas AB and EZ whose axes AΞ and BY and their *latera recta* be AΠ and BΣ. We draw in one of two sections the perpendiculars BM and ΔΞ, and in other section the perpendiculars ZΦ and PY, and let as AM be to AΠ, so EF be to EΣ, and let as ΞA be to AΠ, so EΨ be to EΣ.

Then I say that the segment BAO is similar to the segment ZEO, and that the arc ΔA is similar to the arc PE, and that the arc ΔB is similar to the arc ZP.

[Proof]. Now as to [the statement] that the segment BAO is similar to the segment ZEϘ; this will be proved as we proved [it] in Theorem 11 of this Book. Therefore we join ΔB and PZ and continue them in a straight line to [meet the respective axes at] K and Ω. We bisect ΔB and PZ at Θ and T, and draw through them ΓΘΛ and HTY parallel to the axes, and draw from Γ and H the perpendiculars ΓN and Hι to the axes cutting ΔK and PΩ at I and ς] .

Then the ratio of AΠ to each of AM and AΞ is equal to the ratio of EΣ to each of EΨ [respectively].

Therefore it will be proved from that, as we proved in Theorem 11 of this Book, that as sq.ΔΞ is to sq.BM, so sq.PΨ is to sq.ZΦ. Therefore as ΔΞ is to BM, so PΨ is to ZΦ, and as HK is to KM, so ΨΩ is to ΩΦ.

And convertendo as KΞ is to ΞM, so ΩΨ is to ΨΦ.

Furthermore as sq.ΔΞ is to sq.BM, so sq.PΨ is to sq.ZΦ. Therefore as ΞA is to AM, so ΨE is to EΦ because of what is proved in Theorem 20 of Book I.

And convertendo as AΞ is to ΞM, so EΨ is to ΨΦ.

But we have proved that as KΞ is to ΞM, so ΩΨ is to ΨΦ. Therefore as KΞ is to ΞA, so ΩΨ is to ΨE.

But as ΞA is to ΞΔ, so EΨ is to ΨP. Therefore as KΞ is to ΞΔ, so ΩΨ is to ΨP. And the angles at Ξ and Ψ are right. Therefore the triangle KΞΔ is similar to the triangle ΩΨP, and [hence] the angles at K and Ω are equal, and as ΔK is to KB, so PΩ is to ΩZ.

And convertendo as KΔ is to ΔB,so ΩP is to PZ.

And ΔB was bisected at Θ, and PZ was bisected at T. Therefore ΞΔ is to ΞΛ, so ΨP is to ΨY.

But ΛΞ is equal to ΓN and ΨY is equal to Hι . Therefore as ΔΞ is to ΓN, so ΨP is to Hι.

And therefore as ΞA is to AN, so ΨE is to Eι, axis proved in Theorem 20 of Book I.

And convertendo as AΞ is to ΞN, so EΨ is to Ψι.

But we have proved that as ΚΞ is to ΞΑ, so ΩΨ is to ΨΕ. Therefore as ΚΞ is to ΞΝ, so ΩΨ is to Ψι. And therefore as ΚΔ is to ΔΙ, so ΩΡ is to Ρς.

And *separando* as ΚΙ is to ΙΔ, so Ως is to ςΡ.

But it was shown that as ΚΘ is to ΚΔ , so ΩΤ is to ΤΡ. Therefore as ΚΘ is to ΘΙ, so ΩΤ is to ΤΓ.

But as ΙΘ is to ΘΓ, so ςΤ is to ΤΗ because the triangle ΙΘΓ is similar to the triangle ςΤΗ. Therefore as ΚΘ is to ΘΓ, so ΩΤ is to ΤΗ.

But ΘΚ is equal to the tangent drawn from Γ to the axis because it is parallel to ΘΚ, and they are between parallel straight lines [ΓΛ and ΚΞ].

Similarly too ΩΤ is equal to the tangent drawn from Η to the axis. Therefore the ratio of the tangent drawn from Η to ΗΤ is equal to the ratio of the tangent drawn from Γ to ΓΘ.And it was proved in Theorem 17 of this Book that, when that is the case, and when the angles formed by the tangent and the axis are equal [in both sections], then the segments from the vertices of which the tangents are drawn are similar. Therefore the segments ΔΓΒ and ΡΗΖ are similar and similarly situated.

Furthermore, we make the segment □ α a segment which is not cut off by the mentioned perpendiculars, then I say that it is not similar to the segment ΔΓΒ.

[Proof]. For the segment ΔΓΒ is similar to the segment ΡΗΖ, but the segment ΡΗΖ is dissimilar to the segment □α ,as is proved in Theorem 19 of this Book because it is not cut off by the same pair of perpendiculars [as the segment □α ]. Therefore the segment □α is not similar to the segment ΔΓΒ.

## [Proposition] 22

*For similar hyperbolas and ellipses the same properties hold as we proved hold for parabolas in the preceding theorem* [36].

Let the situation described for the parabola remain the same [for the hyperbola and the ellipse], and let the diameters ΓΘ and ΗΤ end at centers Λ and Υ [respectively].

We draw from Γ and Η tangents ΓΧ and Η□ to the and Η tangents ΓΧ and Η□ the sections. Then they are parallel to ΔΚ and ΡΩ [respectively].

` Now the ratio of ΑΜ to the *latus rectum* [of ΑΒΓ] is equal to the ratio of ΕΦ to the *latus rectum* of other section. Therefore ,since the sections are similar, then their *eidoi* are also similar, as is proved in Theorem 12 of this Book

Therefore the ratio of the transverse diameter of one of the sections to the *latus rectum* is equal to the ratio of the transverse diameter of other section to its *latus rectum*.

And we had made the ratio of two *latera recta* to ΑΜ and ΕΦ [respectively] equal. Therefore, since that is the case, and since the *eidoi* of two sections are similar, then it will be proved, as was proved in Theorem 12 of this Book, that the straight lines can be drawn in the segment ΒΑΟ parallel to ΒΟ, and in the segment ΖΕϘo parallel to ΖϘ, and the number of the straight lines drawn in the segment ΖΕϘ is equal to the number of the straight lines drawn in the segment ΒΑΟ, and their ratios are equal to their ratios, and the ratios of the straight lines drawn in the segment ΖΕϘ, and [also] of ΖϘ to the portions they cut off from the axis adjoining Ε are equal to the ratios of the straight lines drawn in [the segment] ΒΑΟ, and [also] of ΒΟ to the portions they cut off from the axis adjoining Α and [also] the ratios of the portions cut off the axis ΑΜ to the portions cut off from the axis ΕΦ are equal, therefore the segmentσ ΒΑΟ and ΖΕϘ are similar.

Furthermore the ratio ΑΜ to ΑΠ is equal to the ratio ΕΦ to ΕΣ. And also as ΑΞ is to ΑΠ, so ΕΨ is to ΕΣ. Therefore as ΔΞ is to ΑΞ ,so ΡΨ is to ΕΨ, and as ΒΜ is to ΑΜ, so ΖΦ is to ΕΦ. And as ΞΑ is to ΨΕ, so ΑΜ is to ΕΦ, and as AM is to ΜΒ, so ΕΦ is to ΖΦ. Therefore as ΔΞ is to ΒΜ, so ΡΨ is to ΖΦ, and as ΞΚ is to ΚΜ, so ΨΩ is to ΩΦ. And convertendo as ΚΞ is to ΞΜ, so ΩΨ is to ΨΦ.

But as ΞΜ is to ΞΑ, so ΨΦ is to ΨΕ because as ΞΑ is to ΑΜ, so ΨΕ is to ΕΦ. Therefore as ΚΞ is to ΞΑ, so ΩΨ is to ΨΕ.

But as ΞΑ is to ΞΔ, so ΕΨ is to ΨΡ. Therefore as ΚΞ is to ΞΔ, so ΩΨ is to ΨΡ. And the angles at Ξ and Ψ are right. Therefore the angles at Κ and Ω are also equal. Therefore the angles at Χ and □ are equal. And the sections are similar, therefore their *eidoi* are similar.

And ΓΧ and Η□ are tangents. Therefore as pl.ΛΝΧ is to sq.ΓΝ , so pl.Υι□ is to sq.Η ι, because of what is proved in Theorem 37 of Book I. And as sq.ΓΝ is to sq.ΝΧ, so sq.Ηι is to ι□ because of the similarity of the triangles ΓΝΧ and Ηι□ .Therefore as pl.ΛΝΧ is to sq.ΝΧ, so pl.Υι□ is to sq.□ ι .Therefore as is to ΛΝ is to ΝΧ, so Υι is to □ι .

But as ΝΧ is to ΓΝ, so □ι is to Ηι because of the similarity of the triangles [ΓΝΧ and Ηι□ ]. Therefore as ΛΝ is to ΓΝ, so Υι is to Ηι , and the angles [at] Ν and ι are right. Therefore the triangle ΛΝΓ is similar to the triangle ΥιΗ. Therefore the angles at Λ and Υ are equal.

But it was [already] shown that the angles at X and □ are equal. Therefore as XΛ is to ΓΛ, so □Y is to YH, and as XK is to ΓΘ, so Ω□ is to HT because ΓX is parallel to ΘK, and H□ to TΩ.

Furthermore the *eidoi* of two section are similar, therefore as AM is to MB, so EΦ is to ΦZ.

But as MB is to MK, so ΦZ is to ΦΩ. Therefore as AM is to MK, so EΦ is to ΦΩ. And *dividendo* as AM is to AK, so EΦ is to EΩ.

Furthermore as AΛ is to AM, so EY is to EΦ because as AΛ is to AΠ, so EY is to EΣ, and as AΠ is to AM, so EΣ is to EΦ. Therefore as AΛ is to AK, so YE is to EΩ, and as AΛ is to AK, so EY is to YΩ.

Furthermore as ΛN is to NX, so Yι is to ι□ because of the similarity of the triangles. But as NΛ is to ΛX, so sq.AΛ is to sq.ΛX because of what is proved in Theorem 37 of Book I. And likewise as ι ιY is to ι ι□ , so sq.EY is to sq.Y□ .Therefore sq.AΛ is to sq.ΛX, so sq.EY is to sq.Y□ , and [hence] as AΛ is to ΛX, so EY is to Y□ .

But we have proved that as AΛ is to ΛK, so EY is to YΩ. Therefore as ΛX is to ΛK, so Y□ is to YΩ, therefore as ΛX is to XK, so Y□ is to □Ω . And as ΓX is to XΛ, so H□ is to □Y because the triangle ΓXΛ is similar to the triangle H□Y , therefore as ΓX is to XK , so H□ is to □Ω.

But we have proved above that as XK is to ΓΘ, so □Ω is to HT, therefore as ΓX is to ΓΘ, so H□ is to HT .

And the angles at X and □ are equal. Therefore the segments ΔΓB and PHZ are similar and similarly situated, as is proved in Theorem 18 of this Book.

Furthermore we make a segment not cut off by the mentioned perpendiculars, and also [in the case of the ellipse] not cut off by perpendiculars whose distances from the center is equal to that of others perpendiculars, then I say that it is dissimilar to the segment AΓB.

[Proof]. For let, if possible, it be similar to it. Now the segment ΔB is similar to the segment PZ. Therefore the segment Iα is similar to the segment PZ. But it is not cut off by the same perpendiculars [as PZ], nor [in the case of the ellipse] by perpendiculars whose distance from the center is equal to the distance of [those perpendiculars]. But that is impossible, as is proved in Theorems 19 and 20 of this Book. Therefore the segment Iα is not similar to the segment PZ, nor to the segment ΔΓB.

[Proposition] 23

*In sections that are not similar no segment of one of them is similar to an segment of another* [37].

Let there be two dissimilar sections ΑΒ and ΓΔ. And first let them both be hyperbolas or ellipses.

Then I say that no segment of ΑΒ is similar to an segment of ΓΔ.

[Proof]. For let, if that is possible, the segment ΒΕ be similar to the segment ΔΖ.We join ΒΕ and ΔΖ, and bisect them at Η and Θ. Let the centers of the sections be Κ and Λ We join ΗΜΚ and ΘΝΛ, then they are diameter to the sections, as is proved in Theorem 47 of Book I. Now ΗΝΚ and ΘΝΛ are either axes or not. Therefore, if they are axes, and the segments ΒΕ and ΔΖ are similar, then there can be drawn to the axis straight lines parallel to ΕΒ such that the ratios of them and the ratio of ΒΕ to the portions cut off [by these straight lines], and the ratio of ΒΕ to the portions cut off [by these straight lines} from the axis adjacent to its vertex are equal to the ratios of the straight lines equal in number to those [first straight lines] drawn to other axis parallel to ΔΖ and [to the ratio] of ΔΖ to the portions cut off [by them] from the axis of other section adjacent to its vertices, and [such that] the ratios of the segments cut off from one of the axes to the segments cut off from other axis are [all] equal, and the parallel straight lines are perpendiculars to the axes, therefore the sections ΑΒ and ΓΔ will be similar.

But if the diameters ΗΜΚ and ΘΝΛ are not axes then we make the axes ΑΚ and ΓΛ, and draw from Μ and also draw from them [ΜΝ] tangents to the section ΜΣ and ΝΞ. Then, since the segments ΒΕ and ΔΖ are similar, and the tangents ΜΣ and ΝΞ have been drawn from their vertices it will be proved thence, as was proved in Theorem 18 of this Book that the triangle ΜΣΚ is similar to the triangle ΝΞΛ. And ΜΠ and ΝΡ are perpendiculars [to the axes]. Therefore as pl.ΚΠΣ is to sq.ΜΠ, so pl.ΛΡΞ is to sq.ΝΡ [38].

But the ratio pl.ΚΠΣ to sq.ΜΠ is equal to the ratio of the transverse diameter of the section ΑΒ to its *latus rectum*, as is proved in Theorem 37 of Book I. And likewise the ratio pl.ΛΡΞ to sq.ΝΡ is equal to the ratio of the transverse diameter of the section ΓΔ to its *latus rectum*.

Therefore the ratio of the transverse diameter of the section ΑΒ to its *latus rectum* is equal to the ratio of the transverse diameter of the section ΓΔ to its *latus rectum*. Therefore the *eidoi* of the sections ΑΒ and ΓΔ are similar.

But then that is the case, then the sections are similar, as is proved in Theorem 12 of this Book. Therefore the sections ΑΒ and ΓΔ are similar, but we had made them dissimilar, that is impossible. Therefore the segment ΑΕ is not similar to the segment ΔΖ.

[Proposition] 24

Furthermore if we make the section ΑΒ a parabola and the section ΓΔ a hyperbola for an ellipse, then it is evident that one section is not similar to other, because of what we said in Theorem 14 of this Book.

Then I say that the segments ΒΕ and ΔΖ are dissimilar [39].

[Proof]. For if they are similar, then it is possible to draw in them straight lines, equal in number parallel to the straight lines ΒΕ and ΔΖ [respectively] , such that the ratios of these [straight lines] to the portions they cut off from one of the diameters adjacent to the vertices [Μ] of the [first] segment are equal to the ratios of the straight lines cutting other diameter to the portions they cut off from it adjacent to the vertices [Ν] of the segment, and also that the ratio of the base [of the first segment] to [its] diameter is equal to the base [of the second segment] to [its] diameter, and [also that] the ratios of the divisions of one of the diameters [formed by these straight lines] are equal to the ratios of the divisions of other diameter. Then if will be proved, as it was proved for the sections in their entirety in Theorem 14 of this Book, but that impossible. But if one of sections is a hyperbola and other is an ellipse, then impossibility of that will be proved at it was proved in Theorem 16 of this Book.

[Proposition] 25

*It is not possible for a part of any of three conic sections to be an arc of a circle* [40].

Let there be the [conic] section ΑΒΓΔ.

I say that it is not possible for a part of it to be an arc of a circle.

[Proof]. For let, if it is possible, ΑΒΓ be an arc of a circle. We draw in it two straight lines ΑΒ and ΓΒ not parallel to each other in arbitrary positions. We also draw in it ΖΗ not parallel not to them, and draw ΖΘ parallel to ΑΒ and ΗΚ parallel to ΓΕ, and [also] draw ΕΔ parallel to ΖΗ. We bisect the straight lines we draw at Μ, Ν, Ξ, Ο, Π, and Ρ, and join ΜΝ, ΞΟ, and ΠΡ, then these straight lines are diameters to the circle, and they bisect the straight lines drawn by us, therefore they are perpendiculars to them. But they are also diameters to the section because of what was proved in Theorem 28 of Book II. Therefore ΜΝ, ΞΟ, and ΠΡ are axes of the section. But none of them lies on a straight line with its follow because three original straight lines are not parallel. That is impossible for none of sections has more than two axes, as is proved in Theorem 50 of

Book II. Therefore if is not possible for a part of any of sections to be an arc of a circle.

[Proposition] 26

*If ones are cut on one side [of their axes] by parallel planes from the class of planes which, when continued on the side of the vertex of the cone, subtend its exterior angle, then the hyperbolas generated [by these planes] are similar but not equal* [41].

Let there be the cone ΑΒΓ, and let it be cut by two parallel planes, and let their intersections with the base [of the cone] be ΘΜ and ΚΝ. We draw from the center of the base of the cone the perpendicular ΒΛΗΓ to these straight lines. Let the cone be cut by [another] plane passing through ΒΓ and the axis of the cone, and let this plane cut the surface of the cone in ΑΒ and ΑΓ. Let the intersections of this plane with two parallel planes be ΛΔ and ΖΗ, we continue them to [meet continued ΓΑ at] Ο and Ε [respectively].Then I say that the section ΘΖΜ is similar to the section ΚΔΝ, but not equal to it.

[Proof]. We draw from Α a straight line ΑΠ parallel to ΔΛ and ZH. We make the ratio ΟΔ to ΔΞ equal to the ratio sq.ΑΠ to pl.ΒΠΓ, and also the ratio ΕΖ to ΖΙ equal to the ratio sq.ΑΠ to pl.ΒΠΓ. Then since ΒΛ is perpendicular to ΚΝ, the straight lines drawn in the hyperbola ΚΔΝ to ΔΛ parallel to ΚΝ are equal in square to the rectangular planes applied to ΔΞ [which is the *latus rectum*] and in increasing it by a rectangular plane similar to pl.ΟΔΞ as is proved in Theorem 12 of Book I .

Similarly too the straight lines drawn in the hyperbola ΘΖΜ to ΖΗ parallel to ΘΜ are equal in square to the rectangular planes applied to ΖΙ [which is the *latus rectum*] and exceeding it by a rectangular plane similar to pl.ΕΖΙ. And the angles formed by ΚΝ with ΔΛ are equal to the angles formed by ΘΜ with ΖΗ because they are parallel to them.

Therefore the sections are similar, as is proved in Theorem 12 of this Book. And pl.ΟΔΞ ιs smaller than pl.ΕΖΙ. Therefore the sections ΘΖΜ and ΚΔΝ are unequal because of what is proved in Theorem 2 of this Book.

[Proposition] 27

*If a cone is cut by parallel planes that meet two sides of the triangle passing through its axis, but not parallel to the base of the cone and not anti-parallel to it, then the ellipses [by these planes] are similar, but unequal* [42].

Let the cone ΑΒΓ be cut by two parallel planes, and let the intersections of these planes with the plane of the base of the cone be ΘΜ and ΚΝ. We draw through the center of the base of the cone a straight line ΒΓΗΑ which is a perpendicular to ΘΜ and ΚΝ, we cut the cone with [another] plane passing through this straight line and through the axis of the cone, and let the intersections of this plane with two parallels planes be ΖΕΗ and ΔΟΛ.

Then I say that sections ΖΣΕ and ΔΡΟ are similar but not equal.

[Proof]. We draw from A a straight line ΑΠ parallel to ΖΗ and ΔΛ. Let each of the ratios ΟΔ to ΔΞ and ΕΖ to ΖΙ be equal to the ratio sq.ΑΠ to pl.ΒΠΓ. Then since ΒΓΛ is perpendicular to ΚΝ, the straight lines drawn in the ellipse ΔΡΟ to ΔΟ parallel to ΚΝ are equal in square to the rectangular planes applied to ΔΞ [which is the *latus rectum*] and decreasing of it by the rectangular planes similar to pl.ΞΔΟ, as is proved in Theorem 12 of Book I. Similarly too the straight lines drawn in the ellipse ΖΣΕ to ΖΕ parallel to ΘΜ are equal in square the rectangular planes applied to ΖΙ [which is the *latus rectum*] and de creasing of it by the rectangular planes similar to pl.ΕΖΙ. And the angle ΚΛΔ is equal to the angle ΘΗΖ because ΚΛ and ΛΔ are parallel to ΘΗ and ΗΖ [respectively]. And pl.ΟΔΞ is similar to pl.ΕΖΙ. But when that is the case, then two sections are similar, as is proved in Theorem 12 of this Book.

Therefore the sections ΔΡΟ and ΖΣΕ are similar. But then are unequal because pl.ΕΖΙ is greater than pl.ΟΔΞ, and it was proved in Theorem 2 of this Book that, when that is so then two sections are unequal.

[Proposition] 28

*Want to show how to find in a given right cone a parabola equal to a given parabola* [43].

Let the given right cone be the cone with the axial triangle ΑΒΓ. Let the given parabola be the section ΔΕ with axis ΔΛ and the *latus rectum* ΔΖ, and let as *ΔΖ* is to ΑΗ, so sq.ΓΒ is to pl.ΒΑΓ. We draw ΗΘ to ΑΓ. We cut the cone with a plane passing through ΗΘ and erected at right angles to the plane ΑΒΓ, let [this plane] generate the section ΚΗ whose axis is ΗΘ.

Then I say that the section ΚΗ is equal to the section ΔΕ.

[Proof]. The perpendiculars drawn in the section ΚΗ to ΗΘ are equal in square to the rectangular plane applied a straight line whose ratio to ΑΗ is equal to the ratio sq.ΒΓ to pl.ΒΑΓ, as is proved in Theorem 11 of Book I.

But the ratio ΔΖ to ΑΗ also is equal to the ratio sq. ΒΓ to pl.ΒΑΓ. Therefore ΔΖ is equal to the *latus rectum* of the section ΚΗ. And it was proved in

Theorem 1 of this Book that, when that is the case, these two sections are equal. Therefore the section ΔE is equal to the section KH.

Then I say that no other section ,apart from this one, can be found in [this] cone such that the point of its vertex [which is the end of the axis] lies on the straight line AB [and such that it is equal to the section ΔE] 44 for ,if it is possible to find another parabola equal to the section ΔE, then its plane cuts the plane of the axial triangle of the cone at right angles, and the axis of the section lies in the plane of the triangle ABΓ because the cone is a right cone [and similarly for the axis of every section in a right cone].

Therefore if it is possible for another section whose vertex lies on AB to be equal to the section ΔE, then its axis is parallel to AΓ, and the point of its vertex is different from H. And the ratio of its *latus rectum* to the straight line cut off by that section from AB adjacent to A is equal to the ratio sq.BΓ to pl.BAΓ. But this [latter] ratio is equal to the ratio ΔZ to AH. Therefore ΔZ is not equal to the *latus rectum* of that other section.

But these two sections are [supposed to be] equal, that is impossible because of that was proved in Theorem 1 of this Book.

Therefore there cannot be found on AB the vertex of the axis of another section equal to the section ΔE.

[Proposition] 29

*We wait to show how to find in a given right cone a section equal to a given hyperbola, when the ratio of the square on the axis of the cone to the square on the half of the diameter of the base is not greater than the ratio of the transverse diameter [which is the axis of the given section] to the* latus rectum [45].

Let the given cone be the cone on its axial triangle ABΓ, with axis AΘ, and let the given hyperbola be ΔE whose axis ΔϘ and the *eidos* pl.HΔZ.

And first let the ratio sq.AΘ to sq.ΘB is equal to the ratio HΔ to ΔZ. We draw in [exterior] angle BAΠ the straight line ΠN parallel to AΘ and equal to HΔ, And draw through ΠN a plane at right angles to the plane of the triangle ABΓ, then it will cut the cone, and its intersection will be the hyperbola whose axis IN. Then, since AΘ is parallel to ΠN, the ratio of ΠN [which the transverse diameter] to the *latus rectum* of [that] section is equal to the ratio sq.AΘ to pl.ΓΘB, as is proved in Theorem 12 of Book I, and [therefore] it also it equal to the ratio HΔ to ΔZ.

