Commentarii Mathematici Helvetici

Rigidity of measurable structure for $\mathbb{Z}^d$-actions by automorphisms of a torus

Anatole Katok, Svetlana Katok and Klaus Schmidt

Dedicated to the memory of Olga Taussky Todd

Abstract. We show that for certain classes of actions of $\mathbb{Z}^d$, $d \geq 2$, by automorphisms of the torus any measurable conjugacy has to be affine, hence measurable conjugacy implies algebraic conjugacy; similarly any measurable factor is algebraic, and algebraic and affine centralizers provide invariants of measurable conjugacy. Using the algebraic machinery of dual modules and information about class numbers of algebraic number fields we construct various examples of $\mathbb{Z}^d$-actions by Bernoulli automorphisms whose measurable orbit structure is rigid, including actions which are weakly isomorphic but not isomorphic. We show that the structure of the centralizer for these actions may or may not serve as a distinguishing measure-theoretic invariant.

1. Introduction

Description of results

In the course of the last decade various rigidity properties have been found for two different classes of actions by higher-rank abelian groups: on the one hand, certain Anosov and partially hyperbolic actions of $\mathbb{Z}^d$ and $\mathbb{R}^d$, $d \geq 2$, on compact manifolds ([9, 10, 12]) and, on the other, actions of $\mathbb{Z}^d$, $d \geq 2$, by automorphisms of compact abelian groups (cf. e.g. [8, 16]). Among these rigidity phenomena is a relative scarcity of invariant measures which stands in contrast with the classical case $d = 1$ ([11]).

In this paper we make the first step in investigating a different albeit related

phenomenon: rigidity of the measurable orbit structure with respect to the natural smooth invariant measure.

In the classical case of actions by $\mathbb{Z}$ or $\mathbb{R}$ there are certain natural classes of measure-preserving transformations which possess such rigidity: ergodic translations on compact abelian groups give a rather trivial example, while horocycle flows and other homogeneous unipotent systems present a much more interesting one [20, 21, 22]. In contrast to such situations, individual elements of the higher-rank actions mentioned above are Bernoulli automorphisms. The measurable orbit structure of a Bernoulli map can be viewed as very "soft". Recall that the only metric invariant of Bernoulli automorphisms is entropy ([19]); in particular, weak isomorphism is equivalent to isomorphism for Bernoulli maps since it implies equality of entropies. Furthermore, description of centralizers, factors, joinings and other invariant objects associated with a Bernoulli map is impossible in reasonable terms since each of these objects is huge and does not possess any discernible structure.

In this paper we demonstrate that some very natural actions of $\mathbb{Z}^d$, $d \geq 2$, by Bernoulli automorphisms display a remarkable rigidity of their measurable orbit structure. In particular, isomorphisms between such actions, centralizers, and factor maps are very restricted, and a lot of algebraic information is encoded in the measurable structure of such actions (see Section 5).

All these properties occur for broad subclasses of both main classes of actions of higher-rank abelian groups mentioned above: Anosov and partially hyperbolic actions on compact manifolds, and actions by automorphisms of compact abelian groups. However, at present we are unable to present sufficiently definitive general results due to various difficulties of both conceptual and technical nature. Trying to present the most general available results would lead to cumbersome notations and inelegant formulations. To avoid that we chose to restrict our present analysis to a smaller class which in fact represents the intersection of the two, namely the actions of $\mathbb{Z}^d$, $d \geq 2$, by automorphisms of the torus. Thus we study the measurable structure of such actions with respect to Lebesgue (Haar) measure from the point of view of ergodic theory.

Our main purpose is to demonstrate several striking phenomena by means of applying to specific examples general rigidity results which are presented in Section 5 and are based on rigidity of invariant measures developed in [11] (see [7] for further results along these lines including rigidity of joinings). Hence we do not strive for the greatest possible generality even within the class of actions by automorphisms of a torus. The basic algebraic setup for irreducible actions by automorphisms of a torus is presented in Section 3. Then we adapt further necessary algebraic preliminaries to the special but in a sense most representative case of Cartan actions, i.e. to $\mathbb{Z}^{n-1}$-actions by hyperbolic automorphisms of the n-dimensional torus (see Section 4).

The role of entropy for a smooth action of a higher-rank abelian group G on a finite-dimensional manifold is played by the *entropy function* on G whose values

are entropies of individual elements of the action (see Section 2.2 for more details) which is naturally invariant of isomorphism and also of weak isomorphism and is equivariant with respect to a time change.

In Section 6 we produce several kinds of specific examples of actions by ergodic (and hence Bernoulli) automorphisms of tori with the same entropy function. These examples provide concrete instances when general criteria developed in Section 5 can be applied. Our examples include:

   (i) actions which are not weakly isomorphic (Section 6.1),
  (ii) actions which are weakly isomorphic but not isomorphic, such that one
        action is a maximal action by Bernoulli automorphisms and the other is
        not (Section 6.2),
 (iii) weakly isomorphic, but nonisomorphic, maximal actions (Section 6.3).

Once rigidity of conjugacies is established, examples of type (i) appear in a rather simple-minded fashion: one simply constructs actions with the same entropy data which are not isomorphic over $\mathbb{Q}$. This is not surprising since entropy contains only partial information about eigenvalues. Thus one can produce actions with different eigenvalue structure but identical entropy data.

Examples of weakly isomorphic but nonisomorphic actions are more sophisticated. We find them among Cartan actions (see Section 4). The centralizer of a Cartan action in the group of automorphisms of the torus is (isomorphic to) a finite extension of the acting group, and in some cases Cartan actions isomorphic over $\mathbb{Q}$ may be distinguished by looking at the index of the group in its centralizer (type (ii); see Examples 2a and 2b). The underlying cause for this phenomenon is the existence of algebraic number fields $K = \mathbb{Q}(\lambda)$, where λ is a unit, such that the ring of integers $\mathcal{O}_K \neq \mathbb{Z}[\lambda]$. In general finding even simplest possible examples for $n = 3$ involves the use of data from algebraic number theory and rather involved calculations. For examples of type (ii) one may use some special tricks which allow to find some of these and to show nonisomorphism without a serious use of symbolic manipulations on a computer.

A Cartan action α of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ is called maximal if its centralizer in the group of automorphisms of the torus is equal to $\alpha \times \{\pm\mathrm{Id}\}$. A maximal Cartan action turns out to be maximal in the above sense: it cannot be extended to any action of a bigger abelian group by Bernoulli automorphisms.

Examples of maximal Cartan actions isomorphic over $\mathbb{Q}$ but not isomorphic (type (iii)) are the most remarkable. Conjugacy over $\mathbb{Q}$ guarantees that the actions by automorphisms of the torus $\mathbb{T}^n$ arising from their centralizers are weakly isomorphic with finite fibres. The mechanism providing obstructions for algebraic isomorphism in this case involves the connection between the class number of an algebraic number field and $GL(n, \mathbb{Z})$-conjugacy classes of matrices in $SL(n, \mathbb{Z})$ which have the same characteristic polynomial (see Example 3). In finding these examples the use of computational number-theoretic algorithms (which in our case were implemented via the Pari-GP package) has been essential.

One of our central conclusions is that for a broad class of actions of $\mathbb{Z}^d$, $d \geq 2$, (see condition ($\mathcal{R}$) in Section 2.2) the conjugacy class of the centralizer of

the action in the group of *affine* automorphisms of the torus is an invariant of measurable conjugacy. Let $Z_{meas}(\alpha)$ be the centralizer of the action α in the group of measurable automorphisms. As it turns out in all our examples but Example 3b, the conjugacy class of the pair $(Z_{meas}(\alpha), \alpha)$ is a distinguishing invariant of the measurable isomorphism. Thus, in particular, Example 3b shows that there are weakly isomorphic, but nonisomorphic actions for which the affine and hence the measurable centralizers are isomorphic as abstract groups.

We would like to acknowledge a contribution of J.-P. Thouvenot to the early development of ideas which led to this paper. He made an important observation that rigidity of invariant measures can be used to prove rigidity of isomorphisms via a joining construction (see Section 5.1).

2. Preliminaries

2.1. Basic ergodic theory

Any invertible (over $\mathbb{Q}$) integral $n \times n$ matrix $A \in M(n, \mathbb{Z}) \cap GL(n, \mathbb{Q})$ determines an endomorphism of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ which we denote by F_A. Conversely, any endomorphism of $\mathbb{T}^n$ is given by a matrix from $A \in M(n, \mathbb{Z}) \cap GL(n, \mathbb{Q})$. If, in addition, $\det A = \pm 1$, i.e. if A is invertible over $\mathbb{Z}$, then F_A is an automorphism of $\mathbb{T}^n$ (the group of all such A is denoted by $GL(n, \mathbb{Z})$). The map F_A preserves Lebesgue (Haar) measure μ; it is ergodic with respect to μ if and only if there are no roots of unity among the eigenvalues of A, as was first pointed out by Halmos ([6]). Furthermore, in this case there are eigenvalues of absolute value greater than one and (F_A, λ) is an exact endomorphism. If F_A is an automorphism it is in fact Bernoulli ([14]). For simplicity we will call such a map F_A an *ergodic toral endomorphism* (respectively, *automorphism*, if A is invertible). If all eigenvalues of A have absolute values different from one we will call the endomorphism (automorphism) F_A *hyperbolic*.

When it does not lead to a confusion we will not distinguish between a matrix A and corresponding toral endomorphism F_A.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix A, listed with their multiplicities. The entropy $h_\mu(F_A)$ of F_A with respect to Lebesgue measure is equal to

$$\sum_{\{i: |\lambda_i| > 1\}} \log |\lambda_i|.$$

In particular, entropy is determined by the conjugacy class of the matrix A over $\mathbb{Q}$ (or over $\mathbb{C}$). Hence *all ergodic toral automorphisms which are conjugate over $\mathbb{Q}$ are measurably conjugate with respect to Lebesgue measure.*

Classification, up to a conjugacy over $\mathbb{Z}$, of matrices in $SL(n, \mathbb{Z})$, which are irreducible and conjugate over $\mathbb{Q}$ is closely related to the notion of class number of an algebraic number field. A detailed discussion relevant to our purposes appears in Section 4.2. Here we only mention the simplest case $n = 2$ which is not directly

related to rigidity. In this case trace determines conjugacy class over $\mathbb{Q}$ and, in particular, entropy. However if the class number of the corresponding number field is greater than one there are matrices with the given trace which are not conjugate over $\mathbb{Z}$. This algebraic distinctiveness is not reflected in the measurable structure: in fact, in the case of equal entropies the classical Adler–Weiss construction of the Markov partition in [1] yields metric isomorphisms which are more concrete and specific than in the general Ornstein isomorphism theory and yet not algebraic.

2.2. Higher rank actions

Let α be an action by commuting toral automorphisms given by integral matrices $A_1, \dots, A_d$. It defines an embedding $\rho_\alpha : \mathbb{Z}^d \to GL(n, \mathbb{Z})$ by

$$\rho_\alpha^{\mathbf{n}} = A_1^{n_1} \dots A_d^{n_d},$$

where $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, and we have

$$\alpha^{\mathbf{n}} = F_{\rho_\alpha^{\mathbf{n}}}.$$

Similarly, we write $\rho_\alpha : \mathbb{Z}_+^d \to M(n, \mathbb{Z}) \cap GL(n, \mathbb{Q})$ for an action by endomorphisms. Conversely, any embedding $\rho : \mathbb{Z}^d \to GL(n, \mathbb{Z})$ (respectively, $\rho : \mathbb{Z}_+^d \to M(n, \mathbb{Z}) \cap GL(n, \mathbb{Q})$) defines an action by automorphisms (respectively, endomorphisms) of $\mathbb{T}^n$ denoted by α_ρ.

Sometimes we will not explicitly distinguish between an action and the corresponding embedding, e.g. we may talk about "the centralizer of an action in $GL(n, \mathbb{Z})$" etc.