But ΠN is equal to HΔ. Therefore ΔZ is equal to the *latus rectum* of the section whose axis IN. Therefore the *eidos* of the section whose axis IN is equal to the *eidos* of the section ΔE, and the section ΔE and the section whose axis IN are equal because of what is proved in Theorem 2 of this Book.

[Furthermore] no other section can be found equal to the section ΔE with the vertex of its axis on the straight line AB.

[Proof]. For, if that is possible, then the axis of that section lies in the plane of the triangle ABΓ, as is proved in the preceding theorem ,and the triangle ABΓ will be at right angles to the plane in which that other section lies. And since that section is a hyperbola, and is equal to the section ΔE, its axis will meet AΓ beyond A, and the portion of the axis drawn from the triangle to the point where it meets AΓ will be equal to the straight line ΔH, as is proved in Theorem 2 of this Book.

But this [portion] is not ΠN, nor is it parallel to it, for if it were parallel to it, it would be unequal to it. And, when that is the case, if a straight line is drawn from A parallel to that axis, it will fall either between AΘ and AΓ, or between AΘ and AB.

Therefore let the straight line that is parallel to it [the axis of other section] be AM. Then as sq.AM is to pl.BMΓ, so ΔH is to ΔZ, as is proved in Theorem 12 of Book I and Theorem 2 of this Book. But that is impossible for sq.AM is greater than sq.AΘ, and pl.BMΓ is smaller than pl.BΘΓ.

Furthermore we [now] make the ratio sq.AΘ to sq.ΘB smaller than the ratio HΔ to ΔZ, and describe on the triangle ABΓ a circle ABΓ circumscribing it, and continue AΘ to [meet the circle at] Σ, then the ratio AΘ to ΘΣ is smaller than the ratio HΔ to ΔZ.

Therefore let the ratio AΘ to ΘX be equal to the ratio HΔ to ΔZ, and let PΞ be parallel to BΓ. We join AMΞ and AKP. Let each of ΠN and TO be equal to ΔH, and let TO be parallel to AM, and ΠN parallel to AK. We draw through ΠN and TO planes at right angles to the plane of ABΓ, therefore as to generate in the cone two hyperbolas on the axes ΛO and IN.Then the ratio HΔ to ΔZ is equal to the ratio AΘ to ΘX, and to the ratios AM to MΞ and sq.AM to pl.AMΞ. But pl.AMΞ is equal to pl.BMΓ. Therefore as ΔH is to ΔZ, so sq.AM is to pl.BMΓ. But the ratio sq.AM to pl.BMΓ is equal to the ratio of TO [which is the transverse diameter of the *eidos* of the section on the axis OΛ] to its *latus rectum*, as is proved in Theorem 12 of Book I.

Therefore the *eidoi* of the section ΔE and the section on the axis OΛ are equal. And it was proved in Theorem 2 of this Book that, when that is the case, then the section ΔE and the section on the axis NI are equal.

Similarly too it will be proved that the section ΔE is equal to the section on the axis NI.

[Furthermore] no other, third section can be found with the vertex of its axis on one of AB and AΓ equal to the section ΔE.

[Proof]. For, if it is possible to find a section other than those mentioned sections, then its axis lies in the plane of ABΓ, as was proved in the case of the parabola. Therefore we draw AY parallel to that axis then we will prove, as we proved above, that AY does not coincide with AK, nor with AM, and that the ratio ΔH to ΔZ is equal to the ratio sq.AY to pl.BYΓ, and is equal to the ratio sq.AY to pl.AYΩ because pl.AYΩ is equal to pl.BYΓ. But the ratio sq.AY to pl.AYΩ is equal to the ratio AY to YΩ. Therefore as ΔH is to ΔZ , so AY is to YΩ. That is impossible because as ΔH is to ΔZ, so AΘ is to ΘX, and as AΘ is to ΘX, so AY is to YΨ.

Furthermore we [now] make the ratio sq.AΘ to sq.ΘB greater than the ratio ΔH to ΔZ. Then I say that no section can be found in the cone equal to the section ΔE.

[Proof]. For, if it can be found, then we draw AM parallel to the [transverse] diameter of that section. Then as sq.AM is to pl.BMΓ, so ΔH is to ΔZ But the ratio sq.AΘ to pl.BΘΓ is greater than the ratio ΔH to ΔZ. Therefore the ratio sq.AM to pl.BMΓ is smaller than the ratio sq.AΘ to pl.BΘΓ. But sq.AM is greater than sq.AΘ and pl.BMΓ is smaller than pl.BΘΓ. That is impossible , therefore no section can be found in the cone equal to the section ΔE.

## [Proposition] 30

*We want to show how to find in a given right cone a section equal to a given ellipse* [46].

Let there be the given right cone on the axial triangle ABΓ, and let the given ellipse be the section ΔE whose axis ΔH and the *latus rectum* ΔZ.

We draw on the triangle ABΓ the circle ABΓ circumscribing it, and make the ratio AM to MΞ equal to the ratio ΔH to ΔZ, it is evident that this is easily possible, and draw in the triangle ABΓ the straight line OΠ parallel to AM and equal to ΔH. We draw through OΠ a plane cutting the cone and erected at right angles to the plane of the triangle ABΓ. Then this will generate in the cone the ellipse whose axis OΠ, and the ratio of OΠ to its *latus rectum* will be equal to the ratio sq.AM to pl.BMΓ, as is proved in Theorem 13 of Book I.

But pl.BMΓ is equal to pl.AMΞ. Therefore the ratio of OΠ, which is the transverse diameter of that section to its *latus rectum*, is equal to the ratio sq.AM to pl.AMΞ.

But the ratio sq.AM to pl.AMΞ is equal to the ratio AM to MΞ, and as AM is to MΞ, so ΔH is to ΔZ. Therefore the ratio of OΠ to the *latus rectum* of the section with axis OΠ is equal to the ratio ΔH to ΔZ, and the *eidoi* of the section ΔE and of the section with axis OΠ are similar and equal. Therefore the sections themselves are equal, as is proved in Theorem 2 of this Book.

I [also] say that no other section can be found in this cone with that vertex which is closer to A lying on AB, which is equal to the section ΔE.

[Proof].For, if that is possible. Then we will prove, as we proved in Theorem 28 of this Book. That is its axis lies in the plane of the triangle ABΓ , and that its plane is at right angles to the plane of the triangle ABΓ.

And, since that section is an ellipse, its axis will meet BΓ, and since it is equal to the section ΔE, its axis is equal to ΔH, as is proved in Theorem 2 of this Book. And that vertex which is closer to A lies on AB. Therefore its axis does not coincide with OΠ, nor it is parallel to it, and [hence]. When we draw from A a straight line parallel to that axis it will not coincide with AM.

Therefore let it be as AϘΦ. Then AΦ will cut the arc AΓ because it is not parallel to BΓ. And the ratio of the transverse diameter [of the section] to its *latus rectum* will be equal to the ratio sq.AΦ to pl.BΦΓ, as is proved in Theorem 13 of Book I. And it also is equal to the ratio ΔH to ΔZ.

But pl.BΦΓ is equal to pl. AΦϘ . Therefore the ratio sq.AΦ to pl.AΦ Ϙ is equal to the ratio ΔH to ΔZ.

But the ratio sq.AΦ to pl. AΦϘ is equal to the ratio AΦ to ΦϘ, and as ΔH is ΔZ, so AM is to MΞ. Therefore the ratio AΦ to ΦϘ is equal to the ratio AM to MΞ, which is impossible. Therefore no other section besides the section with axis OΠ can be found in this cone equal to the section ΔE with the point of that vertex which is closer to A lying on AB.

## [Proposition] 31

*We want to show how to find a right cone containing a given parabola and similar to a given right cone* [47].

Let the parabola be BAΓ whose axis AΛ, and the *latus rectum* AΔ for that section, and the given one EZK with the axial triangle EZK.

We draw through AΛ a plane ΘΛ at right angles to the plane of the section BAΓ, and draw in that plane the straight line AM, which we make the form together with AΛ the angle equal to the angle EZK. We make the ratio ΔA to AM equal to the ratio KZ to ZE,and draw on AM the triangle AΘM similar to the triangle EZK, and draw ΘA and ΘM from A and M, and construct the cone with vertex Θ and base the circle drawn on AM as its diameter, and perpendicular to the plane AΘM. Then the angle MAΛ is equal to the angle EZK.

But the angle EZK is equal to the angle ΘMA. Therefore the angle MAΛ is equal to the angle ΘMA.Therefore AΛ is parallel to ΘM being a side of the axial triangle [of the cone]. Therefore the plane in which lies the given section generates in the cone a parabola. And the ratio ΔA to AM is equal to the ratio KZ to ZE and to the ratio AM to MΘ. Therefore the ratio AΔ to AM is equal to the ratio AM to AΘ because AΘ is equal to MΘ. Therefore the ratio sq.AM to sq.AΘ is equal to the ratio AΔ to AΘ..But sq.AΘ is equal to pl.AΘM. Therefore the ratio sq.MA to pl.AΘM is equal to the ratio ΔA to AΘ. Therefore the *latus rectum* of the section generated in the cone is ΔA. But it is also the *latus rectum* of the section BAΓ.

And the parabolas with equal *latera recta* are [them selves] equal, as is proved in Theorem 1 of this Book. Therefore the section BAΓ is placed in the cone that we constructed, and the cone that we constructed is similar to the cone EZK because the triangle EZK is similar to the triangle AΘM. Then I say that this section is not found in any other cone a part from this one similar to the cone EZK with its vertex on this side of the plane of the section.

[Proof]. For let, if that is possible, there be another cone containing this section and similar to the cone EZK. The vertex of this cone is I. Let there pass through the axis of [this] cone a plane perpendicular to the plane of the given section, then it will cut it, and the position of the intersection in which this plane cuts that plane will be the axis of the section.

But AΛ is the axis of the section, therefore AΛ is the intersection of these two planes.

But the plane ΘΛ is at right angles to the plane in which lies the section and it passes through AΛ Therefore I lies in the plane ΘΛ. Let IN and IΛ be the sides of the cone. Then IN is parallel to AΛ, and the angle ZEK is equal to the angle AIN and to the angle AΘM. Therefore AI lies on the same straight line as AΘ, and we continue AM to [meet IN at] Ξ. Now the section BAΓ is in the cone with vertex I. Therefore if we make the ratio of some straight line to AI equal to the ratio sq.AΞ to pl.AIΞ, then that straight line will be the *latus rectum* of the section BAΓ.

But AΔ is the *latus rectum* of the section ABΓ. Therefore as sq.AΞ is to pl.AIΞ, so ΔA is to AI. And the ratio sq.AM is to pl.AΘM was shown be equal to the ratio ΔA to AΘ.

But as sq.AM is to pl.AΘM, so sq.AΞ is to pl.AIΞ because of the similarity of the triangles. Therefore as ΔA is to AΘ, so ΔA is to AI, that is impossible.

Therefore no other cone can be found containing that section, similar to the cone ZEK, and such that the point of its vertex is on this side of the plane in which the section lies.

[Proposition] 32

*We want to show how to construct a right cone similar to a given right cone containing a given hyperbola*[48].

[For this problem to be soluble] it is necessary that the ratio of the square on the axis of that cone to the square on the radius of its base be not greater than the ratio of the transverse diameter of the *eidos* corresponding to the axis of the section to its *latus rectum*.

Let there be the given hyperbola BAΓ whose axis AΛ and transverse diameter AN, and let the *eidos* corresponding to the axis of this sections be pl.NAΔ. Let the given cone be the cone with the axial triangle EZK.

We continue KE to Ψ, and draw through AΛ the plane ΘΛ at right angles to the plane in which lies the section. We draw in this plane on NA the segment NΘA of a circle admitting an angle equal to the angle ΨEZ, and complete the circle and bisect the arc NΘA at Θ. We draw from Θ the perpendicular ΘΞ to AN [and continue it to meet the circle again at Σ].

And first let the ratio of the square on EH [which is the axis of the cone] to the square on ZH be equal to the ratio NA to AΔ. We continue NΘ in a straight line from Θ as NM, and draw AM parallel to ΘΣ. Then, since the arc NΣ is equal to the arc ΣA, the angle NΘΣ is equal to the angle ΣΘA.

Therefore the angle MAΘ is equal to the angle ΘMA.

Therefore we construct the equilateral cone with vertex Θ, and base the circle with diameter AM and plane at right angles to the plane ΘAΛ.

Then, when that is so, the plane in which lies the given section generates in [this] cone the hyperbola with whose axis AΛ and the transverse diameter AN. And the angle AΘM is equal to the angle ZEK because the segment AΘN admits an angle equal to the angle ZEΨ. And is equal to ΘM, and ZE is equal to ZK. Therefore we draw the perpendicular ΘΠ [to AM].

Then as sq.EH is to pl.KHZ, so sq.ΘΠ is to pl.MΠA.

But as sq.EH is to pl.KHZ, so NA is to AΔ. Therefore as sq.ΘΠ is to pl.MΠA, so NA is to AΔ. Therefore the ordinates in the generated section falling on AΛ are equal in square to the rectangular planes applied to AΔ and increasing it by a rectangular plane similar to pl.NAΔ as is proved in Theorem 12 of Book I.

And the perpendiculars falling from the section ΒΑΓ on ΑΛ are also equal in square to the rectangular planes applied to ΑΔ and increasing it by a rectangular plane similar to pl.ΝΑΔ. Therefore the section ΒΑΓ is equal to the section generated in the cone with vertex Θ and base the circle on the diameter ΑΜ as is proved in Theorem 2 of this Book.

And it lies in its plane, and its axis coincides with its axis. Therefore the cone with vertex Θ contains the section ΒΑΓ, and it is similar to the cone ΕΖΚ because as ΘΠ is to ΠΜ, so ΕΗ is to ΗΖ. Then I say that no cone, apart from one we constructed which is similar to the cone ΕΖΚ and has the point of its vertex on the same side of the plane in which lies the section ΑΒΓ as Θ, contains this section.

[Proof]. For let, if it is possible, another cone with its vertex at I contain it . Then it will be proved, as we proved in the preceding theorem; that I lies in the plane ΘΑΛ. Therefore let the sides of [that] cone be ΙΟ and ΙΑ. Now that cone is similar to the cone ΖΕΚ. Therefore the angle ΑΙΟ is equal to the angle ΖΕΚ, and the angle ΖΕΨ is equal to the angle ΑΙΝ. Therefore Ι lies on the arc ΑΘΝ, and ΟΙ ,when continued, will pass through Ν. So we join ΣΙ and draw from A the straight line ΑΟ parallel to it, and from Ι the straight line ΤΙ parallel to ΑΝ. Then the section ΒΑΓ lies in the cone with vertex Ι, and its axis ΑΛ has been continued to Ν. Therefore the ratio as sq.ΤΙ is to pl.ΑΤΟ is equal to the ratio of ΝΑ, the transverse diameter, to ΑΔ, the *latus rectum*.

But as ΝΑ is to ΑΔ, so sq.ΕΗ is to pl.ΖΗΚ. Therefore as sq.ΙΤ is to pl.ΟΤΑ, so sq.ΕΗ is to pl.ΖΗΚ, and the angle ΝΙΣ is equal to the angle ΣΙΑ, and they are equal to the angles ΙΑΟ and ΑΟΙ [respectively]. Therefore the angle ΙΑΟ is equal to the angle ΑΟΙ. And the angle ΑΙΟ is equal to the angle ΖΕΚ. Therefore the triangle ΑΙΟ is similar to the triangle ΖΕΚ. And we had proved that as sq.ΙΤ is to pl.ΟΤΑ, so sq.ΕΗ is to pl.ΖΗΚ.

But ΖΗ is equal to ΗΚ. Therefore ΑΤ is equal to ΤΟ. And the ratio ΑΤ to ΤΟ is equal to the ratio ΝΙ to ΙΟ and to the ratio ΝΡ to ΡΑ.Therefore ΝΡ is equal to ΡΑ. But that is impossible because ΘΣ is a diameter of the circle, and has cut ΝΑ at right angles at Ξ. Therefore no cone can be found other than the cone which we constructed, which is similar to the cone ΕΖΚ and contains the section ΒΑΓ. Furthermore we make the ratio sq.ΕΗ to sq.ΖΗ smaller than the ratio ΝΑ to ΑΔ, and carry out the construction as we did before, then as sq.ΕΗ is to pl.ΖΗΚ, so sq.ΘΠ is to pl.ΜΠΑ because of the similarity of two triangles [ΕΖΚ and ΘΑΛ]. And pl.ΜΠΑ is equal to sq.ΠΑ and to sq.ΘΞ. And sq.ΘΠ is equal to sq.ΑΞ. Therefore as sq.ΕΗ is to pl.ΖΗΚ, so sq.ΑΞ is to sq.ΘΞ. But sq.ΑΞ is equal to pl.ΣΞΘ. Therefore the ratio sq.EH to pl.ZHK is equal to the ratio sq.EH to sq.ZH and equal to the ratio pl.ΣΞΘ to sq.ΣΘ, and equal to the ratio ΣΞ to ΞΘ.

But the ratio sq.EH to sq.ZH is smaller than the ratio NA to AΔ. Therefore the ratio ΣΞ to ΞΘ is smaller than the ratio NA to AΔ. Therefore we make the ratio ΣΞ to ΞX equal to the ratio NA to AΔ, and draw through X a straight line IXTo parallel to NA. We join IN, IΣ, and IA, and draw from A the straight line AO parallel to IΣ.

Then it will be proved, as we proved in the preceding theorem, that the triangles OIA and ZEK are isosceles and similar. Therefore if we construct a cone with vertex I and base the circle with the diameter AO and in the plane perpendicular to the plane ΘAΛ, then the plane in which lies the section BAΓ will cut that cone, and from the cutting of the one by the other will result a hyperbola, and the axis of that section will be AΛ,and its transverse diameter AN and the ratio NA to AΔ is equal to the ratio ΣΞ to ΞX and to the ratio ΣP to PI. But the ratio ΣP to PI is equal to the ratio pl.ΣPI to sq.PI, and pl.ΣPI is equal to pl.NPA, therefore as pl.NPA is to sq.IP, so NA is to AΔ.

But as pl.NPA is to sq.IP, so sq.IT is to pl.OTA because the quadrangle ATIP is a parallelogram. Therefore as NA is to AΔ, so sq.IT is to pl.ATO.

Therefore AΔ is the *latus rectum* of the section generated in the cone AIO. Thence it will be proved, as we proved in the preceding part of this theorem, that the cone with the vertex I contains the section BAΓ, and it will also be contained by another equal to this cone, with the vertex Ϙ, when NϘ and AϘ are joined and NϘ continued. And these two cones will be similar to the cone EZK. Then I say that no third cone similar to the cone ZEK, and with the point of its vertex on the same side of the plane in which lies the section BAΓ as I can contain it.

[Proof]. For the point of its vertex will lie on the arc AIN, as we proved if the preceding theorem. Therefore let it be Y, we join YΦΣ. Then we will prove by the converse of the proof we made previously that as NA is to AΔ, so ΣΦ is to ΦY. But that  is impossible because the ratio NA to AΔ was made equal to the ratio ΣΞ to ΞX. Therefore  no third one similar to the cone EZK contains this section.

But if the ratio sq.EH to sq.ZH is greater than the ratio NA to AΔ, then it is not possible for a cone similar to the cone EZK to contain the section BAΓ.

[Proof]. For let, if it is impossible, it be contained by the cone with vertex I. Then we will prove  by a method like the preceding theorem that as ΣP is to PI, so NA is to AΔ. But the ratio NA to AΔ is smaller than the ratio sq.EH to sq.ZH, which we proved  to be equal to the ratio ΣΞ to ΞΘ. Therefore the ratio ΣP to PI is smaller than the ratio ΣΞ to ΣΘ, which is impossible. Therefore no cone [of this kind] similar to the cone ZEK will contain the section BAΓ.

[Proposition] 33

Let the given ellipse be ABΓ whose major axis AΓ, and *latus rectum* AΔ, and let given right cone be the cone EZK.

We want to show how to construct a right cone similar to a given right cone containing a given ellipse [49].

We draw through AΓ a plane at right angles to the plane in which lies the section ABΓ, and draw in it on AΓ the arc AΘΓ [of a circle] admitting an angle equal to the angle ZEK. We bisect it at Θ, and draw from Θ the straight line ΘIA in such way that as ΘΛ is to ΛI , so ΓA is to AΔ.

Similarly too we draw ΘΞ in such way that it is cut [by the circle] in the same ratio. We join AI and ΓI, and draw IΠ parallel to AΓ, and AΠ parallel to ΘΛ [cutting ΓI at M]. We construct the cone whose vertex I and base the circle with diameter AM. Then I say that this cone is similar to the cone EZK, and that it contains the section ABΓ.

[Proof]. The angle ΘIΓ is equal to the angle ΘAΓ because they are in the same arc. But the angle ΘIΓ also is equal to the angle IMΛ because ΘI and ΛM are parallel. But the angle MIA is equal to AΘΓ. Therefore the remaining angle [in the triangle IMA] the angle IAM is equal to the angle ΘΓA. Therefore the triangle AMI is similar to the triangle AΘΓ.

But the triangle AΘΓ is similar to the triangle EZK, and these triangles are isosceles. Therefore the triangle AMI is isosceles and similar to the triangle EZK. Therefore the cone with vertex I and base the circle on diameter AM is similar to the cone EZK. And the plane in which lies the section ABΓ generates in this cone the ellipse whose major axis AΓ. And the ratio ΓA to AΔ is equal to the ratio ΘΛ to ΛI and to the ratio pl.ΘΛI to sq.ΛI. But pl.ΘΛI is equal to pl.ΓAΛ. Therefore as ΓA is to AΔ, so pl.ΓAΛ is to sq.AI.

But as pl.ΓAΛ is to sq.AI, so sq.HI is to pl.AΠM because thee quadrangle ΠAΛI is a parallelogram. Therefore as ΓA is to AΔ, so sq.ΠI is to pl.AΠM. And AΓ is the transverse diameter, therefore AΔ is the *latus rectum* of the section generated in the cone. And it is also the *latus rectum* of the section ABΓ.

Therefore the section ABΓ is contained in the cone that we constructed because of what is proved in Theorem 2 of this Book.

Similarly too it will be proved that it is contained in another cone with vertex N whenever AN and NΓ are drawn.

[Furthermore] no other, third cone similar to the cone ZEK with the point of its vertex on this side of the plane [of ΑΒΓ] contains this section.

[Proof]. For, if it is possible that some other contains it, then we will prove, as we proved in the preceding theorem, that if there is drawn through its axis a plane at right angles to the plane in which the section lies, then that intersection of these two planes is the major of two axes of the section.

And we will also prove, as we proved in the case of the hyperbola in the preceding theorem that the point of vertex of the cone lies on the arc ΑΘΓ. Let this point be Ο, and let the sides of the cone be ΟΑ and ΟΗ. We draw through Ο and Θ the straight line ΘΟΡ and draw ΑΣ parallel to ΘΡ, and ΟΣ parallel to ΑΓ. Then the triangle ΟΑΗ is as isosceles, and as sq.ΟΣ is to pl.ΑΣΗ, so ΓΑ is to ΑΔ. Therefore as sq.ΟΣ is to pl.ΑΣΗ, so pl.ΓΡΑ is to sq.ΟΡ because the quadrangle ΟΣΑΡ is a parallelogram.

But pl.ΓΡΑ is equal to pl.ΘΡΟ. Therefore as ΧΑ is to ΑΔ, so pl.ΘΡΟ is to sq.ΡΟ, and this [latter] ratio is equal to the ratio ΘΡ to ΡΟ. Therefore as ΑΓ is to ΑΔ, so ΘΡ is to ΡΟ.

But the ratio ΑΓ to ΑΔ was also equal to the ratio ΘΛ to ΛΙ. Therefore the ratio ΘΡ to ΡΟ is equal to the ratio ΘΛ to ΛΙ, which is impossible. Therefore it is not possible for there to be a third cone similar to the cone ΕΖΚ containing this section.

# BOOK SEVEN

Apollonius greets Attalus.

Peace be on you. I have sent to you with this letter of mine the seventh book of the treatise on Conics. In this book are many wonderful and beautiful things on the topics of diameters and the *eidoi* corresponding to them[1], set out in detail. All of this is of great use in many types of problems, and there is much need for it in the kind of problems which occur in conic sections which we mentioned, among those which will be discussed and proved in the eighth book of this treatise [2] .