Definitions. Let α and α' be two actions of $\mathbb{Z}^d$ ($\mathbb{Z}_+^d$) by automorphisms (endomorphisms) of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$, respectively. The actions α and α' are *measurably* (or *metrically*, or *measure-theoretically*) *isomorphic* (or *conjugate*) if there exists a Lebesgue measure-preserving bijection $\varphi : \mathbb{T}^n \to \mathbb{T}^{n'}$ such that $\varphi \circ \alpha = \alpha' \circ \varphi$.

The actions α and α' are *measurably isomorphic up to a time change* if there exist a measure-preserving bijection $\varphi : \mathbb{T}^n \to \mathbb{T}^{n'}$ and a $C \in GL(d, \mathbb{Z})$ such that $\varphi \circ \alpha \circ C = \alpha' \circ \varphi$.

The action α' is a *measurable factor* of α if there exists a Lebesgue measure-preserving transformation $\varphi : \mathbb{T}^n \to \mathbb{T}^{n'}$ such that $\varphi \circ \alpha = \alpha' \circ \varphi$. If, in particular, φ is almost everywhere finite-to-one, then α' is called a *finite factor* or a *factor with finite fibres* of α.

Actions α and α' are *weakly measurably isomorphic* if each is a measurable factor of the other.

A *joining* between α and α' is a measure μ on $\mathbb{T}^n \times \mathbb{T}^{n'} = \mathbb{T}^{n+n'}$ invariant under the Cartesian product action $\alpha \times \alpha'$ such that its projections into $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ are Lebesgue measures. As will be explained in Section 5, conjugacies and factors produce special kinds of joinings.

These measure-theoretic notions have natural algebraic counterparts.

Definitions. The actions α and α' are *algebraically isomorphic* (or *conjugate*) if $n = n'$ and if there exists a group automorphism $\varphi : \mathbb{T}^n \to \mathbb{T}^n$ such that $\varphi \circ \alpha = \alpha' \circ \varphi$.

The actions α and α' are *algebraically isomorphic up to a time change* if there exists an automorphism $\varphi : \mathbb{T}^n \to \mathbb{T}^n$ and $C \in GL(d, \mathbb{Z})$ such that $\varphi \circ \alpha \circ C = \alpha' \circ \varphi$.

The action α' is an *algebraic factor* of α if there exists a surjective homomorphism $\varphi : \mathbb{T}^n \to \mathbb{T}^{n'}$ such that $\varphi \circ \alpha = \alpha' \circ \varphi$.

The actions α and α' are *weakly algebraically isomorphic* if each is an algebraic factor of the other. In this case $n = n'$ and each factor map has finite fibres.

Finally, we call a map $\varphi : \mathbb{T}^n \to \mathbb{T}^{n'}$ *affine* if there is a surjective continuous group homomorphism $\psi : \mathbb{T}^n \to \mathbb{T}^{n'}$ and $x' \in \mathbb{T}^{n'}$ s.t. $\varphi(x) = \psi(x) + x'$ for every $x \in \mathbb{T}^n$.

As already mentioned, we intend to show that under certain condition for $d \geq 2$, measure theoretic properties imply their algebraic counterparts.

We will say that an algebraic factor α' of α is a *rank-one factor* if α' is an algebraic factor of α and $\alpha'(\mathbb{Z}_+^d)$ contains a cyclic sub-semigroup of finite index.

The most general situation when certain rigidity phenomena appear is the following :

$(\mathcal{R}')$: *The action α does not possess nontrivial rank-one algebraic factors.*

In the case of actions by automorphisms the condition $(\mathcal{R}')$ is equivalent to the following condition $(\mathcal{R})$ (cf. [27]):

$(\mathcal{R})$: *The action α contains a group, isomorphic to $\mathbb{Z}^2$, which consists of ergodic automorphisms.*

By Proposition 6.6 in [25], Condition $(\mathcal{R})$ is equivalent to saying that *the restriction of α to a subgroup isomorphic to $\mathbb{Z}^2$ is mixing.*

A *Lyapunov exponent* for an action α of $\mathbb{Z}^d$ is a function $\chi : \mathbb{Z}^d \to \mathbb{R}$ which associates to each $\mathbf{n} \in \mathbb{Z}^d$ the logarithm of the absolute value of the eigenvalue for $\rho_\alpha^{\mathbf{n}}$ corresponding to a fixed eigenvector. Any Lyapunov exponent is a linear function; hence it extends uniquely to $\mathbb{R}^d$. The *multiplicity* of an exponent is defined as the sum of multiplicities of eigenvalues corresponding to this exponent. Let χ_i, $i = 1, \ldots, k$, be the different Lyapunov exponents and let m_i be the multiplicity of χ_i. Then the entropy formula for a single toral endomorphism implies that

$$h_\alpha(\mathbf{n}) = h_\mu(\rho_\alpha^{\mathbf{n}}) = \sum_{\{i : \chi_i(\mathbf{n}) > 0\}} m_i \chi_i(\mathbf{n}).$$

The function $h_\alpha : \mathbb{Z}^d \to \mathbb{R}$ is called *the entropy function* of the action α. It naturally extends to a symmetric, convex piecewise linear function of $\mathbb{R}^d$. Any

cone in $\mathbb{R}^d$ where all Lyapunov exponents have constant sign is called a *Weyl chamber*. The entropy function is linear in any Weyl chamber.

The entropy function is a prime invariant of measurable isomorphism; since entropy does not increase for factors the entropy function is also invariant of a weak measurable isomorphism. Furthermore it changes equivariantly with respect to automorphisms of $\mathbb{Z}^d$.

Remark. it is interesting to point out that the convex piecewise linear structure of the entropy function persists in much greater generality, namely for smooth actions on differentiable manifolds with a Borel invariant measure with compact support.

2.3. Finite algebraic factors and invariant lattices

Every algebraic action has many algebraic factors with finite fibres. These factors are in one-to-one correspondence with lattices $\Gamma \subset \mathbb{R}^n$ which contain the standard lattice $\Gamma_0 = \mathbb{Z}^n$, and which satisfy $\rho_\alpha(\Gamma) \subset \Gamma$. The factor-action associated with a particular lattice $\Gamma \supset \Gamma_0$ is denoted by α_Γ. Let us point out that in the case of actions by automorphisms such factors are also invertible: if $\Gamma \supset \Gamma_0$ and $\rho_\alpha(\Gamma) \subset \Gamma$, then $\rho_\alpha(\Gamma) = \Gamma$.