## [Proposition] 1

*If the axis of a parabola is continued in a straight line outside of the section to a point such that the part of it which falls outside of the section is equal to the* latus rectum, *and furthermore a straight line is drawn from the vertex of the section to any point on the section and a perpendicular to the axis dropped from where it meets it, then the straight line which was drawn [from the vertex is equal in square to the rectangular plane under the straight line between the* *foot of the perpendicular and the vertex of the section, and the straight line between of the foot of the perpendicular and the point two which the axis was continued* [3].

Let there be the parabola ΑΒ whose axis ΑΓ. We continue ΓΑ to Δ, let ΑΔ be equal to the *latus rectum*. We draw from Α the straight line ΑΒ in any position [so as to cut the section again at Β], and drop ΒΓ as perpendicular to ΑΓ. Then I say that sq.ΑΒ is equal to pl.ΔΓΑ.

[Proof].ΑΓ is the axis of the section, ΒΓ is perpendicular to it, and ΑΔ is equal to the *latus rectum*. Therefore sq.ΒΓ is equal to pl.ΔΑΓ, as is proved in Theorem 11 of Book I.

Therefore we make sq.ΑΓ common. Then the sum of sq.ΑΓ and sq.ΓΒ is equal to the sum of pl.ΔΑΓ and sq.ΑΓ.

But the sum of sq.ΑΓ and sq.ΓΒ is equal to sq.ΑΒ, and the sum of pl.ΔΑΓ and sq.ΑΓ is equal to pl.ΔΓΑ. Therefore sq.ΑΒ is equal to pl.ΔΓΑ.

[Proposition] 2

*If the axis in a hyperbola is continued in a straight line so that the part of it falling outside of the section in the transverse diameter, and a straight line is cut off adjacent one of the ends of the transverse diameter such that the transverse diameter is divided into two parts in the ratio of the transverse diameter to the* latus rectum, *and the straight line cut off corresponds to the* latus rectum, *and a straight line is drawn from that end of the transverse diameter which is the end of the straight line which was cut of to the section, in any position, and from the place where [that straight line] meets it, a perpendicular is dropped to the axis, then the ratio of the square on the straight line drawn from the end of the transverse diameter to the corresponding plane under two straight lines between the foot of the perpendicular and two ends of the straight line which was cut off is equal to the ratio of the transverse diameter to the excess of it over the straight line which was cut off. And let the straight line that was cut off be called the "homologue"* [4] .

Let the hyperbola be the section whose continued axis ΑΓΕ , and let the *eidos* of the section ΓΔ. Let ΑΘ be cut off from ΑΓ, and let as ΓΘ is to ΘΑ, so ΓΑ is to ΑΔ, which is the *latus rectum*.

We draw from Α to the section the arbitrary straight line ΑΒ, and drop ΒΕ perpendicular to the axis. Then I say that as sq.ΑΒ is to pl.ΘΕΑ, so ΑΓ is to ΓΘ.

[Proof]. We make pl.ΑΕΖ equal to sq.ΒΕ. Therefore as pl.ΑΕΖ is to pl.ΑΕΓ, so sq.ΒΕ is to pl.ΑΕΓ. But the ratio sq.ΒΕ to pl.ΑΕΓ is equal to the ratio of the *latus rectum* [which is ΑΔ] to the transverse diameter [which is ΑΓ], as is proved in Theorem 21 of Book I. Therefore the ratio pl.ΑΕΖ to pl.ΑΕΓ is equal to the ratio ΔΑ to ΑΓ and to the ratio ΖΕ to ΕΓ, and as ΔΑ is to ΑΓ, so ΑΘ is to ΘΓ. Therefore the ratio ΖΕ to ΕΓ is equal to the ratio ΑΘ to ΘΓ. So the ratio ΖΓ to ΓΕ is equal to the ratio ΑΓ to ΓΘ, and the ratio ΖΑ to ΘΕ is equal to the ratio ΑΓ to ΓΘ. But, when we make ΑΕ a common height, as ΖΑ is to ΘΕ , so pl.ΖΑΕ is to pl.ΘΕΑ. Therefore as ΑΓ is to ΓΘ, so pl.ΖΑΕ is to pl.ΑΕΘ. But pl.ΖΑΕ is equal to sq.ΑΒ. Therefore as sq.ΑΒ is to pl.ΑΕΘ, so ΑΓ is to ΓΘ ·

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[Proposition] 3

Let there be the ellipse whose axis ΑΓ and *eidos* ΓΔ. Let the straight line constructed on the continuation of the axis be ΑΘ, and let as ΓΘ is to ΘΑ, so ΓΑ is to ΑΔ.

If a straight line is constructed on the continuation of one of axes of an ellipse, whichever axis it may be, and one of its ends is one of the ends of the transverse diameter, and the other end is outside of the section and the ratio of it to the straight line between its other end and the remaining and of the transverse diameter is equal to the ratio of the *latus rectum* to the transverse diameter, and a straight line is drawn from the common end to the transverse diameter and the straight line constructed on the axis to any point on the section and from the place where its meet the section a perpendicular is dropped to the axis, then the ratio of the square on the straight line which was drawn [to the section] to the pl. two straight lines between the foot of the perpendicular and two ends of the straight line which was constructed on the axis is equal to the ratio of the transverse diameter to the straight line between those two ends of the transverse diameter and the straight line which was constructed that are different from each other. Let the straight line that was constructed be called the "comologue"[6].

From Α let ΑΒ be drawn to the section, and let us drop ΒΕ perpendicular to the axis. Then I say that sq.ΑΒ is to pl.ΘΕΑ, so ΑΓ is to ΓΘ.

[Proof].We make pl.ΑΕΖ equal to sq.ΒΕ. Then as pl.ΑΕΖ to pl.ΑΕΓ, so sq.ΒΕ is to pl.ΑΕΓ.

But the ratio sq.ΒΕ to pl.ΑΕΓ is equal to the ratio of the *latus rectum* which is ΑΔ to the transverse diameter which is ΑΓ, as is proved in Theorem 21 of Book I. Therefore the ratio pl.ΑΕΖ to pl.ΑΕΓ is equal to the ratio ΔΑ to ΑΓ and to the ratio ZE to EΓ, and as ΔA is to AΓ, so AΘ is to ΘΓ. Therefore as ZE is to EΓ, so AΘ is to ΘΓ. And as ZΓ is to ΓE, so AΓ is to ΓΘ, and as ZA is to ΘE, so AΓ is to ΓΘ.

But, when we make AE a common height, as ZA is to ΘE, so pl.ZAE is to pl.ΘEA. Therefore as AΓ is to ΓΘ, so pl.ZAE is to pl.AEΘ. But pl.ZAE is equal to sq.AB. Therefore as sq.AB is to pl.AEΘ, so AΓ is to ΓΘ [7].

[Proposition] 4

*If a straight line is tangent to a hyperbola or an ellipse, so as to fall on one of its diameter, and an ordinate is drawn from the point of contact to that diameter, and from the center a straight line is drawn parallel to the tangent and equal to the half of the diameter conjugate with the diameter passing through the point of contact, then the ratio of the square on the tangent to the square on the straight line parallel to it is equal to the ratio of the straight line between the point of intersection of the tangent and the diameter and the foot of the perpendicular to the straight line between the foot of the perpendicular and the center* [8].

Let the diameter of the hyperbola or the ellipse be AΓ, and its center Θ, and the straight line tangent to the section be BΔ. Let BE be an ordinate to ΓAE and let ΘH be parallel to BΔ, and let ΘH be equal to the half of the diameter conjugate with the diameter passing through B.

Then I say that sq.ΔB is to sq.ΘH, so ΔE is to EΘ.

[Proof]. We draw from B the diameter BΘZ, and draw AΛ and ΔK parallel to BE [and let AΛ meets BΔ at O]. Let the ratio of the straight line M to BΔ be equal to the ratio OB to BΛ. Then M is the half of the straight line such that, when the rectangular planes applied to it in the hyperbola with the addition of a rectangular plane similar to the plane under ZB and the double M, and in the ellipse with the subtraction of a rectangular plane similar to the plane under the double M and ZB, the ordinates falling on BΘ are equal to those rectangular planes. And that has been proved in Theorem 50 of Book I. And BH is the half of the diameter conjugate with the diameter BZ. Therefore pl.ΘB,M is equal to sq.ΘH, as is proved in Theorems 1 and 21 of Book II. And the ratio OB to BΛ is equal to the ratio M to BΔ and to the ratio ΔB to BK. Therefore pl.M,BK is equal to sq.BΔ. But the ratio pl.M,BK to pl.M,BΘ is equal to the ratio BK to BΘ. Therefore the ratio sq.BΔ to pl.BΘ,M is equal to the ratio BK to BΘ.

But as for the ratio BK to BΘ, it is equal to the ratio EΔ to EΘ. And as for the rectangular plane pl.BΘ,M ,it is has we have shown, equal to sq.ΘH.

[Proposition] 5

*If there is a parabola and one of its diameters is drawn in it, and from the vertex of that diameter a perpendicular is dropped to the axis, then the straight line such that straight lines drawn from the section to the diameter parallel to the tangent drawn from the vertex of the diameter [as ordinates] are equal in square to the rectangular planes under the mentioned straight line and the segment cut off from the diameter by ordinates [that straight line is the* latus rectum *corresponding to the diameter] is equal to the* latus rectum *corresponding to the axis larger by the quadruple amount cut off from it by the perpendicular from the axis adjacent to the vertex of the section* [9].

Let there be the parabola whose axis ΑΗ, and one of its diameters ΒΙ, and let the straight lines such that the perpendiculars dropped to ΑΗ are equal in square analogous rectangular planes be ΑΓ – this is corresponding to the axis . We draw from Β the perpendicular ΒΖ to the axis.

Then I say that the straight lines drawn from the section to ΒΙ parallel to the tangent [ΒΔ] from Β are equal in square to the *eidos* applied to the straight line equal to ΑΓ in creased by the quadruple ΑΖ, that straight line is the *latus rectum* corresponding to the diameter ΒΙ

[Proof]. We draw ΕΑ perpendicular to the axis and continue ΙΒ to Ε and draw ΒΔ tangent to the section at Β, and draw ΒΗ so that it forms a right angle with ΒΔ. Then the triangle ΒΔΗ is similar to the triangle ΒΘΕ. Therefore as ΒΘ is to ΒΕ, so ΔΗ is to ΒΔ. Therefore ΔΗ is equal to the half of the *latus rectum* corresponding to the diameter ΒΙ, as is proved in Theorem 49 of Book I.

But pl.ΔΖΗ is equal to sq.ΒΖ because the angle ΑΒΗ is right and ΒΖ is perpendicular [to ΔΗ]. And sq.ΒΖ is equal to pl.ΓΑΖ. Therefore pl.ΑΖΗ is equal to pl.ΓΑΖ.

But ΔΖ is equal to the double ΑΖ, as is proved in Theorem 35 of Book I. Therefore ΑΓ is equal to the double ΖΗ, and the quadruple ΑΖ is equal to the double ΔΖ. Therefore the sum ΑΓ and the quadruple ΑΖ is equal to the double ΔΗ. And we have [already] shown that the double ΔΗ is the *latus rectum* corresponding to the diameter ΒΙ. Therefore the *latus rectum* corresponding to the diameter ΒΙ is equal to the sum of ΑΓ and the quadruple ΑΖ.

[Proposition] 6

*If there are constructed on the continuation of the axis of a hyperbola two straight lines adjacent to two ends of the axis which is the transverse diameter, each of them equal to the straight line which we called "homologue", and placed as it is placed, and two conjugate diameters from among the diameters of the section are drawn, and from the vertex of the section a straight line is drawn parallel to the upright diameter of two opposite hyperbolas to cut the section, and from the place where it meets it a perpendicular is dropped to the axis, then the ratio of the transverse diameter of two conjugate diameters to the upright one is equal in square to the ratio of the straight line between the foot of the perpendicular and the end of the more remote of two homologues to the straight line between the foot of the perpendicular and the end of the nearer of two homologues, and the ratio of the transverse diameter to the* latus rectum *corresponding to it parallel to the second diameter is in length equal to the ratio of two straight lines which we mentioned previously to each other in length* [10].

Let there be the hyperbola whose axis ΕΓ , and transverse diameter ΑΓ, as the continuation of the axis, and center Θ. Let each of two straight lines ΑΝ and ΓΞ be equal to the homologue. Let two conjugate diameters ΖΗ and ΒΚ pass through Θ, and let us draw ΑΛ parallel to ΖΗ, and draw the perpendicular ΛΜ to ΑΜ. Then I say that the ratio of the square on the transverse diameter ΒΚ to the square on the upright diameter ΖΗ is equal to the ratio ΞΜ to ΜΝ.

[Proof]. We join ΓΛ, and draw the perpendicular from Β, and draw from it also ΒΔ parallel to ΖΗ. Then that straight line [ΒΔ] is tangent to the section. And since ΓΘ is equal to ΘΑ, and ΛΟ is equal to ΟΑ, ΓΛ is parallel to ΒΘ. Therefore as ΔΕ is to ΕΘ, so ΑΜ is to ΜΓ because of the similarity of the triangles.

But as ΔΕ is to ΕΘ, so sq.ΔΒ is to sq.ΘΗ, as is proved in Theorem 4 of this Book. Therefore as ΑΜ is to ΜΓ, so sq.ΔΒ is to sq.ΘΗ. And since as sq.ΘΒ is sq.ΔΒ, so sq.ΓΛ is to sq.ΑΛ because of the similarity of the triangles [ΘΒΔ and ΓΛΑ], and as sq.ΒΔ is to sq.ΘΗ, so ΑΜ is to ΜΓ, the ratio sq.ΘΒ to sq.ΘH is compounded of [the ratios] sq.ΓΛ to sq.ΑΛ and ΑΜ to ΜΓ.

But the ratio sq.ΓΛ to sq. ΑΛ is compounded of [the ratios] sq.ΓΛ to pl.ΓΜΞ , pl.ΓΜΞ to pl.ΑΜΝ , and pl.ΑΜΝ to sq.ΑΛ. Therefore the ratio sq.ΘΒ to sq.ΘΗ is compounded of [the ratios] sq.ΓΛ to pl.ΓΜΞ, pl.ΓΜΞ to pl.ΑΜΝ, pl . to sq.ΑΛ, and ΑΜ to ΜΓ. But the ratio sq.ΓΛ to pl.ΓΜΞ is equal to the ratio ΑΓ to ΑΞ, as is proved in Theorem 2 of this Book, and the ratio pl.ΑΜΝ to sq.ΑΛ is equal to the ratio ΓΝ to ΑΓ, as is also proved in Theorem 2 of this Book, and the ratio pl.ΓΜΞ to pl.ΑΜΝ is compounded of [the ratios] ΜΞ to ΜΝ and ΓΜ to ΑΜ. Therefore the ratio sq.ΘΒ to sq.ΘΗ is compounded of [the ratios] ΑΓ to ΑΞ, ΓΝ to ΑΓ, ΓΜ to ΑΜ, ΜΞ to ΜΝ, and ΑΜ to ΜΓ. And the ratio compounded of these ratios which we mentioned is equal to the ratio ΜΞ to ΜΝ because the part of it ΓΝ to ΑΓ, when combined with ΑΓ to ΑΞ, is equal to the ratio ΝΓ to ΑΞ, and ΝΓ is equal to ΑΞ, and as for the part of it ΓΜ to ΑΜ, when combined with ΑΜ to ΓΜ, it is equal to the ratio of ΓΜ to itself. Therefore the ratio compounded of these ratios is equal to the remaining ratio, which is the ratio ΜΞ to ΜΝ. Therefore the ratio sq.ΒΘ to sq.ΘΗ is equal to the ratio ΞΜ to ΜΝ, and [hence] the ratio sq.ΒΚ to sq.ΖΗ is equal to the ratio ΜΞ to ΜΝ.

Furthermore the ratio sq.ΒΚ to sq.ΖΗ is equal to the ratio of ΚΒ to the straight line such that straight lines drawn from the section to ΚΒ parallel to ΖΗ [are equal in square to corresponding rectangular plane] as is proved in Theorems 1 and 21 of Book II. Therefore the ratio of ΚΒ to the mentioned straight line [that is the *latus rectum* corresponding to ΚΒ] is equal to the ratio ΜΞ to ΜΝ.

[Proposition] 7

*If there are constructed on the continuation of the axis of an ellipse two straight lines at two ends of it, each of them equal to the homologue straight lines, and two conjugate diameters are drawn in the section, and from the vertex of the section a straight line is drawn parallel to one of the conjugate diameters so as to meet the section [again], and from the place there it meets [the section] a perpendicular is dropped to the axis, then the ratio of the diameter which is not parallel to the straight line drawn to other diameter is equal in square to the ratio to each other of two parts [of the straight line between the ends of two homologues straight lines which are not the ends of the diameter] into which it is cut by the perpendicular, according to how two homologues are placed, if [they are found on the major axis , they are outside the section, and if in minor axis, then they are on the axis itself. And the ratio of the mentioned diameter to the straight line such that the ordinates dropped on it are equal in square to corresponding rectangular planes is [also] equal to the mentioned ratio* [11].

Let there be the ellipse whose axis ΑΓ. Let two homologues straight lines be ΑΝ and ΓΞ. Let the diameters ΖΗ and ΒΚ be conjugate, in any position. We draw ΑΛ parallel to the diameter ΖΗ, and drop from Λ the perpendicular ΛΜ to the axis. Then I say that the ratio sq.ΒΚ to sq.ΖΗ is equal to the ratio ΜΞ to ΜΝ,and that the ratio of ΚΒ to the straight line such that straight lines drawn to it in the section parallel to ΖΗ are equal in square to corresponding rectangular planes, this straight line is the *latus rectum*, also is equal to the ratio MΞ to MN.

[Proof]. We join ΓΛ, and drop the perpendicular BE from B and draw from it too the straight line BΔ parallel to ZH. Then that line is tangent to the section. And since ΓΘ is equal to ΘA and ΛO is equal to OA, ΓΛ is parallel to BΘ. Therefore as ΔE is to EΘ, so AM is to MΓ because of the similarity of the triangles.

But as ΔE is to EΘ, so sq.ΔB is to sq.ΘH, because of what is proved in Theorem 4 of this Book. Therefore as AM is to MΓ, so sq.ΔB is to sq.ΘH. And since as sq.BΘ is to sq.BΔ, so sq.ΓΛ is to sq.AΛ because of the similarity of two triangles, and as sq.BΔ is to sq.ΘH, so AM is to MΓ.

The ratio sq.ΘB to sq.ΘH is compounded of [the ratios] sq.ΓΛ to sq.AΛ and AM to MΓ.

But the ratio sq.ΓΛ to sq.AΛ is compounded of [the ratios] sq.ΓΛ to pl.ΓMΞ, pl.ΓMΞ to pl. AMN, and pl.AMN to sq.AΛ. Therefore the ratio sq.ΘB to sq.ΘH is compounded of [the ratios sq.ΓΛ to pl.ΓMΞ, pl.ΓMΞ to pl.AMN, pl.AMN to sq.AΛ, and AM to MΓ.

But the ratio sq.ΓΛ to pl.ΓMΞ is equal to the ratio AΓ to AΞ, as is proved in Theorem 3 of this Book, and the ratio pl.AMN to sq.AΛ is equal to the ratio ΓN to AΓ, as is also proved in Theorem 3 of this Book, and the ratio pl.ΓMΞ to pl.AMN is compounded of [the ratios] ΓM to AM and MΞ to MN, therefore the ratio sq.ΘB to sq.ΘH is compounded of [the ratios] AΓ to AΞ, ΓN to AΓ, ΓM to AM, MΞ to MN, and AM to MΓ.

And the ratio compounded of those ratios mentioned by us is equal to the ratio MΞ to MN because the part of it ΓN to AΓ, when combined with AΓ to AΞ is equal to the ratio ΓN to AΞ, and ΓN is equal to AΞ, and as for the part of it ΓM to AM, when combined with AM to ΓM, it is equal to the ratio of ΓM to itself. Therefore the ratio compounded of these ratios is equal to the remaining ratio MΞ to MN. Therefore the ratio sq.BΘ to sq.ΘH is equal to the ratio ΞM to MN. And furthermore the ratio sq.BK to sq.ZH is equal to the ratio of KB to the straight line by which straight lines drawn from the section to KB parallel to ZH are equal in square to corresponding rectangular planes. Therefore the ratio of BK to the *latus rectum* corresponding to it is equal to the ratio MΞ to MN.

Hence it will be proved that if the perpendicular dropped from Λ on the axis passes through the center Θ, then the diameter KB will be equal to the diameter ZH because MΞ is equal to MN [12] .

[Proposition] 8

Furthermore we set the diagram for the hyperbola and the ellipse in the way it was in Theorems 6 and 7 of this Book, then I say that the ratio of the square on ΑΓ which is the transverse diameter to the square on ΒΚ and ΖΗ which are two conjugate diameters, when whey are joined together in a straight line is equal to the ratio of pl.ΝΓ,ΜΞ to the square on the straight line equal to the sum of ΜΞ and the straight line equal in square to pl.ΝΜΞ [13].

[Proof]. We make ΞΙ a mean proportional between ΝΜ and ΜΞ. Then as sq.ΑΓ is to sq.ΒΚ, so sq.ΑΘ is to sq.ΘΒ. But sq.ΑΘ is equal to pl.ΔΘΕ, as is proved in Theorems 37 and 38 of Book I. Therefore as sq.ΑΓ is to sq.ΒΚ, so pl.ΔΘΕ is to sq.ΘΒ.

But as pl.ΔΘΕ is to sq.ΘΒ, so pl.ΑΓΜ is to sq.ΓΛ because ΔΒ and ΒΘ are parallel to ΑΛ and ΛΓ [respectively]. Therefore the ratio pl.ΑΓΜ to sq.ΓΛ is equal to the ratio sq.ΑΓ to sq.ΒΚ. And when we make ΓΜ a common height, as ΓΑ is to ΓΝ, so pl.ΑΓΜ is to ΜΓΝ. And the ratio sq.ΓΛ to pl.ΞΜΓ is equal to the ratio ΑΓ to ΑΞ, as is proved in Theorems 2 and 3 of this Book. And ΓΝ is equal to ΑΞ because ΑΝ and ΓΞ are two homologue straight lines. Therefore as pl.ΑΓΜ is to pl.ΜΓΝ, so sq.ΓΛ is to pl.ΞΜΓ.

Therefore *permutando* as pl.ΑΓΜ is to sq.ΓΛ, so pl.ΜΓΝ is to pl.ΞΜΓ.

But we have [already] proved that as pl.ΑΓΜ is to sq.ΓΛ, so sq.ΑΓ is to sq.ΒΚ. Therefore the ratio sq.ΑΓ to sq.ΒΚ is equal to the ratio pl.ΝΓΜ to pl.ΞΜΓ and is equal to the ratio ΝΓ to ΞΜ. And as ΝΓ is to ΞΜ, so pl.ΝΓ,ΞΜ is to sq.ΜΞ. Therefore as sq.ΛΓ is to sq.ΒΚ, so pl.ΝΓ,ΞΜ is to sq.ΜΞ.

Furthermore as sq.ΒΚ is to sq.ΖΗ, so ΞΜ is to ΜΝ, as was proved in two preceding theorems. Therefore as ΒΚ is to ΖΗ, so ΜΞ is to ΞΙ because ΞΙ is the mean proportional between ΞΜ and ΜΝ. Therefore the ratio ΒΚ to the sum of ΒΚ and ΖΗ is equal to the ratio ΜΞ is to ΜΙ, and the ratio of sq.ΒΚ to the square on the sum of ΒΚ and ΖΗ is equal to the ratio sq.ΜΞ to sq.ΜΙ.

But we have [already] proved that as sq.ΑΓ is to sq.ΒΚ, so pl.ΝΓ,ΞΜ is to sq.ΜΞ. Therefore *ex a equali* the ratio sq.ΑΓ to the square on the sum of ΒΚ and ΖΗ is equal to the ratio pl.ΓΝ,ΞΜ to sq.ΜΙ, and ΜΙ is equal to the sum ΜΞ and the straight line whose square is equal to pl.ΝΜΞ. Therefore the ratio of sq.ΑΓ to the square on the sum of two conjugate diameters ΒΚ and ΖΗ is equal to the ratio of pl.ΝΓ,ΜΞ to the square on ΜΙ which is equal to the sum of ΜΞ and the straight line whose square is equal to pl.ΝΜΞ.

[Proposition] 9

Furthermore we set out what we have mentioned in the situation of Theorems 6 and 7 of this Book, then I say that the ratio sq.ΑΓ to the square on the difference of ΒΚ and ΖΗ is equal to the ratio of pl.ΝΓ,ΜΞ to the square on the difference of ΜΞ and Xi, where ΞΙ is the straight line equal in square to pl.ΝΜΞ.

[Proof]. The ratio of ΚΒ to ΖΗ is equal to the ratio ΜΞ to ΞΙ, as is shown in the proof of the preceding theorem. Therefore the ratio sq.ΒΚ to the square of the difference of ΒΚ and ΖΗ is equal to the ratio sq.ΜΞ to the square of the difference ΜΞ and ΞΙ.

But as sq.ΑΓ is to sq.ΒΚ, so pl.ΝΓ,ΜΞ is to sq.ΜΞ, as is proved in the preceding theorem. Therefore ex the ratio sq.ΑΓ to the square on the difference ΒΚ and ΖΗ is equal to the ratio pl.ΝΓ,ΜΞ to the square on the difference of ΜΞ and ΞΙ. But sq.ΞΙ is equal to pl.ΝΜΞ. Therefore the ratio sq.ΑΓ to the square on the difference of ΒΚ and ΖΗ is equal to the ratio pl.ΝΓ,ΜΞ to the square on the difference of ΜΞ and ΞΙ, where ΞΙ is the straight line equal in square to pl.ΝΜΞ.

[Proposition] 10

We again set the diagram as it was in Theorems 6 and 7 of this Book. Then I say that the ratio sq.ΑΓ to pl.ΒΚ,ΖΗ is equal to the ratio of ΝΓ to the straight line equal in square to pl.ΝΜΞ [15].

[Proof]. It has been shown in the proof of Theorem 8 of this Book that as sq.ΑΓ is to sq.ΒΚ, so ΝΓ is to ΜΞ. And is was proved there also that as sq.ΒΚ is to pl.ΒΚ,ΖΗ , so ΜΞ is to ΞΙ because the ratio ΜΞ to ΞΙ is equal to the ratio ΚΒ to ΖΗ. Therefore as sq.ΑΓ is to pl.ΒΚ,ΖΗ ,so ΝΓ is to ΞΙ.

But sq.ΞΙ is equal to pl.ΝΜΞ. Therefore the ratio sq.ΑΓ to pl.ΒΚ,ΖΗ is equal to the ratio of ΝΓ to the straight line equal in square to pl.ΝΜΞ.

[Proposition] 11

Furthermore we set things in the state that we prescribed for the hyperbola in Theorem 6 of this Book, then I say that the ratio sq.ΑΓ to the sum of sq.ΒΚ and sq.ΖΗ is equal to the ratio ΓΝ to the sum of ΝΜ and ΜΞ [16].

[Proof]. As sq.ΑΓ is to sq.ΒΚ, so ΓΝ is to ΜΞ, as was proved in Theorem 8 of this Book. And the ratio sq.ΒΚ to the sum of sq.ΖΗ and sq.BK is equal to the ratio ΜΞ to the sum of ΜΞ and ΝΜ because it was proved in Theorem 6 of this Book that as sq.ΒΚ is to sq.ΖΗ, so ΜΞ is to ΜΝ. Therefore a equali the ratio sq.ΑΓ to the sum of sq.ΒΚ and sq.ΖΗ is equal to the ratio ΓΝ to the sum of

MΞ and MN.

[Proposition] 12

*In any ellipse the sum of the squares on any two of its conjugate diameters what ever is equal to the sum of the squares on its two axes*[17].

Let the diagram for the ellipse be as it was in Theorem 7 of this Book.

Then the axis is AΓ, two conjugate diameters BK and ZH, and two homologue straight lines AN and XΞ. And the ratio of sq.AΓ to the square on other of two axes of the section is equal to the ratio of AΓ which is the transverse diameter to the *latus rectum* corresponding [to it], as is proved in Theorem 15 of Book I.

But the ratio of AΓ to its *latus rectum* is equal to the ratio ΓN to AN because AN is the homologue straight line. And AN is equal to ΓΞ. Therefore the ratio of sq.AΓ to the square other of two axes of the section is equal to the ratio NΓ to ΓΞ. And for that reason the ratio of sq.AΓ to the sum of sq.AΓ and the square on other of two axes of the section is equal to the ratio NΓ to NΞ.

Furthermore as sq.AΓ is to sq.BK, so NΓ is to MΞ, as is proved in the proof of Theorem 8 of this Book. And the ratio sq.BK to the sum sq.BK and sq.ZH is equal to the ratio MΞ to the sum MΞ and NM because it was proved in Theorem 7 of this Book that as sq.BK is to sq.ZH, so MΞ is to MN.

But the sum of MΞ and NM is equal to ΞN. Therefore the ratio sq.AΓ to the sum of sq.BK and sq.ZH is equal to the ratio NΓ to NΞ. And we had [already] proved that the ratio NΓ to NΞ is equal to the ratio sq.AΓ to the sum of the squares on two axes. Therefore the sum of the squares on two axes is equal to the sum of sq.BK and sq.ZH.

[Proposition] 13

*In every hyperbola the difference between the squares on its axes is equal to the difference between the squares on any pair of its other conjugate diameters whatever*[18].

Let the diagram of the hyperbola be as it was in Theorem 6 of this Book. Then the ratio of the square on AΓ, which is one of the axes to the square on the other of two axes of the section, is equal to the ratio of AΓ to its *latus rectum*, as was proved in Theorem 16 of Book I. But the ratio of AΓ to its *latus rectum* is equal to the ratio ΓN to AN because AN is the homologue straight line. And AN is equal to ΓΞ. Therefore the ratio of sq.AΓ to the square on the other of two axes of the section is equal to the ratio ΝΓ to ΓΞ, and therefore the ratio of sq.ΑΓ to the difference between sq.ΑΓ and the square on the other on two axes of the section is equal to the ratio ΝΓ to ΝΞ.

Furthermore as sq.ΑΓ to is sq.ΒΚ, so ΝΓ is to ΜΞ, as is proved in Theorem 8 of this Book. And the ratio sq.ΒΚ to the difference between sq.ΒΚ and sq.ΖΗ is equal to the ratio ΜΞ to ΝΞ because it was proved in Theorem 6 of this Book that as sq.ΒΚ is to sq.ΖΗ, so ΜΞ is to ΜΝ.

Therefore ex a equali the ratio sq.ΑΓ to the difference between sq.ΚΒ and sq.ΖΗ is equal to the ratio ΝΓ to ΝΞ. And we had [already] proved that the ratio of sq.ΑΓ to the difference between sq.ΑΓ and the square on the other of two axes of the section is equal to that ratio which is the ratio ΝΓ to ΝΞ. Therefore the difference between sq.ΑΓ and the square on the other of two axes of the section is equal to the difference between sq.ΒΚ and sq.ΖΗ.

[Proposition] 14

Furthermore we let the diagram of the ellipse as we represented it in Theorem 7 of this Book, then I say that the ratio of the square on the axis ΑΓ to the difference between the squares on ΒΚ and ΖΗ is equal to the ratio ΝΓ to the double ΜΘ when ΑΛ is parallel to the diameter ΖΗ and ΛΜ is the perpendicular to the axis [19].

[Proof]. The ratio sq.ΑΓ to sq.ΒΚ is equal to the ratio ΝΓ to ΜΞ, as is proved in Theorem 8 of this Book. And the ratio sq.ΒΚ to the difference between sq.ΒΚ and sq.ΖΗ is equal to the ratio ΞΜ to the difference between ΞΜ and ΜΝ because it was proved in Theorem 7 of this Book that as sq.ΒΚ to sq.ΖΗ, so ΜΞ is to ΜΝ. But the difference between ΜΞ and ΜΝ is equal to the double ΜΘ. Therefore the ratio sq.ΑΓ to the difference between sq.ΒΚ and sq.ΖΗ is equal to the ratio ΝΓ to the double ΜΘ.

[Proposition] 15

Furthermore we set the diagram for the hyperbola and the diagram for the ellipse in the situation we represented in Theorems 6 and 7 of this Book, then I say that the ratio of sq.ΑΓ to the square on the straight line which bounds together with the diameter ΒΚ the *eidos* of the section, this straight line is the *latus rectum* corresponding to the diameter ΒΚ, is equal to the ratio of pl.ΝΓ,ΜΞ to sq.ΜΝ [20].

[Proposition] 16

Furthermore we set the diagram as it was in Theorems 6 and 7 of this Book, and let the *latus rectum* corresponding to BK be T, then I say that the ratio sq.ΑΓ to the square on the difference between BK and T is equal to the ratio pl.ΝΓ,ΜΞ to the square on the difference between MN and ΜΞ [21].

[Proof]. The ratio BK to the difference between BK and T is equal to the ratio ΜΞ to the difference between ΜΞ and MN for it was proved in Theorems 6 and 7 of this Book that as BK is to T, so ΜΞ is to MN. Therefore the ratio sq.BK to the square on the difference between BK and T is equal to the ratio sq.ΜΞ to the square on thee difference between ΜΞ and MN.

[Proposition] 17

[Proof]. As BK is to T, so ΜΞ is to MN, as is proved in Theorems 6 and 7 of this Book. Therefore the ratio sq.BK to the square on the sum of BK and T is equal to the ratio sq.ΜΞ to the square on the sum of ΜΞ and MN. But as sq.ΑΓ is to sq.BK, so pl.ΝΓ, ΜΞ is to sq.ΜΞ. Therefore the ratio sq.ΑΓ to the square on the sum of BK and T is equal to the ratio pl.ΝΓ,ΜΞ to the square on the sum of ΜΞ and MN.

[Proposition]18

Furthermore we set the diagram as it was in Theorems of this Book, then I say that as sq.ΑΓ is to pl.BK,T ,so ΝΓ is to NM [23].

[Proof]. As sq.ΑΓ is to sq.BK, so ΝΓ is to ΜΞ, as is proved in the proof of Theorem 8 of this Book. But as sq.BK is to pl.BK,T ,so BK is to T, and as BK is to T, so ΜΞ is to MN, as is proved in Theorems 6 and 7 of this Book. Therefore as sq.ΑΓ is to pl.BK,T , so ΝΓ is to MN.

[Proposition] 19

Furthermore we set the diagram as is was in Theorems 6 and 7 of this Book, then I say that the ratio sq.ΑΓ to the sum of sq.BK and sq.T is equal to the ratio pl.ΝΓ,ΜΞ to the sum of sq.MN and sq.ΜΞ [24].

[Proof]. As sq.ΑΓ is to sq.BK, so pl.ΝΓ,ΜΞ is to sq.ΜΞ , as is proved in Theorem 8 of this Book. But the ratio BK to the sum of sq.BK and sq.T is equal to the ratio sq.ΜΞ to the sum of sq.MN and sq.ΜΞ because it was proved in the proof of Theorems 6 and 7 of this Book that as KB is to T, so ΜΞ is to MN.

Therefore the ratio sq.ΑΓ to the sum of sq.BK and sq.T is equal to the ratio pl.ΝΓ,ΜΞ to the sum of sq.MN and sq.ΜΞ.

[Proposition] 20

Furthermore we set the diagram as is was in Theorems 6 and 7 of this Book, then I say that the ratio sq.ΑΓ to the difference between sq.BK and sq.T is equal to the ratio pl.ΝΓ,ΜΞ to the difference between sq.MN and sq.MN [25].

[Proof]. As sq.ΑΓ is to sq.BK, so pl.ΝΓ,ΜΞ to sq.ΜΞ , as is proved in the proof of Theorem 8 of this Book.

But the ratio sq.BK to the difference between sq.BK and sq.T is equal to the ratio sq.ΜΞ to the difference between sq.ΜΞ and sq.MN because it was proved in Theorems 6 and 7 of this Book that as BK is to T, so ΜΞ is to MN. Therefore the ratio sq.ΑΓ to the difference between sq.BK and sq.T is equal to the ratio pl.ΝΓ,ΜΞ to the difference between sq.ΜΞ and sq.MN.

[Proposition] 21

If there is a hyperbola, and its transverse axis is greater than its upright axis, then the transverse diameter of each pair of conjugate diameters among its other diameters is greater than the upright diameter of that pair, and the ratio of the greater axis to the smaller axis is greater than the ratio of the transverse diameter to the upright diameter among the other conjugate diameters, and the ratio of a transverse diameter nearer to the greater axis to the upright diameter conjugate with it is greater than the ratio of a transverse diameter farther [from that axis] to the upright diameter conjugate with it [26].

Let there be the hyperbola whose axes ΑΓ and IO, and let there be two other transverse diameters BK and ZH, and let ΑΓ be greater than IO.

Then I say that BK is greater than the upright diameter conjugate with it, and that the diameter ZH also is greater than the upright diameter conjugate with it, and that the ratio ΑΓ to OI is greater than the ratio of BK to the upright diameter conjugate with it and than the ratio of ZH to the upright diameter conjugate with it, and that the ratio of BK to the upright diameter conjugate with it is greater than the ratio of ZH to the upright diameter conjugate with it.

[Proof]. We make each of the ratios ΝΓ to AN and ΑΞ to ΓΞ equal to the ratio of ΓΑ to its *latus rectum*. Then AN and ΓΞ belong to the class of straight lines called “homologues”.

Therefore we draw AΔ parallel to the tangent to the section at B, and make AΛ parallel to the tangent to the section at Z, and drop to the greater axis the perpendiculars ΔE and ΛM. Then the ratio of sq.BK to the square on the upright diameter conjugate with it is equal to the ratio ΞE to EN, as is proved in Theorem 6 of this Book.

And likewise the ratio of sq.ZH to the square on the upright diameter conjugate with it is equal to the ratio ΞM to MN. Therefore BK is greater than the upright diameter conjugate with it, and likewise too the diameter ZH is greater than the upright diameter conjugate with it.

Furthermore the ratio of ΓA to its *latus rectum* is equal to the ratio ΓN to AN and is equal to the ratio AΞ to ΞΓ. Therefore ΓN is equal to AΞ, and as ΓN is to AN , so AΞ is to AN. But the ratio ΞE to EM is smaller than the ratio ΞA to AN. Therefore the ratio ΞA to ΓΞ is greater than the ratio ΞE to EN.

Similarly too it will be proved that the ratio ΞA to ΓΞ is greater than the ratio ΞM to MN.

But as ΞA is to ΓΞ, so sq.AΓ is to sq.IO because each of these two ratios is equal to the ratio of AΓ to its *latus rectum*, as is proved in Theorem 16 of Book I. Therefore the ratio sq.AΓ to sq.IO is greater than the ratio ΞE to EN and is greater than ratio ΞM to MN.

But the ratio ΞE to EN is equal to the ratio of sq.BK to the square on the upright diameter conjugate with it, and the ratio MΞ to MN is equal to the ratio of sq.ZH to the square on the upright diameter conjugate with it.

Therefore the ratio sq.AΓ to sq.IO is greater than the ratio of sq.BK to the square on the upright diameter conjugate with it, and is greater than the ratio of sq.ZH to the square on the upright diameter conjugate with it.

Therefore the ratio AΓ to IO is greater than the ratio of BK to the upright diameter conjugate with it, and is greater than the ratio of ZH to the upright diameter conjugate with it.

Furthermore the ratio EΞ to NE which is equal to the ratio of sq.BK to the square on the upright diameter conjugate with it is greater than the ratio ΞM to MN which is equal to the ratio of sq.ZH to the square on the upright diameter conjugate with it. Therefore the ratio of BK to the upright diameter conjugate with it is greater than the ratio of ZH to the upright diameter conjugate with it.

## [Proposition] 22

*If there is a hyperbola and its transverse axis is smaller than its upright axis, then the transverse diameter of each pair of diameters among the other* *conjugate diameters is smaller than the upright diameter of that pair, and the ratio of the smaller axis to the greater axis is smaller than the ratio of any of the other transverse diameters to the upright diameter conjugate with it, and the ratio of a transverse diameter nearer to the smaller axis to the upright diameter conjugate with it is smaller than the ratio of [a transverse diameter] farther [from that axis] to the diameter conjugate with it* [27].

Let there be the hyperbola whose axes ΑΓ and ΟΙ and center Θ, and with two of its diameter ΒΚ and ΖΗ, and let [the transverse axis] ΑΓ be smaller than [the upright axis] ΟΙ.

Then I say that each of ΒΚ and ΖΗ is smaller than the upright diameter conjugate with it, and that the ratio ΑΓ to ΙΟ is smaller than the ratio of ΒΚ to the upright diameter conjugate with it, and [is smaller] than the ratio of ΖΗ to the upright diameter conjugate with it, and that the ratio of ΒΚ to the upright diameter conjugate with it is smaller than the ratio of ΖΗ to the upright diameter conjugate with it.

[Proof]. We make the ratios ΝΓ to ΑΝ equal to the ratio of the diameter ΓΑ to its *latus rectum*, and also equal to the ratio ΑΞ to ΞΓ. Then ΞΓ and ΑΝ belong to the class of straight lines called “homologues”.

We draw ΑΔ parallel to the tangent passing through Β,and ΑΛ parallel to the tangent passing through Ζ, and drop from Δ and Λ the perpendiculars ΔΕ and ΛΜ to the axis. Then the ratio of the square on the diameter ΒΚ to the square on the upright diameter conjugate with it is equal to the ratio ΞΕ to ΕΝ, as is proved in Theorem 6 of this Book.

And likewise the ratio of sq.ΖΗ to the square on the upright diameter conjugate with it is equal to the ratio ΞΜ to ΜΝ. Therefore the diameter BK is smaller than the upright diameter conjugate with it, and the diameter ΖΗ is smaller than the upright diameter conjugate with it.

Furthermore the ratio of ΓΛ to its *latus rectum* is equal to the ratio ΓΝ to ΑΝ and is equal to the ratio ΑΞ to ΞΓ. Therefore ΓΝ is equal to ΑΞ, and as ΓΝ i [28].

For we set the diameter conjugate with it [29] .

[Proof]. Let the major of two axes of the ellipse be ΑΒ, and its minor axis ΓΔ, and [two pairs of] its conjugate diameters be ΕΖ and ΗΚ, and ΝΞ and ΟΠ. Let EZ be greater than ΗΚ, its conjugate, and ΝΞ be greater than ΟΠ, its conjugate, [and let ΕΖ be closer to the major axis than ΝΞ].

We drop from Ε and Ν the perpendiculars ΕΛ and ΝΡ to the axis ΑΒ, and drop from Η and Ο the perpendiculars ΗΜ and ΟΣ to ΓΔ.

Then the ratio pl.AΘB to sq.ΘΓ is equal to the ratio pl.ALB to sq.ΛE, as is proved in Theorem 21 of Book I.

But pl.AΘB is greater than sq.ΘΓ, therefore pl.AΛB is greater than sq.ΛE. Therefore AΘ is greater ΘE, and [hence] AB is greater than EZ.

Furthermore as pl.ΓΘΔ is to sq.ΘB, so pl.ΓMΔ is to sq.MH.

But pl.ΓΘΔ is smaller than sq.ΘB. Therefore pl.ΓMΔ is smaller than sq.MH. Therefore ΘΔ is smaller than ΘH, and [hence] ΓΔ is smaller than KH.

But it was proved that AB is greater than EZ. Therefore the ratio AB to ΓΔ is greater than ratio EZ to KH. And the diameter EZ is conjugate with the diameter KH, and KH is parallel to the tangent to the section at B.

[Furthermore] the diameter ΠO is conjugate with the diameter ΞN, and it [ΠO] is parallel to the tangent to the section at N. And the diameter OΠ is closer to the major axis AB than is the diameter KH.

And as pl.AΛB is to pl.APB, so sq.ΛE is to sq.NP, as is proved in Theorem 21 of Book I.

But pl.APB is greater than pl.AΛB. Therefore sq.NP is greater than sq.EΛ.

And the difference between pl.ARB and pl.AΛB is greater than the difference between sq.NP and sq.EΛ because it has been proved that pl.APB is greater than sq.NP.

But the difference between pl.APB and pl.AΛB is equal to the difference between sq.ΘΛ and sq.ΘP. Therefore the difference between sq.ΘΛ and sq.ΘP is greater than the difference between sq.NP and sq.EΛ. Therefore the sum of sq.ΘΛ and sq.ΛE is greater than the sum of sq.ΘP and sq.PN. Therefore ΘE is greater than ΘN, and [hence] the diameter EZ is greater than the diameter NΞ.

Furthermore as pl.ΓΣΔ is to pl.ΓMΔ, so sq.OΣ is to sq.HM, as is proved in Theorem 21 of Book I. But pl.ΓΣΔ is smaller than sq.OΣ, and pl.ΓMΔ is smaller than sq.MH. Therefore the difference between pl.ΓΣΔ and ΓMΔ is smaller than the difference between sq.OΣ and sq.MH.

But the difference between pl.ΓΣΔ and pl.ΓMΔ is equal to the difference between sq.ΘM and sq.ΘΣ. Therefore the difference between sq.ΘM and sq.ΘΣ is smaller than the difference between sq.OΣ and sq.MH. Therefore the sum of sq.ΘM and sq.MH is smaller than sq.ΘΣ and sq.ΣO. Therefore ΘH is smaller than ΘO, and [hence] the diameter HK is smaller than the diameter OΠ.

And when the diameter EZ conjugate with HK is greater than the diameter ΞN conjugate with OΠ, and the diameter HK is smaller than the diameter OΠ, then the ratio of EZ to its conjugate HK is greater than the ratio of ΞN to its conjugate OΠ.

[Porism 1]

And hence it becomes clear that the difference between AB and ΓΔ is greater than the difference between EZ and HK, and that the difference between EZ and HK is greater than the difference between ΞN and OΠ, and that the difference between sq.AB and sq.ΓΔ s greater the difference between sq.EZ and sq.HK which is greater than the difference between esq. and sq.OΠ.

[Porism 2]

Then I say that the straight line under which and AB the *eidos* of the section is formed is smaller than the straight line under which and EZ the *eidos* of the section is formed, and that the straight line under which and EZ the *eidos* of the section is formed ,is smaller than the straight line under which and ΞN the *eidos* of the section is formed, and that the straight line under which and ΞN the *eidos* of the section is formed is smaller than the straight line under which and ΓΔ the *eidos* of the section is formed [30].

[Proof]. For let AB be greater than OΠ, and OΠ be greater than HK, and HK be greater than ΓΔ, and ΓΔ be smaller than NΞ, and ΞN be smaller than EZ, and EZ be smaller than AB. And sq.AB is equal to the rectangular plane under ΓΔ and the straight line under which and ΓΔ the *eidos* of the section is formed, as is proved in Theorem 15 of Book I. And sq.OΠ is equal to the *eidos* of the section corresponding to NΞ, and sq.HK is equal to the *eidos* of the section corresponding to EZ, and sq.ΓΔ is equal to the *eidos* of the section corresponding to AB.

[Proposition] 25

*In every hyperbola the straight line equal to [the sum of] its two axes is smaller than the straight line equal to [the sum of] any other pair whatever of its conjugate diameters, and the straight line equal to the sum of a transverse diameter closer to the greater axis together with its conjugate diameter is smaller than the straight line equal to the sum of a transverse diameter farther from the greater axis together with its conjugate diameter* [31].

Let there be the hyperbola whose axis AΓ and center Θ, with the some of its conjugate diameters KB and ZH, and OI and YT. Then the axis AΓ is either equal to the other of two axes of the section or it is unequal to it. Now if it is equal to it, then the diameters KB and ZH are equal, as is proved in Theorem 23 of this Book, and likewise the diameter YT is equal to the diameter IO.

But the diameter KB is greater than the axis ΓA, and the diameter YT is greater than diameter KB. Thus what we desired has been proved.

But as form [what happens] if the axis AΓ is unequal to the other of two axes of the section, the difference between sq.AΓ and the square on the other of two axes of the section is equal to the difference between sq.KB and sq.ZH as is proved in Theorem 13 of this Book.

Therefore the straight line equal to [the sum of] two axes is smaller than the straight line equal to [the sum of] diameters BK and ZH. And because the difference between sq.BK and sq.ZH is equal to the difference between sq.YT and sq.OI the straight line equal to [the sum of] diameters BK and ZH is smaller than the straight line equal to [the sum of] the diameters YT and OI.