Let $\Gamma \supset \Gamma_0$ be a lattice. Take any basis in Γ and let $S \in GL(n, \mathbb{Q})$ be the matrix which maps the standard basis in Γ_0 to this basis. Then obviously the factor-action α_Γ is equal to the action $\alpha_{S\rho_\alpha S^{-1}}$. In particular, ρ_α and ρ_{α_Γ} are conjugate over $\mathbb{Q}$, although not necessarily over $\mathbb{Z}$. Notice that conjugacy over $\mathbb{Q}$ is equivalent to conjugacy over $\mathbb{R}$ or over $\mathbb{C}$.

For any positive integer q, the lattice $\frac{1}{q}\Gamma_0$ is invariant under any automorphism in $GL(n, \mathbb{Z})$ and gives rise to a factor which is conjugate to the initial action: one can set $S = \frac{1}{q}\mathrm{Id}$ and obtains that $\rho_\alpha = \rho_{\alpha_{\frac{1}{q}\Gamma_0}}$. On the other hand one can find, for any lattice $\Gamma \supset \Gamma_0$, a positive integer q such that $\frac{1}{q}\Gamma_0 \supset \Gamma$ (take q the least common multiple of denominators of coordinates for a basis of Γ). Thus $\alpha_{\frac{1}{q}\Gamma_0}$ appears as a factor of α_Γ. Summarizing, we have the following properties of finite factors.

Proposition 2.1. *Let α and α' be $\mathbb{Z}^d$-actions by automorphism of the torus $\mathbb{T}^n$. The following are equivalent.*
  (1) *ρ_α and $\rho_{\alpha'}$ are conjugate over $\mathbb{Q}$;*
  (2) *there exists an action α'' such that both α and α' are isomorphic to finite algebraic factors of α'';*
  (3) *α and α' are weakly algebraically isomorphic, i.e. each of them is isomorphic to a finite algebraic factor of the other.*

Obviously, weak algebraic isomorphism implies weak measurable isomorphism.

For $\mathbb{Z}$-actions by Bernoulli automorphisms, weak isomorphism implies isomorphism since it preserves entropy, the only isomorphism invariant for Bernoulli maps. In Section 5 we will show that, for actions by toral automorphisms satisfying Condition $(\mathcal{R})$, measurable isomorphism implies algebraic isomorphism. Hence, existence of such actions which are conjugate over $\mathbb{Q}$ but not over $\mathbb{Z}$ provides examples of actions by Bernoulli maps which are weakly isomorphic but not isomorphic.

2.4. Dual modules

For any action α of $\mathbb{Z}^d$ by automorphisms of a compact abelian group X we denote by $\hat{\alpha}$ the dual action on the discrete group $\hat{X}$ of characters of X. For an element $\chi \in \hat{X}$ we denote $\hat{X}_{\alpha,\chi}$ the subgroup of $\hat{X}$ generated by the orbit $\hat{\alpha}\chi$.

Definition. The action α is called *cyclic* if $\hat{X}_{\alpha,\chi} = \hat{X}$ for some $\chi \in \hat{X}$.

Cyclicity is obviously an invariant of algebraic conjugacy of actions up to a time change.

More generally, the dual group $\hat{X}$ has the structure of a module over the ring $\mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ of Laurent polynomials in d commuting variables. Action by the generators of $\hat{\alpha}$ corresponds to multiplications by independent variables. This module is called *the dual module* of the action α (cf. [24, 25]). Cyclicity of the action corresponds to the condition that this module has a single generator. The structure of the dual module up to isomorphism is an invariant of algebraic conjugacy of the action up to a time change.

In the case of the torus $X = \mathbb{T}^n$ which concerns us in this paper one can slightly modify the construction of the dual module to make it more geometric. A $\mathbb{Z}^d$-action α by automorphisms of the torus $\mathbb{R}^n/\mathbb{Z}^n$ naturally extends to an action on $\mathbb{R}^n$ (this extension coincides with the embedding ρ_α if matrices are identified with linear transformations). This action preserves the lattice $\mathbb{Z}^n$ and furnishes $\mathbb{Z}^n$ with the structure of a module over the ring $\mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. This module is — in an obvious sense — a *transpose* of the dual module defined above. In particular, the condition of cyclicity of the action does not depend on which of these two definitions of dual module one adopts.

2.5. Algebraic and affine centralizers

Let α be an action of $\mathbb{Z}^d$ by toral automorphisms, and let $\rho_\alpha(\mathbb{Z}^d) = \{\rho_\alpha^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$. The *centralizer* of α in the group of automorphisms of $\mathbb{T}^n$ is denoted by $Z(\alpha)$ and is not distinguished from the centralizer of $\rho_\alpha(\mathbb{Z}^d)$ in $GL(n, \mathbb{Z})$.

Similarly, the centralizer of α in the semigroup of all endomorphisms of $\mathbb{T}^n$

(identified with the centralizer of $\rho_\alpha(\mathbb{Z}^d)$ in the semigroup $M(n,\mathbb{Z}) \cap GL(n,\mathbb{Q})$) is denoted by $C(\alpha)$.

The centralizer of α in the group of affine automorphisms of $\mathbb{T}^n$ will be denoted by $Z_{Aff}(\alpha)$.

The centralizer of α in the semigroup of surjective affine maps of $\mathbb{T}^n$ will be denoted by $C_{Aff}(\alpha)$.

3. Irreducible actions

3.1. Definition

The action α on $\mathbb{T}^n$ is called *irreducible* if any nontrivial algebraic factor of α has finite fibres.

The following characterization of irreducible actions is useful (cf. [2]).

Proposition 3.1. *The following conditions are equivalent:*

   (1) α *is irreducible;*
   (2) ρ_α *contains a matrix with characteristic polynomial irreducible over $\mathbb{Q}$;*
   (3) ρ_α *does not have a nontrivial invariant rational subspace or, equivalently, any α-invariant closed subgroup of $\mathbb{T}^n$ is finite.*

Corollary 3.2. *Any irreducible free action α of $\mathbb{Z}_+^d$, $d \geq 2$, satisfies condition $(\mathcal{R}')$.*

Proof. A rank one algebraic factor has to have fibres of positive dimension. Hence the pre-image of the origin under the factor map is a union of finitely many rational tori of positive dimension and by Proposition 3.1 α cannot be irreducible.

3.2. Uniqueness of cyclic actions

Cyclicity uniquely determines an irreducible action up to algebraic conjugacy within a class of weakly algebraically conjugate actions.

Proposition 3.3. *If α is an irreducible cyclic action of $\mathbb{Z}^d$, $d \geq 1$, on $\mathbb{T}^n$ and α' is another cyclic action such that ρ_α and $\rho_{\alpha'}$ are conjugate over $\mathbb{Q}$, then α and α' are algebraically isomorphic.*

For the proof of Proposition 3.3 we need an elementary lemma.

Lemma 3.4. *Let $\rho : \mathbb{Z}^d \to GL(n,\mathbb{Z})$ be an irreducible embedding. The centralizer of ρ in $GL(n,\mathbb{Q})$ acts transitively on $\mathbb{Z}^n \setminus \{0\}$.*

Proof. By diagonalizing ρ over $\mathbb{C}$ and taking the real form of it, one immediately sees that the centralizer of ρ in $GL(n, \mathbb{R})$ acts transitively on vectors with nonzero projections on all eigenspaces and thus has a single open and dense orbit. Since the centralizer over $\mathbb{R}$ is the closure of the centralizer over $\mathbb{Q}$, the $\mathbb{Q}$-linear span of the orbit of any integer or rational vector under the centralizer is an invariant rational subspace. Hence any integer point other than the origin belongs to the single open dense orbit of the centralizer of ρ in $GL(n, \mathbb{R})$. This implies the statement of the lemma.

Proof of Proposition 3.3. Choose $C \in M(n, \mathbb{Z})$ such that $C\rho_{\alpha'}C^{-1} = \rho_\alpha$. Let $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^n$ be cyclic vectors for $\rho_\alpha|_{\mathbb{Z}^n}$ and $\rho_{\alpha'}|_{\mathbb{Z}^n}$, respectively.

Now consider the integer vector $C(\mathbf{l})$ and find $D \in GL(n, \mathbb{Q})$ commuting with ρ_α such that $DC(\mathbf{l}) = \mathbf{k}$. We have $DC\rho_{\alpha'}C^{-1}D^{-1} = \rho_\alpha$. The conjugacy DC maps bijectively the $\mathbb{Z}$-span of the $\rho_{\alpha'}$-orbit of $\mathbf{l}$ to $\mathbb{Z}$-span of the ρ_α-orbit of $\mathbf{k}$. By cyclicity both spans coincide with $\mathbb{Z}^n$, and hence $DC \in GL(n, \mathbb{Z})$.

3.3. Centralizers of integer matrices and algebraic number fields

There is an intimate connection between irreducible actions on $\mathbb{T}^n$ and groups of units in number fields of degree n. Since this connection (in the particular case where the action is Cartan and hence the number field is totally real) plays a central role in the construction of our principal examples (type (ii) and (iii) of the Introduction), we will describe it here in detail even though most of this material is fairly routine from the point of view of algebraic number theory.

Let $A \in GL(n, \mathbb{Z})$ be a matrix with an irreducible characteristic polynomial f and hence distinct eigenvalues. The centralizer of A in $M(n, \mathbb{Q})$ can be identified with the ring of all polynomials in A with rational coefficients modulo the principal ideal generated by the polynomial $f(A)$, and hence with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A, by the map

$$\gamma : p(A) \mapsto p(\lambda) \tag{1}$$

with $p \in \mathbb{Q}[x]$. Notice that if $B = p(A)$ is an integer matrix then $\gamma(B)$ is an algebraic integer, and if $B \in GL(n, \mathbb{Z})$ then $\gamma(B)$ is an algebraic unit (converse is not necessarily true).

Lemma 3.5. *The map γ in (1) is injective.*

Proof. If $\gamma(p(A)) = 1$ for $p(A) \neq \mathrm{Id}$, then $p(A)$ has 1 as an eigenvalue, and hence has a rational subspace consisting of all invariant vectors. This subspace must be invariant under A which contradicts its irreducibility.

Denote by $\mathcal{O}_K$ the ring of integers in K, by $\mathcal{U}_K$ the group of units in $\mathcal{O}_K$, by $C(A)$ the centralizer of A in $M(n, \mathbb{Z})$ and by $Z(A)$ the centralizer of A in the group

$GL(n, \mathbb{Z})$.

Lemma 3.6. $\gamma(C(A))$ *is a ring in K such that $\mathbb{Z}[\lambda] \subset \gamma(C(A)) \subset \mathcal{O}_K$, and* $\gamma(Z(A)) = \mathcal{U}_K \cap \gamma(C(A))$.

Proof. $\gamma(C(A))$ is a ring because $C(A)$ is a ring. As we pointed out above images of integer matrices are algebraic integers and images of matrices with determinant ± 1 are algebraic units. Hence $\gamma(C(A)) \subset \mathcal{O}_K$. Finally, for every polynomial p with integer coefficients, $p(A)$ is an integer matrix, hence $\mathbb{Z}[\lambda] \subset \gamma(C(A))$.

Notice that $\mathbb{Z}(\lambda)$ is a finite index subring of $\mathcal{O}_K$; hence $\gamma(C(A))$ has the same property.

Remark. The groups of units in two different rings, say $\mathcal{O}_1 \subset \mathcal{O}_2$, may coincide. Examples can be found in the table of totally real cubic fields in [4].

Proposition 3.7. $Z(A)$ *is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times F$ where r_1 is the number the real embeddings, r_2 is the number of pairs of complex conjugate embeddings of the field K into $\mathbb{C}$, and F is a finite cyclic group.*

Proof. By Lemma 3.6, $Z(A)$ is isomorphic to the group of units in the order $\gamma(C(A))$, the statement follows from the Dirichlet Unit Theorem ([3], Ch.2, §4.3).