[Proposition] 26

*In every ellipse the sum of its two axes is smaller than [the sum] of any conjugate pair of its diameters, and the sum of any conjugate pair of its diameters which is closer to two axes is smaller than the sum of any conjugate pair of its diameters farther from two axes, and the sum of the conjugate pair of its diameter each of which is equal to the other is greater than that of any [other] conjugate pair of its diameter* [32].

Let there be the ellipse whose major axis AB and minor axis ΓΔ, and conjugate diameters EZ and KH, and NΞ and OΠ, and YT and PΣ, and let EZ be greater than [its conjugate KH, and let ΞN be greater than [its conjugate] OΠ, and let PΣ be equal to [its conjugate] YT.

Then I say that the straight line equal to [the sum of] two axes AB and ΓΔ is smaller than the straight line equal to [the sum of] two diameters EZ and HK, and that the straight line equal to [the sum of] two diameters NΞ and OΠ, and that the greatest of them [the sums of the pairs of conjugate diameters] is the straight line equal to [the sum of] two diameters PΣ and YT.

[Proof]. The ratio AB to ΓΔ is greater than the ratio EZ to KH, as is proved in Theorem 24 of this Book. Therefore the ratio of the square on the sum AB and ΓΔ to the sum of sq.AB and sq.ΓΔ is smaller than the square on the sum EZ and KH to the sum of sq.EZ and sq.KH. But the sum of sq.EZ and sq.KH is equal to the sum of sq.AB and sq.ΓΔ, as is proved in Theorem 12 of this Book. Therefore the square on the sum AB and ΓΔ is smaller than the square on the sum of EZ and KH. Therefore the straight line equal to the sum of two axes

AB and ΓΔ is smaller than the straight line equal to the sum of two diameters EZ and KH.

Similarly too if will be proved that the straight line equal to [the sum of] EZ and HK is smaller than the straight line equal to the sum of two diameters ΡΣ and YT.

[Proposition] 27

*In every ellipse or hyperbola in which two axes are unequal the increment of the greater axis over the smaller is greater than the increment of [the greater of] any conjugate diameter among its diameters over the diameter conjugate with it, and the increment of [the greater of a pair of] them nearer to the greater axis over the diameter conjugate with it is greater than the increment of [the greater of a pair of them] farther [from the major axis] over the diameter conjugate with it* [33] .

Now it has been proved in Theorem 24 of this Book that in case of the ellipse that is as we stated, but as for the hyperbola it will be proved as follows. We make the axis of the hyperbola AΓ. Let some of its conjugate diameters be KB and ZH, and TY and IO.

Then I say that the difference between AΓ and the other axis is greater than the difference between KB and ZH, and that the difference between KB and ZH is greater than the difference between TY and IO.

[Proof]. The difference between sq.AΓ and the square on the other of two axes of the section is equal to the difference between sq.KB and sq.ZH, as is proved in Theorem 13 of this Book. And the diameter BK is greater than the axis AΓ. Therefore the difference between AΓ and the axis conjugate with it is greater than the difference between KB and ZH.

Similarly too it will be proved that the difference between KB and ZH is greater than the difference between TY and IO.

[Proposition] 28

*In every hyperbola or ellipse the rectangular plane under its two axes is smaller than the rectangular plane under any conjugate pair whatever of its diameters, and of the conjugate diameters for those in which the greater [of the pair] is closer to the greater axis ,the rectangular plane under the diameter and the diameter conjugate with it is smaller than rectangular plane under one of those in which it is farther from it [the greater axis] and the diameter conjugate with it* [34] .

Now as for the case of the hyperbola, that will be proved from what we said in that precedes. For each of two axes is smaller than the diameter adjacent to it of any pair of conjugate diameters, and those of the [diameters] closer two axes are smaller than those farther.

But as for the case of the ellipse we make its major axis AB and the minor ΓΔ, and let some of its conjugate diameters be EZ and KH, NΞ and OΠ, and PΣ and YT, then I say that pl.AB,ΓΔ is smaller than pl.EZ,KH and that pl.EZ,KH is smaller than pl.NΞ,ΠO, and pl.NΞ,ΠO is smaller than pl.TY,PΣ.

[Proof].The sum of two axes AB and ΓΔ is smaller than the sum of two diameters EZ and HK, as is proved in Theorem 26 of this Book, and [hence] the square on the sum AB and ΓΔ is smaller than the square on the sum EZ and HK.

But the sum sq.AB and sq.ΓΔ is equal to the sum of sq.EZ and sq.HK, as is proved in Theorem 12 of this Book. Therefore the by subtraction the double pl.AB,ΓΔ is smaller than the double pl.EZ,KH . Therefore pl.AB,ΓΔ is smaller than pl.EZ,KH .

Similarly too it will be proved that pl.EZ,KH is smaller than pl.NΞ,OP ,and pl.NΞ,OΠ is smaller than pl.YT,PΣ .

[Proposition] 29

*The differences between the* eidoi *corresponding to [each of] the diameters of any hyperbola and [each of] the squares onthose diameters are equal* [35] .

Let there be the hyperbola whose axis AΓ and center Θ, and let some of its conjugate diameters be KB and TY, and OY and ZH, then I say that the difference between the *eidos* of the section corresponding to AΓ and sq.AΓ is equal to the difference between the *eidos* of the section corresponding to KB and sq.KB, and [also is equal to] the difference between the *eidos* corresponding to TY and sq.TY.

[Proof]. The difference between sq.AΓ and the square on the other of the two axes of the section is equal to the difference between sq.KB and sq.ZH, and [also is equal to] the difference between sq.YT and sq.IO, as was proved in Theorem 13 in this Book.

But as for the *eidos* of the section corresponding to AΓ, it is equal to the square on the other of two axes of the section, has we stated in Theorem 16 of Book I. And as for the *eidos* of the section corresponding to KB, it is equal to sq.ZH, and as for the *eidos* of the section corresponding to TY, it is equal to sq.OI. Therefore the difference between the *eidos* of the section corresponding to ΑΓ and sq.ΑΓ is equal to the difference between the *eidos* of the section corresponding to BK and sq.BK, and [also is equal to] the difference between the *eidos* of the section corresponding to TY and sq.TY.

[Proposition] 30

*If there is added to [one of] the* eidoi *corresponding to any of the diameters of an ellipse the square of that diameter [the sum always] comes out equal* [36].

Let the center of the ellipse be Θ, and some of its conjugate diameters be BK and ZH, and TY and OI.

Then I say that the *eidos* of the section corresponding to BK together with sq.BK is equal to the *eidos* of the section corresponding to TY together with sq.TY.

[Proof]. The sum of sq.BK and sq.HZ is equal to the sum of sq.YT and sq.OI, as is proved in Theorem 12 of this Book.

But as for the *eidos* of the section corresponding to BK, is equal to sq.ZH, and as for sq.OI, it is equal to the *eidos* of the section corresponding to TY, as is proved in Theorem 15 of Book I.

Therefore the *eidos* of the section corresponding to BK together with sq.BK is equal to the *eidos* of the section corresponding to TY together with sq.TY

[Proposition] 31

*When a pair of conjugate diameters is drawn in an ellipse or between conjugate opposite hyperbolas, then the parallelogram under that pair of diameters with angles equal to the angles under the diameter at the center is equal to the rectangular plane under two axes* [37].

Let there be the ellipse or the conjugate opposite hyperbolas whose center Θ and axes AB and ΓΔ , and with one pair of its conjugate diameters ZΛ and ΞN.

Let the tangents [to these section] pass through Z and Λ, and Ξ and N be HP and KM, and HK and PM. Then HP and KM are parallel to the diameter ΞN, and HK and PM are parallel to the diameter ZΛ, as is proved in Theorems 5 and 20 of Book II. Therefore the quadrangle HM is a parallelogram, and its angles are equal to the angles under the diameters ZΛ and ΞN at the center Θ.

Then I say that the quadrangle MH is equal to the rectangular plane under two axes AB and ΓΔ.

[Proof]. We drop from Z the perpendicular ZΠ to BΘA, and make the straight line ΠO a mean proportional between EΠ and ΠΘ. Then as sq.AΘ is to sq.ΘΓ, so pl.ΘΠE is to sq.ZΠ, as is proved in Theorem 37 of Book I. But pl.ΘΠE is equal to sq.ΠO. Therefore as sq.AΘ is to sq.ΘΓ, so sq.ΠO is to sq.ZΠ, and as AΘ is to ΘΓ, so ΠO is to ZΠ, and as sq.AΘ is to pl.AΘΓ, so pl.OΠ,ΘE is to pl.ZΠ,ΘE .
And *permutando* as sq.AΘ is to pl.OΠ,ΘE , so pl.AΘΓ is to pl.ZΠ,ΘE .

But sq.AΘ is equal to pl.EΘΠ, as is proved in Theorem 37 of Book I. Therefore as pl.EΘΠ is to pl.OΠ,ΘE , so pl.AΘΓ is to pl.ZΠ.ΘE . And ΘΞ is parallel to ZE. Therefore as sq.ZE is to sq.ΘΞ, so EΠ is to ΠΘ, as is proved in Theorem 4 of this Book. And as the triangle ΘZE is to the triangle ΞΘT, so sq.ZE is to sq.ΘΞ because two triangles are similar. Therefore as the triangle ΘZE is to the triangle ΞΘT, so EΠ is to ΠΘ, and as the double triangle ΘZE is to the double the triangle ΞΘT, so EΠ is to ΠΘ. But the quadrangle ΞΘZH is a mean proportional between the double triangle ΘZE and the double triangle ΞΘT.

And similarly OΠ is a mean proportional between EΠ and ΠΘ. Therefore as the double triangle ΘZE is to the parallelogram ΘH, so OΠ is to ΠΘ.

But as OΠ is to ΠΘ, so pl.OΠ,ΘE is to pl.ΠΘE. Therefore as the double triangle ΘZE is to the quadrangle ΘH, so pl.OΠ,ΘE is to pl.ΠΘE.

And we had [already] proved that as pl.OΠ,ΘE is to pl.ΠΘE, so pl.ZΠ,ΘE is to pl.AΘΓ. Therefore as the double triangle ΘZE is to the quadrangle ΘH, so pl.ZΠ,ΘE is to pl.AΘΓ. But the double triangle ΘZE is equal to pl.ZΠ,ΘE. Therefore, the quadrangle ΘH is equal to pl.AΘΓ, and [hence] the quadruple quadrangle ΘH with is [the quadrangle] HM is equal to the quadruple pl.AΘΓ with is equal to the rectangular plane under two axes AB and ΓΔ. Therefore the quadrangle MH is equal to the rectangular plane under two axes AB and ΓΔ.

[Porisms]

*Thus it has been shown from the preceding theorems that:*

*1) in every hyperbola the sum of the squares on its two axes is smaller than [the sum of] the squares on any conjugate pair whatever of its diameter , and [the sum is] the squares on a pair of conjugate diameters closer to two axes is smaller than [the sum of] the squares on a pair of conjugate diameters farther from two axes* [38], *2) and that in every ellipse the difference between the squares on its two axes is greater than the difference between the squares on any conjugate pair whatever of its diameters ,and the difference between the squares on [a pair of] conjugate diameters close to two axes is grater than the difference between the squares on [a pair of] conjugate farther from two axes* [39],

3) *and that if there is a hyperbola in which the transverse diameter of the sides of the* eidos *of the section corresponding to the axis is greater than the* latus rectum, *then the transverse diameter of [each of]* eidoi *of the section corresponding to the other diameters is greater than its* latus rectum *and [in that case] the rate of the transverse diameter of the* eidos *corresponding to that axis to the* latus rectum *is greater than the ratio of every [other] transverse diameter to the* latus rectum *of the* eidos *corresponding to it , this ratio in the* eidoi *corresponding to those transverse diameters closer to the axis is greater than in those corresponding to transverse diameters farther from the axis* [40],

*4) but if the transverse diameter of the* eidos *corresponding to the axis of the hyperbola is smaller than the* latus rectum, *then other transverse diameters of other* eidoi *are smaller than their latera recta, and the ratio of the transverse diameter of the* eidos *corresponding to that axis to its* latus rectum *is smaller than the ratio of every [other] transverse diameter to the* latus rectum *of the* eidos *corresponding to it, and this ratio in the* eidoi *corresponding to those transverse diameters closer to the axis is smaller than in those corresponding to transverse diameter farther from the axis* [41],

*5) and if the* eidos *of the hyperbola corresponding to the axis is equilateral, then the* eidoi *of the section corresponding to other diameters are equilateral* [42],

It has also been shown that

*6) in every ellipse the transverse diameter of the* eidos *of the section corresponding to the diameters drawn between the major axis and two equal conjugate diameters is greater than their* latus rectum, *and the ratio of it [the transverse diameter] to it [the* latus rectum *in the* eidoi *corresponding to these diameters closer to the major axis is greater than in those corresponding to transverse diameters farther from it* [43] ,

*7) but as for the transverse diameter of the eidoi of the ellipse corresponding to the diameters between the minor axis and two equal conjugate diameters, it is smaller than* latus rectum, *and the ratio of it [the transverse diameter] to it [the* latus rectum in these *eidoi* corresponding to those diameters closer to the minor axis is smaller than *in those corresponding the diameters farther from it* [44].

These are theorems which can be proved from what we proved in the treatment of the diameters and *eidoi* of sections and their sides, and the ratios of the conjugate diameters and corresponding *latera recta*.

[Proposition] 32

*In every parabola the* latus rectum *which is the straight line such that the ordinates dropped to the axis are equal in square to the rectangular planes under that straight line and the segments of the axis cut off by ordinates is the smallest of the latera recta which are the straight lines such that the ordinates dropped on the other diameters are equal in square to corresponding rectangular planes, and the* latus rectum *corresponding to [one of] those diameters closer to the axis is smaller than the* latus rectum *corresponding to the diameter farther* [45].

Let there be the parabola AB whose axis AZ and with two other of its diameters BΘ and ΓH, and let the *latera recta* [correspondingly to the diameters AZ, ΓH, and BΘ] be AK, ΓΛ and BM [respectively] .

I say that AK is smaller than ΓΛ, and that ΓΛ is smaller than BM.

[Proof]. We drop from B and Γ the perpendiculars BΔ and ΓE to the axis. Then ΓΛ is equal to the sum of AK and the quadruple EA, as is proved in Theorem 5 of this Book. And similarly BM is equal to the sum of AK and the quadruple AΔ. Therefore AK is smaller than ΓΛ, and ΓΛ is smaller than BM .

[Proposition ] 33

*If there is a hyperbola, and the transverse diameter of the* eidos *corresponding to the axis is not smaller than its* latus rectum, *then the* latus rectum *of the* eidos *corresponding to the axis is smaller than the* latus rectum *of [any of] the* eidoi *corresponding to other diameters of the section, and the* latus rectum *of [any of] the eidoi corresponding to diameters closer to the axis is smaller than the* latus rectum *of the* eidoi *corresponding to the diameters farther from the axis* [46] .

Let there be the hyperbola whose axis AΓ and center Θ, and with two of its diameters KB and YT.

Then I say that the *latus rectum* of the *eidos* of the section corresponding to AΓ is smaller than the *latus rectum* of the *eidos* of the section corresponding to KB, and that the *latus rectum* of the *eidos* of the section corresponding to KB is smaller than the *latus rectum* of the *eidos* of the section corresponding to YT.

[Proof]. First we make the axis ΑΓ equal to the *latus rectum* to the *eidos* corresponding to it. Then the diameter ΒΚ is equal to the *latus rectum* of the *eidos* corresponding to it, which can be proved from Theorem 23 of this Book and Theorem 16 of Book I.

But ΑΓ is smaller than ΒΚ. Therefore the *latus rectum* of the *eidos* corresponding to ΑΓ is smaller than the *latus rectum* of the *eidos* corresponding to ΚΒ.

Furthermore the diameter ΤΥ is equal to the *latus rectum* of the *eidos* of the section corresponding to it. But the diameter ΚΒ is smaller than the diameter ΤΥ. Therefore the *latus rectum* of the *eidos* of the section corresponding to ΚΒ is smaller than the *latus rectum* of the *eidos* of the section corresponding to ΥΤ.

Furthermore we make the axis ΑΓ greater than the *latus rectum* of the *eidos* of the section corresponding to it, and [then] the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it is greater than the ratio of ΚΒ to its *latus rectum*, as is proved from Theorem 21 of this Book and Theorem 16 of Book I. And similarly the ratio of ΚΒ to its *latus rectum* is greater than the ratio of ΥΤ to its *latus rectum*. But the axis ΑΓ is smaller than the diameter ΚΒ, and the diameter ΒΚ is smaller than the diameter ΤΥ. Therefore the *latus rectum* of the axis ΑΓ is smaller than the *latus rectum* of the diameter ΚΒ, and the *latus rectum* of the diameter BK is smaller than the *latus rectum* of the diameter ΥΤ.

## [Proposition] 34

Furthermore we make ΑΓ smaller than the *latus rectum* of the *eidos* corresponding to it, but not smaller than the half of the *latus rectum* of the *eidos* corresponding to it, then I say that again the *latus rectum* of the *eidos* corresponding to ΑΓ is smaller than the *latus rectum* of the *eidos* corresponding to ΚΒ, and that the *latus rectum* of the *eidos* corresponding to ΚΒ is smaller than the *latus rectum* of the *eidos* corresponding to ΤΥ [47] .

[Proof]. We make each of the ratios ΓΝ to ΑΝ and ΑΞ to ΞΓ equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it , and draw from Γ the straight line ΓΛ parallel to ΚΒ, and the straight line ΓΔ parallel to ΤΥ, and drop from Δ and Λ the perpendiculars ΔΕ and ΛΜ to the axis. Then, since each of the ratios ΓΝ to ΑΝ and ΑΞ to ΓΞ is equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to if. ΓΝ is equal to ΑΞ and ΓΞ equal to ΑΝ.

Therefore the ratio of sq.ΑΓ to the square on the *latus rectum* of the *eidos* corresponding to it is equal to the ratio pl.ΓΝ,ΑΞ to sq.ΑΝ.

But the diameter ΑΓ is smaller than ΑΝ its *latus rectum* . But not smaller than the half of the *latus rectum*. Therefore ΑΝ is greater than ΑΞ but not greater than the double ΑΞ. And the sum of ΜΝ and ΑΝ is greater than the double ΑΝ. Therefore the rectangular plane under ΑΜ and the sum ΜΝ and ΑΝ to the rectangular plane under ΑΞ and the sum of ΜΝ and ΑΝ is smaller than the rectangular plane under ΑΜ and the sum ΜΝ and ΑΝ to sq.ΑΝ. Therefore the ratio ΑΜ to ΑΞ is smaller than the rectangular plane under ΑΜ and the sum ΜΝ and ΑΝ to sq.ΑΝ, and [hence] the ratio ΜΞ to ΑΞ is smaller than the ratio of the sum of sq.ΑΝ and the rectangular plane under ΑΜ and the sum of ΜΝ and ΑΝ to sq.ΑΝ. But the sum of sq.ΑΝ and the rectangular plane under ΑΜ and the sum of ΜΝ and ΑΝ is equal to sq.ΜΝ. Therefore the ratio ΜΞ to ΑΞ is smaller than the ratio sq.ΜΝ to sq.ΑΝ.

But the ratio ΜΞ to ΑΞ is equal to the ratio pl.ΓΝ,ΜΞ to pl. ΓΝ,ΑΞ. Therefore the ratio pl.ΓΝ,ΜΞ to pl.ΓΝ.ΑΞ is smaller than the ratio sq.ΜΝ to sq.ΑΝ. And *permutando* the ratio pl.ΓΝ,ΜΞ to sq.ΜΝ is smaller than the ratio pl.ΓΝ,ΑΞ to sq.ΑΝ.

Now as for the ratio pl.ΓΝ,ΞΜ to sq.ΜΝ, is equal to the ratio of sq.ΓΑ to the square on the *latus rectum* of the diameter ΚΒ, as is proved in Theorem 15 of this Book, and as for the ratio pl.ΓΝ,ΑΞ to sq.ΑΝ, we have [already] proved that it is equal to the ratio of sq.ΑΓ to the square of the diameter ΑΓ.

Therefore the ratio of sq.ΑΓ to the square of the diameter ΒΚ is smaller than the ratio of sq.ΑΓ to the square on the *latus rectum* of the *eidos* corresponding to it. Therefore the *latus rectum* of the diameter ΑΓ is smaller than the *latus rectum* of the diameter ΒΚ.

Furthermore ΑΝ is not greater than the double ΑΞ. Therefore ΜΝ is smaller than the double ΜΞ. And the sum of ΕΝ and ΜΝ is greater than the double ΜΝ. Therefore pl.ΕΜ, the sum of ΕΝ and ΜΝ is greater than sq.ΜΝ. Therefore the ratio pl. ΕΜ, the sum of ΕΝ and ΜΝ.to pl.ΜΞ, the ratio [the rectangular plane] under ΜΞ and the sum ΜΝ and ΕΝ is smaller than the ratio of [the rectangular plane] under ΕΜ and the sum ΕΝ and ΜΝ to sq.ΜΝ. But the ratio [the rectangular plane] under ΕΜ and the sum ΕΝ and ΜΝ to [the rectangular plane] under ΜΞ and the sum of ΜΝ and ΕΝ is equal to the ratio ΕΜ to ΜΞ. Therefore the ratio ΕΜ to ΜΞ is smaller than the ratio [the rectangular plane] under ΜΕ and the sum ΕΝ and ΜΝ to sq.ΜΝ. Therefore the ratio ΕΞ to ΜΞ is smaller than the ratio of the sum sq.ΜΝ and [rectangular plane] under ΜΕ and the sum ΕΝ and ΜΝ to sq.ΜΝ. But the sum of sq.ΜΝ and [the rectangular plane] under ΜΕ and the sum of ΕΝ and ΜΝ is equal to sq.ΕΝ. Therefore the ratio ΕΞ to ΜΞ is smaller than the ratio sq.ΕΝ to sq.ΜΝ.

But the ratio ΕΞ to ΜΞ is equal to the ratio pl.ΓΝ,ΕΞ to pl.ΓΝ,ΜΞ.

Therefore the ratio pl.ΓΝ,ΕΞ to pl.ΓΝ,ΜΞ is smaller than sq.ΕΝ to sq.ΜΝ.

And *permutando* the ratio pl.ΓΝ,ΕΞ to sq.ΕΝ is smaller than pl.ΓΝ,ΜΞ to sq.ΜΞ. But as for the ratio pl.ΓΝ,ΕΞ to sq.ΕΝ, it is equal to the ratio of sq.ΑΓ to the square on the *latus rectum* of the diameter ΤΥ, as is proved in Theorem 15 of this Book, and as for the ratio pl.ΓΝ,ΜΞ to sq.ΜΝ, it is equal to the ratio of sq.ΑΓ to the square on the *latus rectum* of the diameter ΚΒ, as is proved in Theorem 15 of this Book.

Therefore the ratio of sq.ΑΓ to the square on the *latus rectum* of the diameter ΤΥ is smaller than the ratio of it [sq.ΑΓ] to the square on the *latus rectum* of the diameter ΚΒ.

Therefore the *latus rectum* of the diameter ΚΒ is smaller than the *latus rectum* of the diameter ΤΥ. And it has already been shown that the *latus rectum* of the diameter ΑΓ is smaller than the *latus rectum* of the diameter ΚΒ.

## [Proposition] 35

Furthermore we make ΑΓ smaller than the half of the *latus rectum* of the *eidos* of the section corresponding to it, then I say that there are two diameters [one] on either side of this axis such that the *latus rectum* of the *eidos* corresponding to each of them is the double that [diameter], and that [*latus rectum*] is smaller than the *latus rectum* of the *eidos* corresponding to any other of the diameters on that side [of the axis],and the *latus rectum* of *eidoi* corresponding to the diameters closer to those two diameters is smaller than the *latus rectum* of the *eidos* corresponding to a diameter farther from them[48] .

[Proof]. ΑΓ has been cut into two parts Ξ such that the ratio Ξ to ΞΓ is equal to the ratio of ΑΓ to its *latus rectum*, and likewise the ratio ΓΝ to ΝΑ [is the same ratio]. And the diameter ΑΓ is smaller than the half of its *latus rectum*. Therefore ΑΝ is greater than the double ΑΞ. Therefore ΝΞ is greater than ΞΑ.

Therefore let ΞΜ be equal to ΞΝ, and let ΜΛ be the perpendicular to the axis meeting the section at Λ. We join ΓΛ and draw the diameter ΚΒ parallel to ΓΛ. Then the ratio ΕΜ to ΜΝ is equal to the ratio of ΒΚ to the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 6 of this Book.