Now consider an irreducible action α of $\mathbb{Z}^d$ on $\mathbb{T}^n$. Denote $\rho_\alpha(\mathbb{Z}^d)$ by Γ, and let λ be an eigenvalue of a matrix $A \in \Gamma$ with an irreducible characteristic polynomial. The centralizers of Γ in $M(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ coincide with $C(A)$ and $Z(A)$ correspondingly. The field $K = \mathbb{Q}(\lambda)$ has degree n and we can consider the map γ as above. By Lemma 3.6 $\gamma(\Gamma) \subset \mathcal{U}_K$.

For the purposes of purely algebraic considerations in this and the next section it is convenient to consider actions of integer $n \times n$ matrices on $\mathbb{Q}^n$ rather than on $\mathbb{R}^n$ and correspondingly to think of α as an action by automorphisms of the rational torus $\mathbb{T}^n_\mathbb{Q} = \mathbb{Q}^n / \mathbb{Z}^n$.

Let $v = (v_1, \ldots, v_n)$ be an eigenvector of A with eigenvalue λ whose coordinates belong to K. Consider the "projection" $\pi : \mathbb{Q}^n \to K$ defined by $\pi(r_1, \ldots r_n) = \sum_{i=1}^n r_i v_i$. It is a bijection ([29], Prop. 8) which conjugates the action of the group Γ with the action on K given by multiplication by corresponding eigenvalues $\prod_{i=1}^d \lambda_i^{k_i}$, $k_1, \ldots, k_d \in \mathbb{Z}$. Here $A_1, \ldots, A_d \in \Gamma$ are the images of the generators of the action α, and $A_i v = \lambda_i v$, $i = 1, \ldots, d$. The lattice $\pi \mathbb{Z}^n \subset K$ is a module over the ring $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$.

Conversely, any such data, consisting of an algebraic number field $K = \mathbb{Q}(\lambda)$ of degree n, a d-tuple $\bar{\lambda} = (\lambda_1, \ldots, \lambda_d)$ of multiplicatively independent units in K, and a lattice $\mathcal{L} \subset K$ which is a module over $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$, determine an $\mathbb{Z}^d$-action $\alpha_{\bar{\lambda}, \mathcal{L}}$ by automorphisms of $\mathbb{T}^n$ up to algebraic conjugacy (corresponding to

a choice of a basis in the lattice $\mathcal{L}$). This action is generated by multiplications by $\lambda_1, \ldots, \lambda_d$ (which preserve $\mathcal{L}$ by assumption). The action $\alpha_{\bar\lambda, \mathcal{L}}$ diagonalizes over $\mathbb{C}$ as follows. Let $\phi_1 = \text{id}, \phi_2, \ldots, \phi_n$ be different embeddings of K into $\mathbb{C}$. The multiplications by λ_i, $i = 1, \ldots, d$, are simultaneously conjugate over $\mathbb{C}$ to the respective matrices

$$\begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 0 & \phi_2(\lambda_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \phi_n(\lambda_i) \end{pmatrix}, \quad i = 1, \ldots, d.$$

We will assume that the action is irreducible which in many interesting cases can be easily checked.

Thus, all actions $\alpha_{\bar\lambda, \mathcal{L}}$ with fixed $\bar\lambda$ are weakly algebraically isomorphic since the corresponding embeddings are conjugate over $\mathbb{Q}$ (Proposition 2.1). Actions produced with different sets of units in the same field, say $\bar\lambda$ and $\bar\mu = (\mu_1, \ldots, \mu_d)$, are weakly algebraically isomorphic if and only if there is an automorphism g of K such that $\mu_i = g\lambda_i$, $i = 1, \ldots, d$. By Proposition 3.3 there is a unique cyclic action (up to algebraic isomorphism) within any class of weakly algebraically isomorphic actions: it corresponds to setting $\mathcal{L} = \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$; we will denote this action by $\alpha_{\bar\lambda}^{\min}$. Cyclicity of the action $\alpha_{\bar\lambda}^{\min}$ is obvious since the whole lattice is obtained from its single element 1 by the action of the ring $\mathbb{Z}[\lambda_1^{\pm 1}, \ldots, \lambda_d^{\pm 1}]$.

Let us summarize this discussion.

Proposition 3.8. *Any irreducible action α of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ is algebraically conjugate to an action of the form $\alpha_{\bar\lambda, \mathcal{L}}$. It is weakly algebraically conjugate to the cyclic action $\alpha_{\bar\lambda}^{\min}$. The field $K = \mathbb{Q}[\lambda_1, \ldots, \lambda_d]$ has degree n, and the vector of units $\bar\lambda = (\lambda_1, \ldots, \lambda_d)$ is defined up to an automorphism of K.*

Apart from the cyclic model $\alpha_{\bar\lambda}^{\min}$ there is another canonical choice of the lattice $\mathcal{L}$, namely the ring of integers $\mathcal{O}_K$. We will denote the action $\alpha_{\bar\lambda, \mathcal{O}_K}$ by $\alpha_{\bar\lambda}^{\max}$. More generally, one can choose as the lattice $\mathcal{L}$ any subring $\mathcal{O}$ such that $\mathbb{Z}[\lambda_1, \ldots, \lambda_d] \subset \mathcal{O} \subset \mathcal{O}_K$.

Proposition 3.9. *Assume that $\mathcal{O} \supsetneq \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. Then the action $\alpha_{\bar\lambda, \mathcal{O}}$ is not algebraically isomorphic up to a time change to $\alpha_{\bar\lambda}^{\min}$. In particular, if $\mathcal{O}_K \neq \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$, then the actions $\alpha_{\bar\lambda}^{\max}$ and $\alpha_{\bar\lambda}^{\min}$ are not algebraically isomorphic up to a time change.*

Proof. Let us denote the centralizers in $M(n, \mathbb{Z})$ of the actions $\alpha_{\bar\lambda, \mathcal{O}}$ and $\alpha_{\bar\lambda}^{\min}$ by C_1 and C_2, respectively. The centralizer C_1 contains multiplications by all elements of $\mathcal{O}$. For, if one takes any basis in $\mathcal{O}$, the multiplication by an element $\mu \in \mathcal{O}$ takes elements of the basis into elements of $\mathcal{O}$, which are linear combinations with integral coefficients of the basis elements; hence the multiplication is given by an integer matrix. On the other hand any element of each centralizer is a multiplication by an integer in K (Lemma 3.6).

Now assume that the multiplication by $\mu \in \mathcal{O}_K$ belongs to C_2. This means that this multiplication preserves $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$; in particular, $\mu = \mu \cdot 1 \in \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. Thus C_2 consists of multiplication by elements of $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. An algebraic isomorphism up to a time change has to preserve both the module of polynomials with integer coefficients in the generators of the action and the centralizer of the action in $M(n, \mathbb{Z})$, which is impossible.

The central question which appears in connection with our examples is the classification of weakly algebraically isomorphic Cartan actions up to algebraic isomorphism.

Proposition 3.9 is useful in distinguishing weakly algebraically isomorphic actions when $\mathcal{O}_K \neq \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. Cyclicity also can serve as a distinguishing invariant.

Corollary 3.10. *The action $\alpha_{\bar{\lambda}, \mathcal{O}}$ is cyclic if and only if $\mathcal{O} = \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$.*

Proof. The action $\alpha_{\bar{\lambda}}^{\min}$ corresponding to the ring $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$ is cyclic by definition since the ring coincides with the orbit of 1. By Proposition 3.3, if $\alpha_{\bar{\lambda}, \mathcal{O}}$ were cyclic, it would be algebraically conjugate to $\alpha_{\bar{\lambda}}^{\min}$, which, by Proposition 3.9, implies that $\mathcal{O} = \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$.

The property common to all actions of the $\alpha_{\bar{\lambda}, \mathcal{O}}$ is transitivity of the action of the centralizer $C(\alpha_{\bar{\lambda}, \mathcal{O}})$ on the lattice. Similarly to cyclicity this property is obviously an invariant of algebraic conjugacy up to a time change.

Proposition 3.11. *Any irreducible action α of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ whose centralizer $C(\alpha)$ in $M(n, \mathbb{Z})$ acts transitively on $\mathbb{Z}^n$ is algebraically isomorphic to an action $\alpha_{\bar{\lambda}, \mathcal{O}}$, where $\mathcal{O} \subset \mathcal{O}_K$ is a ring which contains $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$.*

Proof. By Proposition 3.8 any irreducible action α of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ is algebraically conjugate to an action of the form $\alpha_{\bar{\lambda}, \mathcal{L}}$ for a lattice $\mathcal{L} \subset K$. Let C be the centralizer of $\alpha_{\bar{\lambda}, \mathcal{L}}$ in the semigroup of linear endomorphisms of $\mathcal{L}$. We fix an element $\beta \in \mathcal{L}$ with $C\beta = \mathcal{L}$ and consider conjugation of the action $\alpha_{\bar{\lambda}, \mathcal{L}}$ by multiplication by β^{-1}; this is simply $\alpha_{\bar{\lambda}, \beta^{-1}\mathcal{L}}$. The centralizer of $\alpha_{\bar{\lambda}, \beta^{-1}\mathcal{L}}$ acts on the element $1 \in \beta^{-1}\mathcal{L}$ transitively. By Lemma 3.6 the centralizer consists of all multiplications by elements of a certain subring $\mathcal{O} \subset \mathcal{O}_K$ which contains $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. Thus $1 \in \beta^{-1}\mathcal{L} = \mathcal{O}$.

3.4. Structure of algebraic and affine centralizers for irreducible actions

By Lemma 3.6, the centralizer $C(\alpha)$, as an additive group, is isomorphic to $\mathbb{Z}^n$ and has an additional ring structure. In the terminology of Proposition 3.7, the

centralizer $Z(\alpha)$ for an irreducible action α by toral automorphisms is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times F$.

An irreducible action α has *maximal rank* if $d = r_1 + r_2 - 1$. In this case $Z(\alpha)$ is a finite extension of α.

Notice that any affine map commuting with an action α by toral automorphisms preserves the set $\mathrm{Fix}(\alpha)$ of fixed points of the action. This set is always a subgroup of the torus and hence, for an irreducible action, always finite. The translation by any element of $\mathrm{Fix}(\alpha)$ commutes with α and thus belongs to $Z_{Aff}(\alpha)$. Furthermore, the affine centralizers $Z_{Aff}(\alpha)$ and $C_{Aff}(\alpha)$ are generated by these translations and, respectively, $Z(\alpha)$ and $C(\alpha)$.

Remark. Most of the material of this section extends to general irreducible actions of $\mathbb{Z}^d$ by automorphisms of compact connected abelian groups; a group possessing such an action must be a torus or a solenoid ([25, 26]). In the solenoid case, which includes natural extensions of $\mathbb{Z}^d$-actions by toral endomorphisms, the algebraic numbers $\lambda_1, \ldots, \lambda_d$ which appear in the constructions are not in general integers. As we mentioned in the introduction we restrict our algebraic setting here since we are able to exhibit some of the most interesting and striking new phenomena using Cartan actions and certain actions directly derived from them. However, other interesting examples appear for actions on the torus connected with not totally real algebraic number fields, actions on solenoids, and actions on zero-dimensional abelian groups (cf. e.g. [16, 24, 25, 26]).

One can also extend the setup of this section to certain classes of reducible actions. Since some of these satisfy condition $(\mathcal{R})$ basic rigidity results still hold and a number of further interesting examples can be constructed.

4. Cartan actions

4.1. Structure of Cartan actions

Of particular interest for our study are abelian groups of ergodic automorphisms of $\mathbb{T}^n$ of maximal possible rank $n-1$ (in agreement with the real rank of the Lie group $SL(n, \mathbb{R})$).

Definition. An action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ for $n \geq 3$ by ergodic automorphisms is called a *Cartan action*.