Therefore the diameter ΒΚ is the half of the *latus rectum* of the section corresponding to it.

Therefore we draw between Α and Β the diameters ΔΕ and ΥΤ, and draw from Γ the straight line ΓΡ parallel to the diameter ΔΕ and the straight line ΓΟ parallel to the diameter YT, and drop from P and O the perpendiculars Pι and OΠ to the axis.

Now MΞ is equal to EN. Therefore pl.MΞι is smaller than sq.ΞN we make [the rectangular plane] under ιΞ and the sum of ιN and NΞ common [to both sides], then [rectangular plane] under ιΞ and the sum of MN and Nι is smaller than sq.Nι. Therefore the ratio [the rectangular plane] under Mι and the sum of MN and Nι to [the rectangular plane] under ιΞ and the sum of MN and Nι is greater than the ratio [the rectangular plane] under Mι and the sum of MN and Nι to sq.Nι. But the ratio [the rectangular plane] under Mι and the sum of MN and Nι to [the rectangular plane] under Ξι and the sum of MN and Nι is equal to the ratio Mι to Ξι. Therefore the ratio Mι to Ξι is greater than [the rectangular plane] under Mι and the sum of MN and Nι to sq.Nι. Therefore the ratio MΞ to Ξι is greater than the ratio the sum sq.Nι and [the rectangular plane] under Mι and the sum of MN and Nι to sq.Nι.

But the sum of sq.Nι and [the rectangular plane] under Mι and the sum of MN and Nι is equal to sq.MN. Therefore the ratio MΞ to Ξι is greater than the ratio sq.MN to sq.Nι.

But the ratio MΞ to Ξι is equal to the ratio pl.ΓN,MΞ to pl.ΓN,Ξι. Therefore the ratio pl.ΓN,MΞ to pl.ΓNΞι is greater than the ratio sq.MN to sq.Nι

And *permutando* the ratio pl.ΓN,MΞ to sq.MN is greater than pl.ΓN,ιΞ to sq,Nι.

But as for the ratio pl.ΓN,MΞ to sq.MN, it is equal to the ratio of sq.AΓ to the square on the *latus rectum* of the *eidos* corresponding to KB as is proved in Theorem 15 of this Book. And as for the ratio pl.ΓN,Ξι to sq.Nι , it is equal to the ratio of sq.AΓ to the square on the *latus rectum* of the *eidos* corresponding to ΔE as is proved in Theorem 15 of this Book.

Therefore the ratio of sq.AΓ to the square on the *latus rectum* of the *eidos* corresponding to KB is greater than the ratio of sq.AΓ to the *latus rectum* of the *eidos* corresponding to ΔE . Therefore the *latus rectum* of the *eidos* corresponding to KB is smaller than the *latus rectum* of the *eidos* corresponding to ΔE.

Furthermore pl.ιΞΠ is smaller than sq.NΞ. Therefore it will be proved from that, as we proved previously that the *latus rectum* of the *eidos* corresponding to ΔE is smaller than the *latus rectum* of the *eidos* corresponding to YT.

Furthermore pl.ΠΞA is smaller than sq.NΞ. Therefore the *latus rectum* of the *eidos* corresponding to YT is smaller than the *latus rectum* of the *eidos* corresponding to AΓ.

Furthermore we draw two diameters ZH and ΦX farther from the axis than is the diameter BK, then I say that the *latus rectum* of the *eidos* corresponding to BK is smaller than the *latus rectum* of the *eidos* corresponding to ZH, and that the *latus rectum* of the *eidos* corresponding to ZH is smaller than the *latus rectum* of the *eidos* corresponding to ΦX.

[Proof]. Now we draw from Γ two straight lines ΓΨ and ΓϘ parallel to ZH and ΦX, and drop from Y and Ϙ the perpendiculars ΨΩ and ϘΣ to the axis. Then pl.ΣΞM is greater than sq.NΞ . Therefore when we go through a procedure like the preceding one, it is shown that the ratio pl.ΓN,ΞΣ to sq.NΣ is smaller than the ratio pl.NΓ,MΞ to sq.MN, and from that it will be proved that the *latus rectum* of the *eidos* corresponding to ZH is greater than the *latus rectum* of the *eidos* corresponding to KB. And because pl.ΩΞΣ is greater than sq.NΞ the *latus rectum* of the *eidos* corresponding to ΦX is greater than the *latus rectum* of the *eidos* corresponding to ZH.

[Proposition] 36

Let there be the hyperbola whose axis AΓ and center Θ, and with two other of its diameters ΔE and BK.

If there is a hyperbola, and the *eidos* corresponding to its axis is not equilateral, then the difference between two sides of the *eidos* corresponding to its axis is greater than the difference between the sides of [any of] the *eidoi* corresponding to other diameters , and the difference between the sides of the *eidoi* corresponding to those diameters closer to the axis is greater than the difference between the sides of the *eidoi* corresponding to those diameters farther from it [49].

Then I say that the difference between two sides of the *eidos* corresponding to AΓ is greater than the difference between two sides of the *eidos* corresponding to ΔE, and that this [latter] difference is greater than the difference between two sides of the *eidos* corresponding to BK.

But we draw ΓZ and ΓΛ parallel to the diameters ΔE and BK, and drop from Λ and Z the perpendiculars ZΠ and ΛM to the axis and make each of the ratios ΓN to NA and AΞ to ΓΞ equal to the ratio of AΓ to the *latus rectum* of the *eidos* corresponding to it. Then the ratio of sq.AΓ to the square on the difference between AΓ and the *latus rectum* of the *eidos* corresponding to it is equal to the ratio pl.ΓN,AΞ to sq.NΞ. And ΓZ is parallel to the diameter ΔE, and ZΠ is the perpendicular to the axis. Therefore the ratio pl.ΓN,ΞΠ to the square on the difference between ΠΞ and ΠN is equal to the ratio of sq.AΓ to the square of the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 16 of this Book.

But the difference between ΠΞ and ΠN is equal to ΞN. Therefore the ratio of sq.ΑΓ to the square on the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it is equal to the ratio pl.ΓN,ΞΠ to sq.ΞN.
And the ratio pl.ΓN,ΞΠ to sq.ΞN is greater than the ratio pl.ΓN,ΑΞ to sq. ΞN.

Therefore the ratio of sq.ΑΓ to the square on the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it is greater than ΑN the ratio of sq.ΑΓ to the square of the difference between it and the *latus rectum* of the *eidos* corresponding to it. Therefore the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it is smaller than the difference between ΑΓ and the *latus rectum* of the *eidos* corresponding to it.

Furthermore ΛΓ is parallel to the diameter KB, and ΛM is the perpendicular to the axis. Therefore the ratio pl.ΓN,ΞM to the square on the difference between MΞ and MN is equal to the ratio of sq.ΑΓ to the square on the difference between BK and the *latus rectum* of the *eidos* corresponding to it as is proved in Theorem 16 of this Book.

And the ratio pl.ΓN,ΞM to sq.NΞ is greater than the ratio pl.ΓN,ΠΞ to sq.NΞ .Therefore the ratio of sq.ΑΓ to the square on the difference between KB and the *latus rectum* of the *eidos* corresponding to it is greater than the ratio of sq.ΑΓ to the square on the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it.

Therefore the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between BK and the *latus rectum* of the *eidos* corresponding to it.

[Proposition] 37

*In every ellipse for the* eidoi *of the section corresponding to the diameters greater than their [corresponding] latera recta the difference between two sides of the* eidos *corresponding to the major axis is greater than the difference between two sides of [any of] the* eidoi *corresponding to the remaining diameters, and the difference between two sides of those* eidoi *corresponding to the diameters closer to the major axis is greater than the difference between two sides of those* eidoi *corresponding to the diameters farther [from the major axis].*

*But in the case when the diameters on which the which the corresponding* eidoi *are smaller than the latera recta, the difference between* *two sides of the* eidos *corresponding to the minor axis is greater than difference between two sides of the others of these* eidoi *and the difference between two sides of those of the* eidoi *corresponding to the diameters closer to the minor axis is greater than the difference between two sides of those* eidoi *corresponding to the diameters farther from it.*

*And the difference between two sides of the* eidos *corresponding to the major axis is greater than the difference between two sides of the* eidos *corresponding to the minor axis* [50].

Let there be the ellipse whose major axis ΑΓ and minor axis ΕΔ, and with two of its diameters ΚΒ and ΖΗ, both ΖΗ and ΚΒ being greater than the *latus rectum* of the *eidos* corresponding to it.

Then I say that the difference between ΑΓ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between ΒΚ and the *latus rectum* of the *eidos* corresponding to it, and that the difference between ΒΚ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between ΖΗ and the *latus rectum* of the *eidos* corresponding to it.

[Proof]. ΑΓ is greater than the *latus rectum* of the *eidos* corresponding to it, and ΚΒ also is greater than the *latus rectum* of the *eidos* corresponding to it, and also the *latus rectum* of the *eidos* corresponding to ΚΒ is greater than the *latus rectum* of the *eidos* corresponding to ΑΓ, as is proved in Theorem 24 of this Book. Therefore the difference between ΑΓ and the *latus rectum* of the *eidos* constructed to it is greater than the difference between ΚΒ and the *latus rectum* of the *eidos* corresponding to it.

Similarly too it will be proved that the difference between ΚΒ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between ΖΗ and the *latus rectum* of the *eidos* corresponding to it.

Furthermore, we make each of ΒΚ and ΖΗ smaller than the *latus rectum* of the *eidos* corresponding on it, then I say that the difference between ΔΕ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between ΖΗ and the *latus rectum* of the *eidos* corresponding to it, and that the difference between ΖΗ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between KB and the *latus rectum* of the *eidos* corresponding to it.

[Proof]. ΔΕ is smaller than ΖΗ, and the *latus rectum* of the *eidos* corresponding to ΔΕ is grater than the *latus rectum* of the *eidos* corresponding to ΖΗ, as is proved in this Book. Therefore the difference between ΔΕ and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between ΖΗ and the *latus rectum* of the *eidos* corresponding to it.

Similarly too it will be proved that the difference between ZH and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between KB and the *latus rectum* of the *eidos* corresponding to it.

Furthermore the ratio of the *latus rectum* of the *eidos* corresponding to ΔE to ΔE is equal to the ratio of AΓ to the *latus rectum* of the *eidos* corresponding to AΓ, as is proved in Theorem 15 of Book I. And the *latus rectum* of the *eidos* corresponding to ΔE is greater than AΓ, as is proved from Theorem 15 of Book I. Therefore the difference between ΔE and the *latus rectum* of the *eidos* corresponding to it is greater than the difference between AΓ and the *latus rectum* of the *eidos* corresponding to it.

[Proposition] 38

*If there is a hyperbola, and the transverse side of the* eidos *corresponding to its axis is not smaller than one third of its* latus rectum, *then the sum of the straight lines bounding each of the* eidoi *corresponding to its diameters which are nor the axes is greater than the sum of the straight lines bounding the* eidos *corresponding to its axis, and the sum the straight lines bounding the* eidoi *corresponding to those diameters closer to the axis is smaller than [the sum of] the sides bounding the* eidoi *corresponding those diameters farther from it* [51] .

Let there be the hyperbola whose axis AΓ, AΓ being not smaller then one third of the *latus rectum* of the *eidos* corresponding to it. Let two of its diameters be KB and TY.

Then I say that [the sum of] the sides bounding the *eidos* corresponding to AΓ is smaller than [the sum of] the sides bounding the *eidos* corresponding to KB, and that [the sum of] the sides bounding the *eidos* corresponding to KB, is smaller than [ the sum of] the sides bounding the *eidos* corresponding to YT.

[Proof]. We make first the axis AΓ not smaller than the *latus rectum* of the *eidos* corresponding to it.

Now the diameter KB is greater than the axis AΓ, and the diameter TY is greater than the diameter KB, and the *latus rectum* of the *eidos* corresponding to TY is greater than the *latus rectum* of the *eidos* corresponding to KB, as is proved in Theorem 33 of this Book, and likewise too the *latus rectum* of the *eidos* corresponding to KB is greater than the *latus rectum* of the *eidos* corresponding two AΓ. Therefore the sum of the diameter YT and the *latus rectum* of the *eidos* corresponding to it is greater than the sum of the diameter KB and the *latus rectum* of the *eidos* corresponding to it, and the sum of the diameter KB and the *latus rectum* of the *eidos* corresponding to it is greater than the sum of the diameter ΑΓ and the *latus rectum* of the *eidos* corresponding to it. Therefore the sum of the sides bounding the *eidos* corresponding to TY is greater than the sum of the sides bounding the *eidos* corresponding to KB, and the sum of these [latter] sides is greater than the sum of the sides bounding the *eidos* corresponding to ΑΓ.

[Proposition] 39

Furthermore we make ΑΓ smaller than the *latus rectum* of the *eidos* corresponding to it, but not smaller than one third of the *latus rectum* of the *eidos* corresponding to it, and let each of the ratios ΓΝ to ΑΝ and ΑΞ to ΓΞ be equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it, and draw from Γ two straight lines ΓΔ and ΓΛ parallel to the diameters YT and KB [respectively], and drop from Δ and Λ the perpendiculars ΔΕ and ΛΜ to the axis. Then the ratio of ΑΧ to the *latus rectum* of the *eidos* corresponding to it is equal to the ratio ΑΞ to ΞΓ, and ΑΓ is not smaller than one third of the *latus rectum* of the *eidos* corresponding to it. Therefore ΑΞ is not smaller than one third of ΑΝ. Therefore ΑΞ is not smaller than the quarter of the sum of ΝΑ and ΑΞ. Therefore [the rectangular plane] under the quadruple ΑΞ and the sum of ΝΑ and ΑΞ is not smaller than the square of the sum of ΝΑ and ΑΞ. Therefore the ratio the quadruple [the rectangular plane] under ΑΜ and the sum ΝΑ and ΑΞ to the quadruple [the rectangular plane] under ΑΞ and the sum of ΝΑ and ΑΞ is not greater than the quadruple [the rectangular plane] under ΑΜ and the sum p\of ΝΑ and ΑΞ to the square on the sum of ΝΑ and ΑΞ. Therefore the ratio ΑΜ to ΑΞ is not greater than the ratio the quadruple [the rectangular plane] under ΑΜ and the sum of ΝΑ and ΑΞ to the square on the sum ΝΑ and ΑΞ. And *componendo* the ratio ΜΞ to ΞΑ is not greater than the ratio the quadruple sum of the square on the sum of ΝΑ and ΑΞ and [the rectangular plane] under ΑΜ and the sum of ΝΑ and ΑΞ to the square on the sum of ΝΑ and ΑΞ.

But the quadruple sum of the square of the sum of ΝΑ and ΑΞ and [the corresponding plane] under ΑΜ and the sum of ΝΑ and ΑΞ is smaller than the square on the sum of ΜΝ and ΜΞ. Therefore the ratio ΜΞ to ΞΑ is smaller than the ratio of the square on the sum of ΜΝ and ΜΞ to the square on the sum of ΑΝ and ΑΖ.

But the ratio ΜΞ to ΑΞ is equal to the ratio of pl.ΓΝ,ΜΞ to pl.ΓΝ,ΑΞ .

Therefore the ratio pl.ΓN,MΞ to pl.ΓN,AΞ is smaller than the ratio square on the sum of MN and MΞ to the square on the sum of AN and AΞ.

And the ratio pl.ΓN,MΞ to the square on the sum of MN and MΞ is smaller than the ratio pl.ΓN,AΞ to the square of the sum of AN and AΞ.

But as for the ratio pl.ΓN,MΞ to the square on the sum of MN and MΞ, it is equal to the ratio of sq.AΓ to the square on the diameter KB together with the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 17 of this Book, and as for the ratio pl.ΓN,AΞ to the square on the sum of AΞ and AN, it is equal to the ratio of sq.AΓ to the square on the diameter AX together with the *latus rectum* of the *eidos* corresponding to it.

Therefore the ratio of sq.AΓ to the square on [the sum of] two sides of the *eidos* corresponding to KB is  smaller than the ratio of sq.AΓ to the square on [the sum of] two sides of the *eidos* corresponding to AΓ. Therefore the sum of two sides of the *eidos* corresponding to KB is greater than the sum of two sides of the *eidos* corresponding to AΓ. And therefore the sum of the sides bounding the *eidos* corresponding to KB is greater than the sum of the sides bounding the *eidos* corresponding to AΓ.

Furthermore MΞ is greater than the quarter of the sum of MN and MΞ, therefore the quadruple [the rectangular plane] under MΞ and the sum NM and MΞ is greater then the square on the sum of MN and MΞ. Therefore it will be proved from that, as it was proved above, that the ratio pl.ΓN,ΞE to the square on the sum of NE and EΞ is smaller than the ratio pl.ΓN,MΞ to the square for the sum of MN and MΞ.

But as for the ratio pl.ΓN,EΞ to the square on the sum of NE and EΞ, it is equal to the ratio of sq.AΓ  to the square on [the sum of] two sides of the *eidos* corresponding to TY, as is proved in Theorem 17 of this Book. And for that reason the ratio pl.ΓN,MΞ to the square on the sum of MN and MΞ is equal to the ratio of sq.AΓ to the square on [the sum of] two sides of the *eidos* corresponding to KB. Therefore the ratio of sq.AΓ to the square on [the sum of] two sides  of the *eidos* corresponding to TY is smaller than its ratio to the square on [the sum of] two sides of the *eidos* corresponding to KB. Therefore the sum of two sides of the *eidos* corresponding to TY is greater than the sum of two sides of the *eidos* corresponding to KB. And therefore the sum of [four] sides of the *eidos* corresponding to TY is greater than the sum of [four] sides of the *eidos* corresponding to KB.

[Proposition] 40

*If there is a hyperbola, and its transverse axis is smaller than one third of its* latus rectum, *then there are two diameters, [one] on either side of its axis, each of which is equal to one third of the* latus rectum *of the diameter, and the sum of the sides bounding the* eidos *corresponding to each of two diameters is smaller than [the sum of] sides bounding any of the* eidoi *corresponding to the diameters on that side [of the axis], and sum of the sides bounding the* eidoi *constructed on the diameters closer to [that diameter] is smaller than [the sum of] the sides bounding the* eidoi *corresponding to [the diameters] farther from it* [53].

Therefore we make the diagram in Theorem 35 in the same way as it was. Then ΑΞ is smaller than ΑΝ, and therefore ΑΞ is smaller than one the half of ΞΝ. Therefore we make ΜΞ equal to the half of ΞΝ, and drop from Μ the perpendicular ΜΛ to the axis, and join ΓΛ and draw the diameter ΚΒ parallel to ΓΛ. Then the ratio ΜΞ to ΜΝ is equal to the ratio of ΚΒ to the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 6 of this Book.

But ΜΞ is equal to one third of ΜΝ. Therefore ΚΒ is one third of the *latus rectum* of the *eidos* corresponding to it.

Therefore let two diameters ΔΕ and ΤΥ fall anywhere between Α and Β, we draw ΓΡ and ΓΟ [respectively] parallel to them, and drop Ρι and ΟΠ as perpendiculars to the axis. Then ΜΞ is equal to the quarter of the sum ΜΞ and ΜΝ. Therefore the square of the sum of ΜΝ and ΜΞ is greater than the quadruple [rectangular plane] under ΜΞ and the sum of ΜΝ and Ξι. Therefore we subtract the quadruple [rectangular plane] under Μι and the sum of ΜΝ and Ξι from both of two [sides] and there remains the square on the sum of Νι and Ξι is greater than the quadruple [rectangular plane] under Ξι and the sum of ΜΝ and Ξι. Therefore the ratio of the quadruple [rectangular plane] under Ξι and the sum of ΜΝ and Ξι to the quadruple [rectangular plane] under Μι and the sum of ΜΝ and Ξι is greater than its ratio to the square on the sum of Νι and Ξι.

But the ratio the quadruple [rectangular plane] under Μι and the sum of ΜΝ and Ξι to the quadruple [rectangular plane] under Ξι and the sum of ΜΝ and Ξι is equal to the ratio Μι to Ξι. Therefore the ratio Μι to Ξι is greater than the ratio the quadruple [rectangular plane] under Μι and the sum of ΜΝ and Ξι to the square on the sum of Νι and Ξι.

And *componendo* the ratio ΜΞ to Ξι is greater than the ratio of the sum of the square on the sum of ΝΜ and Ξι and the quadruple [rectangular plane} under Μι and the sum of ΜΝ and Ξι to the square on the sum of Νι and Ξι.

But the sum of the square on the sum of Νι and Ξι and the quadruple [rectangular plane] under Mι and the sum of MN and Ξι is equal to the square on the sum of MN and MΞ. Therefore the ratio MΞ to Ξι is greater than the ratio the square on the sum of MN and MΞ to the square on the sum of Nι and Ξι.

But the ratio NΞ to Ξι is equal to pl.ΓN,MΞ to pl.ΓNΞι. Therefore the ratio pl.ΓN,MΞ to pl.ΓN,Ξι is greater than the ratio of the square on the sum of MN and MΞ to the square on the sum of Nι and Ξι.

And *permutando* the ratio pl.ΓN,MΞ to the square on the sum of NM and MΞ is greater than pl.ΓN,Ξι to the square on the sum of Nι and Ξι.

Bur as for the ratio pl.ΓN,MΞ to the square on the sum of NM and MΞ, it is equal to the ratio of sq.AΓ to the square on [the sum of] two sides of the *eidos* corresponding to KB, as is proved in Theorem 17 of this Book, and as for the ratio pl.ΓN,Ξι to the square on the sum of Nι and Ξι, it is equal to the ratio of sq.AΓ to the square on the sum of two sides of the *eidos* corresponding to ΔE, as is also proved in Theorem 17 of this Book. Therefore the ratio of sq.AΓ to the square on the sum of two sides of the *eidos* corresponding to KB is greater than its ratio to the square on the sum of two sides of the *eidos* corresponding to ΔE.

Therefore the sum of the sides bounding the *eidos* corresponding to KB is smaller than the sum of the sides of the *eidos* corresponding to ΔE.

Furthermore the square on the sum of Ξι and Nι is greater than the quadruple [rectangular plane] under ΞΠ and the sum of Nι and ΞΠ. Therefore it will be proved thence, as we proved previously, that the sum of the straight lines bounding the *eidos* corresponding to ΔE is smaller than the sum of the sides bounding the *eidos* corresponding to TY.

Furthermore the quadruple [rectangular plane] under AΞ and the sum of NΞ and ΞA is smaller the square on the sum of NΠ and ΠΞ. Therefore it will be proved thence also as we proved [previously] that the sum of the straight lines bounding the *eidos* corresponding to TY is smaller than the sum of the sides bounding the *eidos* corresponding to AΓ.

Furthermore we draw the diameters ZH and ΦX making them farther from AΓ than is the diameter KB, and draw from Γ two straight lines ΓΨ and ΓϘ parallel to XΦ and HZ [respectively], and drop from Ψ and Ϙ the perpendiculars ΨΩ and ϘΣ to the axis. Then the quadruple [rectangular plane] under MΞ and the sum of ΣN and MΞ is greater than the square on the sum MN and MΞ. Therefore when we make the sum of MΞ and the quadruple [rectangular plane] under ΣM and ΣN common [to both sides], it will be proved from that, as we proved previously, that the sum of the straight lines bounding the *eidos* corresponding to ZH is greater than the sum of the straight lines bounding the *eidos* corresponding to BK.

Furthermore the quadruple [rectangular plane] under ΣΞ and the sum of ΩΣ and ΣΞ is greater than the square on the sum of ΣN and ΣΞ. Therefore it will be proved thence also that the sum of the straight lines bounding the *eidos* corresponding to ΦX is the greater than the sum of the sides bounding the *eidos* corresponding to ZH.

## [Proposition] 41

*In every ellipse the sum of [four] sides bounding the* eidos *corresponding to its major axis is smaller than the sum of the sides bounding any* eidos *corresponding to another of its diameter, and the sum of the sides bounding [one of] the* eidoi *corresponding to those diameters closer to the major axis is smaller than the sum of the sides bounding an* eidos *corresponding to a diameter farther from it, and the sum of the sides bounding the* eidos *corresponding to the minor axis is greater than the sum of the sides bounding the* eidoi *corresponding to other diameters* [54].

[Proof]. Let the major of two axes of the ellipse be ΑΓ, and its minor axis be ΔΕ, and let there be other diameters BK and ZH.

Let ΓΛ and ΓΙ be parallel to these two diameters and let us drop two perpendiculars ΛΜ and ΙΟ to the [major] axis. Let the ratio ΓΝ to ΑΝ be equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it, and likewise we make the ratio ΑΞ to ΞΓ [equal to that ratio].

Then the ratio of sq.ΑΓ to the square of the straight line equal to the sum of the diameter ΑΓ and the *latus rectum* of the *eidos* corresponding to it is equal to the ratio sq.ΝΓ to sq.ΝΞ, and is equal to the ratio pl.ΝΓ,ΑΞ to sq.ΝΞ because pl.ΝΓ,ΑΞ is equal to sq.ΝΓ.

And the ratio sq.ΑΓ to sq.ΕΔ is equal to the ratio ΝΓ to ΓΞ because it was proved in Theorem 15 of Book I that the ratio sq.ΑΓ to sq.ΔΕ is equal to the ratio of ΑΓ to its *latus rectum*, and the ratio ΓΝ to ΓΞ is equal to the ratio pl.ΝΓΞ to sq.ΓΞ, and the ratio of sq.ΔΕ to square on the straight line equal to the sum of ΔΕ and the *latus rectum* of the *eidos* corresponding to it is equal to the ratio sq.ΓΞ to sq.ΝΞ also because of what was proved in Theorem 15 of Book I. Therefore the ratio of sq.ΑΓ to the square on the straight line equal to the sum of the diameter ΔΕ and the *latus rectum* of the *eidos* corresponding to it is equal to the ratio pl.ΝΓΞ to sq.ΝΞ.

And it was shown that the ratio pl.ΝΓ,ΑΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on the straight line equal to the sum of ΑΓ and the *latus rectum* of the *eidos* corresponding to it.

Therefore the ratio of ΑΓ to the sum of ΑΓ and its *latus rectum* is greater than the ratio of ΑΓ to the sum of ΔΕ and its *latus rectum*. Therefore the sum of the sides bounding the *eidos* corresponding to ΑΓ is smaller than the sum of the sides of the *eidos* corresponding to ΔΕ.

And [also] the ratio of pl.ΝΓ,ΜΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on the straight line equal to the sum of the diameter ΚΒ and the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 17 of this Book.

Therefore the ratio of ΑΓ to the sum of ΑΓ and its *latus rectum* is greater than the ratio of ΑΓ to the sum of ΚΒ and its *latus rectum*. Therefore the sum of the sides bounding the *eidos* corresponding to ΑΓ is smaller than the sum of the sides of the *eidos* corresponding to ΚΒ.

Furthermore the ratio pl.ΝΓ,ΜΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on the straight line equal to the sum of the diameter ΚΒ and the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 17 of this Book, and likewise also the ratio pl.ΝΓ,ΟΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on the straight line equal to the sum of the diameter ΖΗ and its *latus rectum*.

Therefore the ratio of ΑΓ to the sum of ΚΒ and its *latus rectum* is greater than the ratio of ΑΓ to the sum of ΖΗ and its *latus rectum*. Therefore the sum of the sides bounding the *eidos* corresponding to ΚΒ is smaller than the sum of the sides of the *eidos* corresponding to ΖΗ.

Furthermore the ratio pl.ΓΝ, ΞΟ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on the straight line equal to the sum of the diameter ΖΗ and the *latus rectum* of the *eidos* corresponding to it, as is proved in Theorem 17 of this Book.

And we have [already] proved that the ratio pl.ΝΓΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ to the square on  the sum of ΔΕ and its *latus rectum*.

Therefore the ratio to the sum of ΖΗ and its *latus rectum* is greater than the ratio of ΑΓ to the sum of ΔΕ and its *latus rectum*.

Therefore the sum of the sides bounding the *eidos* corresponding to ΖΗ is smaller than the sum of the sides of the *eidos* corresponding to ΔΕ.

[Proposition] 42

*The smallest of the* eidoi *corresponding to the diameters of a hyperbola is the* eidos *corresponding to its axis, and those* eidoi *corresponding to the diameters closer to the axis are smaller than those* eidoi *corresponding to the diameters farther from it* [55] .

Let there be the hyperbola whose axis ΑΓ and two of its diameters ΚΒ and ΤΥ.

Then I say that the *eidos* corresponding to ΑΓ is smaller than the *eidoi* corresponding to other diameters of the section, and that the *eidos* corresponding to ΚΒ is smaller than the *eidos* corresponding to ΤΥ.

[Proof]. We draw the straight lines ΓΛ and ΓΔ parallel to the diameters ΚΒ and ΤΥ [respectively], and drop to the axis the perpendiculars ΔΕ and ΛΜ, and make the ratio ΓΝ to ΑΝ equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it. Then the ratio ΓΝ to ΑΝ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΑΓ. And the ratio ΓΝ to ΝΜ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΚΒ, as is proved in Theorem 18 of this Book.

And the ratio ΓΝ to ΑΝ is greater than the ratio ΓΝ to ΜΝ.

Therefore the ratio of sq.ΑΓ to the *eidos* corresponding to ΑΓ is greater than its ratio to the *eidos* corresponding to ΚΒ.

Therefore the *eidos* corresponding to ΑΓ is smaller than the *eidos* corresponding to ΚΒ.

Furthermore the ratio ΓΝ to ΝΕ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΤΥ, as is proved in Theorem 18 of this Book.

And likewise also the ratio ΓΝ to ΜΝ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΚΒ.

And the ratio ΓΝ to ΝΜ is greater than the ratio ΓΝ to ΕΝ. Therefore the ratio of sq.ΑΓ to the *eidos* corresponding to ΚΒ is greater than its ratio to the *eidos* corresponding to ΤΥ.

## [Proposition] 43

*The smallest of the* eidoi *constructed to the diameters on an ellipse is the* eidos *corresponding to the major axis, and the greatest of them is the* eidos *corresponding to the minor axis, and those* eidoi *corresponding to the diameters closer to the major axis are smaller than those corresponding to the diameters farther from it* [56] .

Let there be the ellipse whose major axis ΑΓ and minor axis ΔΕ, and with two other of its diameters ΚΒ and ΤΥ.

Then, I say that the *eidos* corresponding to ΑΓ is smaller than the *eidos* corresponding to ΚΒ, and that the *eidos* corresponding to ΚΒ is smaller than the *eidos* corresponding to ΤΥ, and that the *eidos* corresponding to ΤΥ is smaller than the *eidos* corresponding to ΔΕ.

[Proof]. We draw ΓΛ and ΓΙ parallel to the diameters ΚΒ and ΤΥ [respectively], and drop as perpendicular to the axis ΛΜ and ΙΟ. We make the ratio ΓΝ to ΝΑ equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it. Then the ratio of sq.ΑΓ to the *eidos* corresponding to ΑΓ is equal to the ratio ΝΓ to ΝΑ.

But sq.ΑΓ to equal to the *eidos* corresponding to ΔΕ, as is proved in Theorem 15 of Book I. Therefore the *eidos* corresponding to ΑΓ is smaller than the *eidos* corresponding to ΔΕ.

Now the ratio ΓΝ to ΜΝ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΚΒ. As is proved in Theorem 18 of this Book. And likewise the ratio ΓΝ to ΝΟ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΤΥ.

And the ratio ΓΝ to ΧΝ is equal to the ratio of sq.ΑΓ to the *eidos* corresponding to ΔΕ. But ΑΝ is smaller than ΝΜ, and ΝΜ is smaller than ΝΟ, and ΝΟ is smaller than ΝΓ. Therefore the *eidos* corresponding to ΑΓ is smaller than the *eidos* corresponding to ΚΒ, and the *eidos* constructed on ΚΒ is smaller than the *eidos* corresponding to ΤΥ, and the *eidos* corresponding to ΤΥ is smaller than the *eidos* corresponding to ΔΕ.

[Proposition] 44

*If there is a hyperbola, and the transverse side of the* eidos *corresponding to its axis is either [1] not smaller than its* latus rectum, *or [2] smaller than it, but [such that] its square is not smaller than the half of the square of the difference between it [the transverse side] and it [the* latus rectum], *then the sum of the squares of two sides of the eidos corresponding to the axis is smaller than [the sum of] the squares of two sides of any* eidos *corresponding to one of its other diameter* [57].

Let ther be the hyperbola whose axis is ΑΓ, and with two of its diameters ΚΒ and ΤΥ. Let ΑΓ be either not smaller than the *latus rectum* of the *eidos* corresponding to it, or let ΑΓ be smaller than it, but let sq.ΑΓ be not smaller than the half of the square of the difference between it [ΑΓ] and it [its *latus rectum*].

Then I say that the sum of the squares of two sides of the *eidos* corresponding to ΑΓ is smaller than [the sum of] the squares of two sides of the *eidos* corresponding to ΚΒ, and that [the sum of] the squares of two sides of the *eidos* corresponding to ΚΒ is smaller that [the sum of] the squares of two sides of the *eidos* corresponding to ΤΥ.

[Proof]. First we make ΑΓ not smaller than the *latus rectum* of the *eidos* corresponding to it. Then the *latus rectum* of the *eidos* corresponding to ΚΒ is greater than the *latus rectum* of the *eidos* corresponding to ΑΓ, as is proved in Theorem 33 of this Book. And likewise the *latus rectum* of the *eidos* corresponding to ΤΥ is greater than the *latus rectum* of the *eidos* corresponding to ΚΒ. And ΑΓ is smaller than ΚΒ, and ΚΒ is smaller than ΤΥ. Therefore [the sum of] the squares on two sides of the *eidos* corresponding ΑΓ is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΚΒ, and [the sum of] the squares on two sides of the *eidos* corresponding to ΚΒ is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΤΥ.

## [Proposition] 45

Furthermore we make ΑΓ smaller than the *latus rectum* of the *eidos* corresponding to it, but [such that] its square is not smaller than the half of the square on the difference between it [ΑΓ] and it [its *latus rectum*] and set the diagram as it was in the preceding theorem, and let each of two ratios ΓΝ to ΑΝ and ΑΞ to ΓΞ be equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it, then the double sq.ΑΞ is not smaller than sq.ΝΞ because ΑΞ is equal to ΓΝ, and the ratio of ΑΓ to its *latus rectum* is equal to the ratio ΑΞ to ΞΓ, and sq.ΑΓ is not smaller than the half of the square on the difference between its *latus rectum*. We draw two diameters ΚΒ and ΤΥ, and draw ΓΔ and ΓΛ parallel to them, and drop to the axis the perpendiculars ΔΕ and ΛΜ [58].

Then the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it is equal to the ratio ΓΝ to ΑΝ and is equal to the ratio ΑΞ to ΞΓ. And the double sq.ΑΞ is not smaller than sq.ΞΝ, and [hence] the double pl.ΜΞΑ is greater than sq.ΞΝ . Therefore we make the double pl.ΝΑΞ common [to both sides]. Therefore the double pl.ΑΞ sum of the double pl ΝΑΞ and sq.ΝΞ is greater than the sum of the double pl.ΝΑΞ and sq.ΝΞ. Therefore the double pl.ΑΞ and the sum of ΝΜ and ΑΞ is greater than the sum of sq.ΝΑ and sq.ΑΞ is greater than the sum of sq ΝΑ and sq ΑΞ Therefore the double [rectangular plane] under ΑΞ and the sum ΝΜ and ΑΞ is greater than the sum of the double pl.ΝΑΞ and sq.ΝΞ. Therefore the double [rectangular plane] under ΑΞ and the sum of ΝΜ and ΑΞ is greater than the sum of sq.ΝΑ and sq.ΑΞ.

Therefore the ratio the double [rectangular plane] under AM and the sum of NM and AΞ to the double [rectangular plane] under AΞ and the sum of NM and AΞ is smaller than the ratio the double [rectangular plane] under AM and the sum of NM and AΞ to the sum of sq.AN and sq.AΞ. But the ratio the double [rectangular plane] under AM and the sum NM and AΞ to the double [rectangular plane] under AΞ and the sum of MN and AΞ is equal to the ratio AM to AΞ. Therefore the ratio AM to AΞ is smaller than the ratio the double [rectangular plane] under AΞ and the sum of NM and AΞ to the sum of sq.AN and sq.AΞ. [And *componendo* the ratio MΞ to ΞA is smaller than the ratio the sum of the double [the rectangular plane] under AM and the sum of (NM and AΞ) and sq.NA and sq.AΞ to the sum of sq.NA and sq.AΞ][59]

And the sum of sq.NM and sq.MΞ is smaller than the sum of sq.NA, sq.AΞ, and the double [rectangular plane] under AM and the sum of NM and AΞ. Therefore the ratio MΞ to AΞ is smaller than the ratio the sum of sq.NM and sq.MΞ to sum of sq.AN and sq.AΞ.

But the ratio MΞ to AΞ is equal to the ratio pl.ΓN,MΞ to pl.ΓN,AΞ. Therefore the ratio pl.ΓN,MΞ to pl.ΓN,AΞ is smaller than the ratio the sum of sq.NM and sq.MΞ to the sum of sq.AN and sq.AΞ. And *permutando* the ratio pl.ΓN,MΞ to the sum of sq.MN and sq.MΞ is smaller than pl.ΓN,AΞ to the sum of sq.AN and sq.AΞ.

But the ratio pl.ΓN,MΞ to thee sum of sq.NM and sq.MΞ is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to KB, as is proved in Theorem 19 of this Book. And the ratio pl.ΓN,AΞ to the sum of sq.AN and sq.AΞ is equal to the ratio of sq.AΓ to the [sum of the] squares on two sides of the *eidos* corresponding to AΓ, as is proved from the preceding topic in this theorem. Therefore the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* constructed on KB is smaller than its ratio to [the sum of] the squares on two sides of the *eidos* corresponding to AΓ. Therefore [the sum of] the squares on two sides of the *eidos* corresponding to KB is greater than [thee sum of] the squares on two sides of the *eidos* corresponding to AΓ.

Furthermore the double sq.MΞ is greater than sq.NΞ, and [hence] the double pl.EΞM is greater than sq.NΞ. Therefore it will be proved, as we proved in the preceding, that [the sum of] the squares on two sides of the *eidos* corresponding to TY is greater than [the sum of] the squares on two sides of the *eidos* corresponding to KB.

[Proposition] 46

But if the square on the transverse diameter [ΑΓ] is less than the half of the square on the difference between it and the *latus rectum* of the *eidos* corresponding to it, then on either side of the axis are two diameters, the square on each of which is equal to the half of the square on the difference between it and the *latus rectum* of the *eidos* corresponding to it, and the sum of the squares of two sides of the *eidos* corresponding to it is smaller than [the sum of] the squares of two sides of any *eidos* corresponding to [one of] the diameters drawn on the side [of the axis] on which it lies, and [the sum of] the squares of two sides of those *eidoi* corresponding to the diameters on its side [of the axis] closer to it is smaller than [the sum of] the squares of two sides [of *eidoi*] corresponding to those diameters farter from it [60].

Let the axis of the section be ΑΓ, and let sq.ΑΓ be smaller than the half of the square on the difference between it and the *latus rectum* of the *eidos* corresponding to it. Let each of the ratios ΓΝ to ΑΝ and ΑΞ to ΞΓ be equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it. Then the double sq.ΑΞ is smaller than sq.ΝΞ. We make the double sq.ΜΞ equal to sq.ΝΞ, and drop from Μ the perpendicular ΜΛ to the axis, and join ΛΓ and draw the diameter ΚΒ parallel to ΓΛ. Then the ratio ΜΞ to ΜΝ is equal to the ratio of ΚΒ to the *latus rectum* of the *eidos* constructed on it, as is proved in Theorem 6 of this Book. And hence sq.ΚΒ is equal to the half of the square on the difference between it the *latus rectum* of the *eidos* corresponding to it.

So we draw between Α and Β two diameters ΔΕ and ΤΥ, and draw ΓΡ and ΓΟ parallel to them [respectively], drop the perpendiculars Ρι and ΟΠ to the axis.

Now the double sq.ΜΞ is equal to sq. ΞΝ. Therefore the double pl. ΜΞι is smaller than sq.ΝΞ. We make the double pl.ΝιΞ common [to both sides]. Then the double [rectangular plane] under ιΞ and the sum of ΜΝ and ιΞ is smaller than the sum of sq.Νι and sq.ιΞ.

Thence it will be proved, as we proved in the preceding theorem that [the sum of] the squares on two sides of the *eidos* corresponding to ΚΒ is less than [the sum of] the squares on two sides of the *eidos* corresponding to ΔΕ.

Furthermore the double pl.ιΞΠ is smaller than sq.ΞΝ. Therefore we make the double pl.ΝΠΞ common [to both sides]. Then the double [rectangular plane] under ΞΠ and the sum of ιΝ and ΞΠ than the sum of sq.ΝΠ and sq.ΠΞ, and it will be proved thence also, as it was proved in the preceding theorem that [the sum of] the squares on two sides of the *eidos* constructed on ΔΕ is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΤΥ.

Furthermore the double pl.ΠΞι is smaller than sq.ΝΞ, and it will proved thence also, as we proved previously, that [the sum of] the squares on two sides of the *eidos* corresponding to ΤΥ is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΑΓ.

Furthermore we draw two diameters ΖΗ and ΦΧ, and let them be farther from the axis than is the diameter ΚΒ, and we draw ΓΨ and ΓΙ parallel to them, and drop to the axis to the perpendiculars ΨϘ and ΣΙ, then the double pl.ΣΞΜ is greater than sq.ΝΞ, therefore it will be proved thence also, as we proved previously, that [the sum of] the squares on two sides of the *eidos* corresponding to ΖΗ is greater than [the sum of] the squares on two sides of the *eidos* corresponding to ΚΒ.

Furthermore the double pl. ϘΞΣ is greater than sq.ΝΞ, therefore it will be proved thence, as we proved previously, that [the sum of] the squares on two sides of the *eidos* corresponding to ΦΧ is greater than [the sum of] the squares on two sides of the *eidos* corresponding to ΖΗ.

[Proposition] 47

*If there is an ellipse, and the square on the transverse side of the* eidos *corresponding to its major axis is not greater than the half of the square on the sum of two sides of the* eidos *corresponding to it, then [the sum of] the squares on two sides of the* eidos *corresponding to the major axis is smaller than [the sum of] the squares on two sides of [all] other* eidoi *corresponding to its diameters, and [the sum of] the squares and two sides of those* eidoi *corresponding to diameters closer to it is smaller than [the sum of] the squares on two sides of those* eidoi *corresponding to the diameters farther from it, and the greatest of them is [the sum of] the squares on two sides of the* eidos *corresponding to the minor axis* [61].

Let there be the ellipse whose major axis ΑΓ and minor axis ΔΕ. Let sq.ΑΓ not be greater than the half of the square on [the sum of] two sides of the *eidos* corresponding to it, and let there be in the section two other diameters KB and ΤΥ. We draw ΓΛ and ΓΙ parallel to them [respectively], and drop to the axis the perpendiculars ΛΜ and ΙΟ, and make each of the ratios ΓΝ to ΑΝ and ΑΞ to ΞΓ equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it. Then the ratio pl.ΝΓ,ΑΞ to the sum of sq.ΝΓ and sq.ΓΞ is equal to the ratio of sq.ΑΓ to [the sum of] the squares on two sides of the *eidos* corresponding to ΑΓ. And the ratio of the *latus rectum* of the *eidos* corresponding to ΔΕ to ΔΕ is equal to the ratio ΝΓ to ΓΞ because the ratio ΝΓ to ΓΞ is equal to the ratio of

ΑΓ to its *latus rectum*, and the ratio of ΑΓ to its *latus rectum* is equal to the ratio of the *latus rectum* of the diameter ΔΕ to ΔΕ because of what is proved in Theorem 15 of Book I.

Similarly too the ratio of the *latus rectum* of the *eidos* corresponding to ΔΕ to ΔΕ is equal to the ratio of the square on the *latus rectum* of the *eidos* corresponding to ΔΕ to sq.ΑΓ. And the ratio ΝΓ to ΓΞ is equal to the ratio pl.ΝΓΞ to sq.ΓΞ. Therefore the ratio of the *latus rectum* of the *eidos* corresponding to ΔΕ to ΔΕ is equal to the ratio pl.ΝΓΞ to sq.ΓΞ, and is equal to the ratio of the square on the *latus rectum* of the *eidos* corresponding to ΔΕ to sq.ΑΓ. [And the ratio of the square on the *latus rectum* of the *eidos* corresponding to ΔΕ to sq.ΑΓ is equal to the ratio sq.ΑΓ to sq.ΔΕ].

And the ratio of sq.ΔΕ to [the sum of] the squares on two sides of the *eidos* corresponding to ΔΕ is equal to the ratio sq.ΓΞ to the sum of sq.ΝΓ and sq.ΓΞ. Therefore the ratio pl.ΝΓΞ to the sum of sq.ΝΓ and sq.ΓΞ is equal to the ratio of sq.ΑΓ to [the sum of] the squares on two sides of the *eidos* corresponding to ΔΕ.

And the ratio pl.ΝΓ,ΑΞ to sq.ΝΞ is equal to the ratio of sq.ΑΓ [to the sum of] the squares on two sides of the *eidos* corresponding to it.

[Therefore the ratio sq.ΑΓ to the sum of the squares on two sides of the *eidos* corresponding to ΑΓ is greater than the ratio sq.ΑΓ to the sum of the squares on two sides of the *eidos* corresponding to ΔΕ. Therefore the sum of the squares on two sides of the *eidos* corresponding to ΑΓ is smaller than the sum of the squares on two sides of the *eidos* corresponding to ΔΕ][62].

Now sq.ΑΓ is not greater than the half of the square on [the sum of] two sides of the *eidos* corresponding to ΑΓ. Therefore the double pl.ΝΓ,ΑΞ is not greater than sq.ΝΞ , and [hence] the double pl.ΝΓ,ΜΞ is smaller than sq.ΝΞ. Therefore we subtract the double pl.ΝΜΞ from both [sides] alike, and there remains the double pl.ΓΜΞ is smaller than the sum of sq.ΝΜ and sq.ΜΞ. Therefore the ratio the double pl.ΑΜΓ to the double pl.ΞΜΓ is greater than the ratio double pl.ΑΜΓ to the sum of sq.ΝΜ and sq.ΜΞ. Therefore the ratio ΑΜ to ΜΞ is greater than the ratio the double pl.ΑΜΓ to the sum of sq.ΜΝ and sq.ΜΞ.

But the sum of the double pl.ΑΜΓ, sq.ΝΜ, and sq.ΜΞ is equal to the sum of sq.ΝΓ and sq.ΓΞ because ΑΝ is equal to ΓΞ. Therefore *componendo* the ratio ΑΞ to ΜΞ is greater than the ratio of the sum of sq.ΝΓ and sq.ΓΞ to the sum of sq.ΝΜ and sq.ΜΞ. But the ratio ΑΞ to ΜΞ is equal to the ratio pl.ΝΓ,ΑΞ to pl.ΝΓ,ΜΞ. Therefore the ratio pl.ΝΓ,ΑΞ to pl.ΝΓ,ΜΞ is greater than the ratio the sum of sq.ΝΓ and sq.ΓΞ to the sum of sq.ΝΜ and sq.ΜΞ.

And *permutando* the ratio pl.NΓ,AΞ to the sum of sq.NΓ and sq.ΓΞ is greater than the ratio pl.NΓ,MΞ to the sum of sq.NM and sq.MΞ.

But as for the ratio pl.NΓ,AΞ to the sum of sq.NΓ and sq.ΓΞ, we have proved that it is equal to the ratio of sq.AΓ to [the sum of] the square on two sides of the *eidos* corresponding to it, and as for the ratio pl.NG,MX to the sum of sq.NM and sq.MΞ it is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to KB, as is proved in Theorem 19 of this Book. Therefore the ratio  of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to it is greater than its ratio to [the sum of] the squares on two sides of the *eidos* corresponding to KB. Therefore [the sum of] the squares on two sides of the *eidos* corresponding to AΓ is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to KB.

Furthermore MN is either smaller than OΞ or it is not  smaller than it.

Therefore first let it be smaller than it. Then the sum of sq.NM and sq.MΞ is greater than the sum sq.NO and sq.OΞ. But the sum of sq.OΞ is greater than the double [rectangular plane] under OΞ and the difference between OΞ and MN. Therefore the ratio  the double [rectangular plane] under MO and the difference between OΞ and MN to the double [rectangular plane] under OΞ and the difference between OΞ and MN is greater than the ratio the double [rectangular plane] under MO and the difference between OΞ and MN to the sum of sq.OΞ and sq.ON. Therefore the ratio MO to OΞ is greater than the ratio the double [rectangular plane] under MO and the difference between OΞ and MN to the sum of sq.ON and sq.OΞ. But the sum of the double [rectangular plane] under MO and the difference between OΞ and MN, sq.ON, and sq.OΞ is equal to sq.MN and sq.MΞ because the difference between (the sum of sq.MΞ and sq.MN) and (the sum sq.NO and sq.OΞ) is equal to the difference between the double sq.MΘ and sq.ΘO. Therefore *componendo* the ratio MΞ to ΞO is greater than the ratio the sum of sq.MN and sq.MΞ to the sum of sq.ON and sq.OΞ.  But the ratio MΞ to ΞO is equal to the ratio pl.NΓ,MΞ to pl.NΓ,ΞO.  Therefore the ratio pl.NΓ,MΞ to pl.NΓ,OΞ is greater than the ratio the sum of sq.MN and sq.MΞ to the sum of sq.ON and sq.OΞ.

And *permutando* the ratio pl.NΓ,MΞ to the sum MN and sq.MΞ is greater than pl.NΓ,ΞO to the sum of sq.ON and sq.OΞ.

But as for the ratio pl.NΓ,MΞ to the sum of sq.MN and sq.MΞ, it is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to KB, as is proved in Theorem 19 of this Book, and as for the ratio pl.NΓ,ΞO to the sum of sq.ON and sq.ΞO, it is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to TY.

Furthermore we make MN not smaller than ΞO, then the sum sq.MN and sq.MΞ is notgreater the sum of sq.NO and sq.OΞ. Therefore the ratio pl.NΓ.MΞ to the sum of sq.NM and sq.MΞ is greater than the ratio pl.NΓ,ΞO to the sum of sq.NO and sq.OΞ. Therefore it will be proved thence also, as we proved in the preceding part of this theorem, that [the sum of] the squares on two sides of the *eidos* corresponding to KB is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to TY.

Similarly too what we stated will be proved if the perpendicular drawn from I falls between M and Θ or on Θ itself for in every case NM turns out to be smaller than the distance which the perpendicular [IO] cuts off from it [the major axis towards N and A].

Now the ratio pl.NΓΞ to the sum of sq.NΓ and sq.ΓΞ is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to ΔE, as we proved in the first part of this theorem, and the ratio pl.NΓ,OΞ to the sum of sq.NO and sq.OΞ is equal to the ratio of sq.AΓ to [the sum of] the squares on two sides of the *eidos* corresponding to TY, as is proved in Theorem 19 of this Book. Therefore it will be proved thence, as we proved above, that [the sum of] the squares on two sides of the *eidos* corresponding to TY is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΔE.

## [Proposition] 48

*If there is an ellipse, and the square on its major axis is greater than the half of the square on the sum of two sides of the* eidos *corresponding to it, then there are two diameters [one] on either side of the axis, such that the square on each of them is equal to the half of the square on the sum of two sides of the eidos corresponding to it, and [the sum of] the square on two sides of the* eidos *corresponding to it is smaller thin [the sum of] the squares on two sides of [any of] other* eidoi *corresponding to diameters drawn in that quadrant in which [that diameter] is, and [the sum of] the squares on two sides of* eidoi *corresponding to those diameters in that quadrant closer to it is smaller than [the sum of] the squares on two sides of* eidoi *corresponding to those diameters farther [from it]* [63].

Let the diagram be as we drew it in the theorem preceding this one.

Then it will be proved, as it was proved there, that the double sq.AΞ is greater than sq.NΞ. We make the double sq.MΞ equal to sq.NΞ, and drop from

M the perpendicular ΜΛ to the axis to meet the section, and join ΓΛ, and draw in the section the diameter KB parallel to ΓΛ.

Then the ratio MΞ to ΞN is equal to the ratio of KB to [the sum of] two sides of the *eidos* corresponding to it, as is drawn from the proof of Theorem 7 of this Book. And therefore the ratio sq.MΞ to sq.ΞN is equal to the ratio of sq.KB to the square on the sum of two sides of the *eidos* corresponding to it. But sq.MΞ is equal to the half of sq.ΞN .Therefore sq.KB is equal to the half of the square on [the sum of] two sides of the sides of the *eidos* corresponding to it.

Therefore we draw two diameters ΔE and TY between A and B, and draw from Γ two straight lines ΓO and Γς [respectively] parallel to them, and drop to the axis the perpendiculars Oι and ςΠ .

Now sq.MΞ is equal to the half sq.ΞN, and pl.NΞΘ also is equal to the half of sq.NΞ. Therefore pl.NΞΘ is equal to sq.MΞ. Therefore pl.NΞM is equal to pl.MΞΘ. And when we subtract two smaller [members] from two greater [members] , we get the ratio of the remainder NM to the remainder MΘ equal to the ratio of the whole NΞ to the whole MΞ. Therefore pl.NΞ,MΘ is equal to pl.NMΞ. Therefore pl.NΞ,MΘ is greater than pl.Nι,MΞ ,and the double pl.NΞ,MΘ is greater than the double pl.Nι,MΞ. Therefore the quadruple pl.MΘΞ is greater than the double pl.Nι,MΞ .

We make the double pl.ιMΞ common [to both sides], then the sum of the quadruple pl.ΞΘM and the double pl.ιMΞ is greater than the double pl.IMΞ.

Furthermore we make the quadruple sq.MΘ common [to both sides], then the sum of the quadruple pl.ΞΘM , the double pl.ιMΞ, and the quadruple sq.MΘ is greater than the sum of the double pl.NMΞ and the quadruple sq.MΘ.

But the sum of the quadruple ΞΘM, the double pl.ιMΞ, and the quadruple sq.MΘ is equal to the double [rectangular plane] under MΞ and the sum of Θι and ΘM, and the sum of the double pl.NMΞ and the quadruple sq.MΘ is equal to the sum of sq.MN and sq.MΞ. Therefore the double [rectangular plane] under MΞ and the sum of Θι and ΘM is greater than the sum of sq.NM and sq.MΞ. And therefore the ratio the double [rectangular plane] under Mi and the sum of Θι and ΘM to the double [rectangular plane] under MΞ and the sum of Θι and ΘM is smaller than the double [rectangular plane] under Mι and the sum of Θι and ΘM to the sum of sq.NM and sq.MΞ. Therefore the ratio Mι to MΞ is smaller than the double [rectangular plane] under Mι and the sum of Θι and ΘM to the sum of sq.NM and sq.MΞ.

But the sum of sq.Nι and sq.Ξι is greater than the sum of sq.NM and sq.MΞ by an amount equal to the double the [rectangular plane] under Mι and the sum of Θι and ΘM.

Therefore *componendo* the ratio ιΞ to MΞ is smaller than the ratio the sum of sq.Nι and sq.Ξι to the sum of sq.NM and sq.MΞ. Then it will be proved thence, as it was proved in the preceding theorem, that [the sum of] the squares on two sides of the *eidos* corresponding to BK is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΔE.

Furthermore the double pl.NΞ,ιΘ is greater than the double pl.NΠ,ιΞ, therefore it will be proved thence, as we proved in the preceding part of this theorem, that the sum of the squares on two sides of the *eidos* corresponding to ΔE is smaller than the sum of the squares on two sides of the *eidos* corresponding to TY.

Furthermore the double pl.NΞ,ΠΘ is greater than the double pl.NAΞ, therefore it will be proved thence that the ratio AΞ to ΞΠ is smaller than the ratio the sum of sq.NA and sq.AΞ to the sum of sq.NΠ and sq.ΠΞ.

But the ratio AH to ΞΠ is equal to the ratio pl.NΓ,AΞ to pl.NΓ,ΞΠ. Therefore the ratio pl.NΓ,AΞ to pl.NΓ,ΞΠ is smaller than the ratio the sum of sq.NA and sq.AΞ to the sum of sq.NΠ and sq.ΠΞ. Therefore it will be proved thence, as we proved previously, that [the sum of] the squares on two sides of the *eidos* corresponding to TY is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to AΓ.

Furthermore we draw in the section in those two quadrants [in which the diameters are already drawn] two other diameters ZH and ΦX farther from the major axis than is the diameter KB, and draw from Γ two straight lines ΓΨ and ΓP parallel to them, and drop to the axis two perpendiculars ΨΩ and PΣ, it will be proved by means of a procedure like the preceding, that [the sum of] the squares on two sides of the *eidos* corresponding to KB is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ZH, and that [the sum of] these [latter] two squares is smaller than [the sum of] the squares on two sides of the *eidos* corresponding to ΦX, whether Σ and Ω are both between M and Θ, or whether one of them is on the center Θ and the other between M and Θ or between Θ and Γ.

Hence [the sum of] the squares on two sides of the *eidos* corresponding to KB equal in square to the half of the square on [the sum of] two sides of the *eidos* corresponding to it is smaller than [the sum of] the squares on two sides of any of the *eidoi* corresponding to other diameters drawn in the two quadrants ΑϘ and Γ□, and [the sum of] the squares on two sides of those *eidoi* corresponding to the diameters drawn in two quadrants ΑϘ and Γ□ closer to it [ΚΒ] is smaller than [the sum of] the squares on two sides of those *eidoi* corresponding to the diameters farther [from it].

Therefore [the sum of] the squares on two sides of the *eidos* corresponding to Ϙ□ turns out to be greater than the sum of the squares on two sides of the *eidoi* corresponding to any of the remaining diameters.

## [Proposition] 49

*If there is a hyperbola, and the transverse side of the* eidos *corresponding to its axis is greater than its* latus rectum*, then the difference between the squares on two sides of that eidos is smaller than the difference between the squares on two sides of any of the* eidoi *corresponding to other diameters, and the difference between the squares on two sides of those* eidoi *corresponding to diameters closer [to the axis] is smaller than the difference between the squares on two sides of those* eidoi *corresponding to diameters farther from it, and the difference between the squares on two sides of any of the* eidoi *corresponding to diameters which are not axes is greater than the difference between the square on the axis and the* eidos[64] *corresponding to it, but smaller than double that difference.*

Let there be the hyperbola whose axis ΑΓ and center Θ, and let ΑΓ be greater than the *latus rectum* of the *eidos* corresponding to it.

And let each of the ratios ΓΝ to ΝΑ and ΑΞ to ΓΞ be equal to the ratio of ΑΓ to the *latus rectum* of the *eidos* corresponding to it. We draw two diameters ΚΒ and ΤΥ.

Then I say that the difference between sq.ΑΓ and the square on its *latus rectum* is smaller than the difference between sq.ΚΒ and the square on the *latus rectum* of the *eidos* corresponding to ΚΒ, and that the difference between sq.ΚΒ and the square on its *latus rectum* is smaller than the difference between sq.ΤΥ and the square on its *latus rectum*.

[Proof]. We draw ΓΛ and ΓΔ parallel to the diameters ΚΒ and ΤΥ [respectively], and drop to the axis the perpendiculars ΔΕ and ΛΜ. Then the ratio of ΑΓ to its *latus rectum* is equal to the ratio ΒΝ to ΑΝ and also is equal to the ratio ΑΞ to ΞΓ. Therefore the ratio pl.ΝΓ,ΑΞ to the difference between sq.ΑΞ and sq.ΑΝ is equal to the ratio of sq.ΑΓ to the difference between it [sq.ΑΓ] and the square on its *latus rectum*.

Now the ratio MΞ to AΞ is smaller than the ratio MN to NA. Therefore the ratio MΞ to AΞ is smaller than the ratio the sum of MΞ and MN to the sum of AΞ and AN which is smaller than the ratio [the rectangular plane] under ΞN and the sum of ΞM and MN to [the rectangular plane] under ΞN and the sum of AΞ and AN. But the ratio MΞ to AΞ is equal to the ratio pl.ΓN,MΞ to pl.ΓN,AΞ.

Therefore the ratio pl.ΓN,MΞ to pl.ΓN,AΞ is smaller than [the rectangular plane] under ΞN and the sum of MΞ and MN to [the rectangular plane] under ΞN and the sum of AΞ and AN.

Now as for [the rectangular plane] under ΞN and the sum MΞ and MN, it is equal to the difference between sq.MΞ and sq.MN, and as for [the rectangular plane] under ΞN and the sum AΞ and AN, it is equal to the difference between sq.AΞ and sq.AN. Therefore the ratio pl.ΓN,MΞ to pl.ΓN,AΞ is smaller than the ratio the difference between sq.MΞ and sq.MN to the difference between sq.AΞ and sq.AN

And *permutando* the ratio pl.ΓN,MΞ to the difference between sq.MΞ and sq.MN is smaller than pl.ΓN,AΞ to the difference between sq.AΞ and sq.AN. But as for the ratio pl.ΓN,MΞ to the difference between sq.MΞ and sq.MN, it is equal to the ratio of sq.AΓ to the difference between the squares on two sides of the *eidos* corresponding to KB, as is proved in Theorem 20 of this Book, and as for the ratio pl.ΓN,AΞ to the difference between sq.AΞ and sq.AN, we have shown that it is equal to the ratio of sq.AΓ to the difference between the square on it [AΓ] and the square on the *latus rectum* of the *eidos* corresponding to it. Therefore the ratio of sq.AΓ to the difference between the squares on two sides of the *eidos* corresponding to KB is smaller than its ratio to the difference between the squares on two sides of the *eidos* corresponding to AΓ. Therefore, the difference between the squares on two sides of the *eidos* corresponding to KB is greater than the difference between the squares on two sides of the *eidos* corresponding to AΓ.

Furthermore, the ratio EΞ to MΞ is smaller than EN to MN; therefore the ratio EΞ to MΞ is smaller than the ratio of the sum of EΞ and EN to the sum of MΞ and MN. Therefore it will be proved thence, as we proved above, that the difference between the squares on two sides of the *eidos* corresponding to TY is greater than the difference between the squares on two sides of the *eidos* corresponding to KB.

Furthermore we make the straight line BO equal to the *latus rectum* of the *eidos* corresponding to KB, then the difference between sq.KB and sq.BO is equal to the sum of the double pl.BOK and sq.OK. Therefore the difference between sq.KB and sq.BO is greater than pl.BKO and is smaller than the double pl.BKO. But pl.BKO is equal to the difference between sq.BK and the *eidos* corresponding to it, and the difference between sq.BK and the *eidos* corresponding to it is equal to the difference between sq.ΑΓ and the *eidos* corresponding ΑΓ, as is proved in Theorem 29 of this Book.

Therefore de difference between sq.BK and the square on the *latus rectum* of the *eidos* corresponding to it is greater than the difference between sq.ΑΓ and the *eidos* corresponding to it, but is smaller than the double that difference.

[Proposition] 50

*If there is a hyperbola, and the transverse side of the* eidos *corresponding to its axis is smaller than its* latus rectum, *then the difference between the squares on two sides of the* eidos *corresponding to the axis is greater than the difference between the squares on two sides of any of the* eidoi *corresponding to the diameters other than it, and the difference between the squares on two sides of those* eidoi *corresponding to the diameters closer to the axis is greater than the difference between the squares on two sides of those* eidoi *corresponding to the diameters farther from it, and the difference between the square on any of those diameters and the square on the* latus rectum *of the* eidos *corresponding to it is greater than the double difference between the square on the axis and the* eidos *corresponding to the axis* [65].

Let the axis of the hyperbola be ΑΓ, and let each of the ratios ΓΝ to ΑΝ and ΑΞ to ΞΓ be equal to the ratio of ΑΓ to its *latus rectum*, and we make the rest of the diagram which preceded in the theorem before this remain the same.

Then the ratio pl.ΓΝ,ΑΞ to the difference between sq.ΑΝ and sq.ΑΞ is equal to the ratio of sq.ΑΓ to the difference between sq.ΑΓ and the square on the *latus rectum* of the *eidos* corresponding to it. And the ratio ΜΞ to ΑΞ is greater than the ratio ΜΝ to ΑΝ. Therefore the ratio ΜΞ to ΑΞ is greater than the ratio of the sum ΜΞ and ΜΝ to the sum of ΑΞ and ΑΝ. Therefore the ratio pl.ΓΝ,ΜΞ to pl.ΓΝ,ΑΞ is greater than the ratio of the sum ΜΞ and ΜΝ to the sum of ΑΞ and ΑΝ.

But the ratio of the sum of ΜΞ and ΜΝ to the sum of ΑΞ and ΑΝ is equal to the ratio pl.ΞΝ, the sum of ΜΞ and ΜΝ to pl.ΞΝ, the sum of ΑΞ and ΑΝ. Therefore the ratio pl.ΓΝ,ΜΞ to pl.ΓΝ,ΑΞ is greater than the ratio pl.ΞΝ, the sum of ΜΞ and ΜΝ to pl.ΞΝ, the sum of ΑΞ and ΑΝ.

Therefore it will proved thence by [a method] similar to that which we used above that the difference between sq.KB and the square on the *latus rectum* of the *eidos* corresponding to it is smaller than the difference between sq.ΑΓ and the square on the *latus rectum* of the *eidos* corresponding to it.

Then we make BO equal to the *latus rectum* of the *eidos* corresponding to KB. Therefore pl.BKO is equal to the difference between sq.ΑΓ and the *eidos* corresponding to ΑΓ because of what was proved in Theorem 29 of this Book.

And the difference between sq.BO and sq.KB equal to the sum of the double pl.BKO and sq.KO, which is greater than the double pl.OKB.

Therefore the difference between the squares on two sides of the *eidos* corresponding to KB is greater than the double difference between sq.ΑΓ and the *eidos* corresponding to ΑΓ.

## [Proposition] 51

*The difference between the squares on two sides of the* eidos *corresponding to the major axis of an ellipse is greater than the difference between the squares on two sides of any* eidos *corresponding to other diameters which are greater than the* latus rectum *of the* eidoi *corresponding to them, and the difference between the squares on two sides of those* eidoi *constructed to those of these diameters closer to the major axis is greater than the difference between the squares on two sides of those* eidoi *corresponding to those diameters farther from it, and the difference between the squares on two sides of the* eidos *corresponding to its minor axis is greater than the difference between the squares on two sides of any* eidos *corresponding to other diameters which are smaller than the latera recta of the* eidoi *corresponding to them, and the difference between the squares on two sides of those* eidoi *corresponding to those of these diameters closer to the minor axis is greater than the difference between the squares on two sides on those* eidoi *corresponding to the diameters farther from it.*

Let there be the ellipse whose major axis ΑΓ and minor axis ΔΕ, and one of two equal conjugate diameters ΤΥ. Let two diameters ΒΚ and ΛΜ be drawn between Α and Τ, and let ΓΠ and ΓΡ [respectively] be parallel to them, and let there be dropped to the axis the perpendiculars ΠΧ and Ρι.

We construct in the diagram [elements] corresponding to the constructions in the hyperbola in the theorem preceding this.

Then I sat that the amount by which sq.ΑΓ is greater than the square on the *latus rectum* of the *eidos* corresponding to it is greater than the amount by which sq.KB is greater than the *latus rectum* of the *eidos* corresponding to it, and that the latter amount is greater than the amount by which sq.ΛM is greater than the square on the *latus rectum* of the *eidos* corresponding to it.

[Proof]. The ratio AΞ to ΞX is smaller than the ratio AΘ to ΘX. Therefore the ratio pl.NΓ,AΞ to pl.NΓ,ΞX is smaller than the ratio the double pl.ΞN.AΘ to the double pl.ΞN,ΘX.

But as for the double pl.ΞN,AΘ , it is equal to the difference between sq.ΞA and sq.AN, and as for the double pl.ΞN,ΘX , it is equal to the difference between sq.ΞX and sq.XN. Therefore the ratio pl.ΓN,AΞ to pl.ΓN,ΞX is smaller than the ratio the difference between sq.ΞA and sq.AN to the difference between sq.ΞX and sq.XN.

And *permutando* the ratio pl.ΓN,AΞ to the difference between sq.ΞA and sq.AN is smaller than pl.ΓN,ΞX to the difference between sq.ΞX and sq.XN .

But as for the ratio pl.ΓN,AΞ to the difference between sq.ΞA and sq.AN, it is equal to the ratio of sq.AΓ to the difference between it [sq.AΓ] and the square on the *latus rectum* of the *eidos* corresponding to it because each of the ratios ΓN to AN and AΞ to ΞΓ is equal to the ratio of AΓ to its *latus rectum* because both AN and ΞΓ are homologues. And as for the ratio pl.ΓN.ΞX to the difference between sq.ΞX and sq.XN, it is equal to the ratio of sq.AΓ to the difference between sq.BK and the square on the *latus rectum* on the *eidos* corresponding to it, as is proved in Theorem 20 of this Book. Therefore the ratio of sq.AΓ to the difference between it and the square on the *latus rectum* of the *eidos* corresponding to it is smaller than the ratio of sq.AΓ to the difference between sq.KB and the square on the *latus rectum* of the *eidos* corresponding to it therefore the difference between the squares on two sides of the *eidos* corresponding to AΓ is greater than the difference between the squares on two sides of the *eidos* corresponding to KB.

Furthermore we will prove, as we proved in the preceding part of this theorem, that the ratio pl.ΓN,ΞX to pl.ΓN,Ξι is smaller than the ratio the difference between sq.ΞX and sq.XN to the difference between sq.Ξι and sq.ιN.

And *permutando* the ratio pl.ΓN,ΞX to the difference between sq.ΞX and sq.XN is smaller than the ratio pl.ΓN,Ξι to the difference between sq.Ξι and sq.ιN .

And it will be proved thence that the difference between the squares on two sides of the *eidos* corresponding to BK is greater than the difference between the squares on two sides of the *eidos* corresponding to MΛ.

Furthermore we draw two diameters ΩΨ and ΦΣ between A and T, and draw from Γ two straight lines ΓH and ΓO parallel to them, and drop to the axis perpendiculars Hς and OϘ, then I say that the difference between sq.ΔE and the square on the latus  rectum of the *eidos* corresponding to it is greater than the difference between sq.ΩΨ and the square on the *latus rectum* of the *eidos* corresponding to it, and that this [latter] difference is greater than the difference between sq.ΦΣ and the square on the *latus rectum* of the *eidos* corresponding to it.

[Proof].  The ratio pl.ΓN,Ξς to pl.ΓN,Ξ Ϙ is greater than the ratio ςΘ to ΘϘ because Ξς is greater than ΞϘ and ςΘ is smaller than ϘΘ, and the ratio ςΘ to ϘΘ is equal to the ratio the double pl.ΞN,ςΘ to the double pl.ΞN, ϘΘ .

Now as for the double pl.ΞN,ςΘ , it is equal to the difference between sq.NΓ and sq.ςΞ, and as for the double pl.ΞN, ϘΘ , it is equal to the difference between sq.N Ϙ and sq. ϘΞ. Therefore the ratio pl.ΓN,Ξς to pl.ΓN,Ξ Ϙ is greater than the ratio the difference between sq.Nς and sq.XΞ to the difference between sq.N Ϙ and sq. ϘΞ .

And *permutando* the ratio pl.ΓN,ςΞ to the difference between sq.Nς and sq.ςΞ is greater than the ratio pl.ΓN, ϘΞ to the difference  between sq.N Ϙ and sq. ϘΞ.

Therefore it will be proved thence, by [a method] similar to that which we used above, that the ratio of sq.AΓ to the difference between sq.ΦΣ and the square on the *latus rectum* of the *eidos* corresponding to ΦΣ is greater than the ratio of sq.AΓ to the difference between sq.ΩΨ and the square on the *latus rectum* of the *eidos* corresponding to it [ΩΨ]. Therefore the difference between sq. ΩΨ and the square on the *latus rectum* of the *eidos* corresponding to it is greater than the difference between sq.ΦΣ and the square on the *latus rectum* of the *eidos* corresponding to it.

Furthermore the ratio ϘΞ to ΦΓ is greater than the ratio ϘΘ to ΘΓ because ϘΞ is greater than ΞΓ and ϘΘ is smaller than ΘΓ, therefore the ratio pl.ΓN, ϘΞ to pl.NΓΞ is greater than the ratio the double pl.NΞ, ϘΘ to the double pl.NΞ,ΘΓ, and it will be proved thence, as we proved previously, that the difference between sq.ΔE and the square on the *latus rectum* of the *eidos* corresponding to it is greater than the difference between sq.ΩΨ and the square on the *latus rectum* of the *eidos* corresponding to it.