Proposition 4.1. *Let α be a Cartan action on $\mathbb{T}^n$.*

(1) *Any element of ρ_α other than identity has real eigenvalues and is hyperbolic and thus Bernoulli.*
(2) *α is irreducible.*
(3) *The centralizer of $Z(\alpha)$ is a finite extension of $\rho_\alpha(\mathbb{Z}^{n-1})$.*

Proof. First, let us point out that it is sufficient to prove the proposition for irreducible actions. For, if α is not irreducible, it has a nontrivial irreducible algebraic factor of dimension, say, $m \leq n - 1$. Since every factor of an ergodic automorphism is ergodic, we thus obtain an action of $\mathbb{Z}^{n-1}$ in $\mathbb{T}^m$ by ergodic automorphisms. By considering a restriction of this action to a subgroup of rank $m - 1$ which contains an irreducible matrix, we obtain a Cartan action on $\mathbb{T}^m$. By Statement 3 for irreducible actions, the centralizer of this Cartan action is a finite extension of $\mathbb{Z}^{m-1}$, and thus cannot contain $\mathbb{Z}^{n-1}$, a contradiction.

Now assuming that α is irreducible, take a matrix $A \in \rho_\alpha(\mathbb{Z}^{n-1})$ with irreducible characteristic polynomial f. Such a matrix exists by Proposition 3.1. It has distinct eigenvalues, say $\lambda = \lambda_1, \ldots, \lambda_n$. Consider the correspondence γ defined in (1). By Lemma 3.6 for every $B \in \rho_\alpha(\mathbb{Z}^{n-1})$ we have $\gamma(B) \in \mathcal{U}_K$, hence the group of units $\mathcal{U}_K$ in K contains a subgroup isomorphic to $\mathbb{Z}^{n-1}$. By the Dirichlet Unit Theorem the rank of the group of units in K is equal to $r_1 + r_2 - 1$, where r_1 is the number of real embeddings and r_2 is the number of pairs of complex conjugate embeddings of K into $\mathbb{C}$. Since $r_1 + 2r_2 = n$ we deduce that $r_2 = 0$, so the field K is totally real, that is all eigenvalues of A, and hence of any matrix in $\rho_\alpha(\mathbb{Z}^{n-1})$, are real. The same argument gives Statement 3, since any element of the centralizer of $\rho_\alpha(\mathbb{Z}^{n-1})$ in $GL(n, \mathbb{Z})$ corresponds to a unit in K. Hyperbolicity of matrices in $\rho_\alpha(\mathbb{Z}^{n-1})$ is proved in the same way as Lemma 3.5.

Lemma 4.2. *Let A be a hyperbolic matrix in $SL(n, \mathbb{Z})$ with irreducible characteristic polynomial and distinct real eigenvalues. Then every element of the centralizer $Z(A)$ other than $\{\pm 1\}$ is hyperbolic.*

Proof. Assume that $B \in Z(A)$ is not hyperbolic. As B is simultaneously diagonalizable with A and has real eigenvalues, it has an eigenvalue $+1$ or -1. The corresponding eigenspace is rational and A-invariant. Since A is irreducible, this eigenspace has to coincide with the whole space and hence $B = \pm 1$.

Corollary 4.3. *Cartan actions are exactly the maximal rank irreducible actions corresponding to totally real number fields.*

Corollary 4.4. *The centralizer $Z(\alpha)$ for a Cartan action α is isomorphic to $\mathbb{Z}^{n-1} \times \{\pm 1\}$.*

We will call a Cartan action α *maximal* if α is an index two subgroup in $Z(\alpha)$.

Let us point out that $Z_{Aff}(\alpha)$ is isomorphic $Z(\alpha) \times \text{Fix}(\alpha)$. Thus, the factor of $Z_{Aff}(\alpha)$ by the subgroup of finite order elements is always isomorphic to $\mathbb{Z}^{n-1}$. If α is maximal, this factor is identified with α itself. In the next Section we will show (Corollary 5.4) that for a Cartan action α on $\mathbb{T}^n$, $n \geq 3$ the isomorphism type of the pair $(Z_{Aff}(\alpha), \alpha)$ is an invariant of the measurable isomorphism. Thus, in particular, for a maximal Cartan action the order of the group $\text{Fix}(\alpha)$ is a measurable invariant.

Remark. An important geometric distinction between Cartan actions and general irreducible actions by hyperbolic automorphisms is the absence of multiple Lyapunov exponents. This greatly simplifies proofs of various rigidity properties both in the differentiable and measurable context.

4.2. Algebraically nonisomorphic maximal Cartan actions

In Section 3.3 we described a particular class of irreducible actions $\alpha_{\bar{\lambda}, \mathcal{O}}$ which is characterized by the transitivity of the action of the centralizer $C(\alpha_{\bar{\lambda}, \mathcal{O}})$ on the lattice (Proposition 3.11). In the case $\mathcal{O}_K = \mathbb{Z}[\lambda]$ there is only one such action, namely the cyclic one (Corollary 3.10). Now we will analyze this special case for totally real fields in detail and show how information about the class number of the field helps to construct algebraically nonisomorphic maximal Cartan actions. This will in particular provide examples of Cartan actions not isomorphic up to a time change to any action of the form $\alpha_{\bar{\lambda}, \mathcal{O}}$.

It is well-known that for $n = 2$ there are natural bijections between conjugacy classes of hyperbolic elements in $SL(2, \mathbb{Z})$ of a given trace, ideal classes in the corresponding real quadratic field, and congruence classes of primitive integral indefinite quadratic forms of the corresponding discriminant. This has been used by Sarnak [23] in his proof of the Prime Geodesic Theorem for surfaces of constant negative curvature (see also [13]). It follows from an old Theorem of Latimer and MacDuffee (see [17], [28], and a more modern account in [29]), that the first bijection persists for $n > 2$. Let A a hyperbolic matrix $A \in SL(n, \mathbb{Z})$ with irreducible characteristic polynomial f and distinct real eigenvalues, $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A, and $\mathcal{O}_K = \mathbb{Z}[\lambda]$. To each matrix A' with the same eigenvalues, we assign the eigenvector $v = (v_1, \ldots, v_n)$ with eigenvalue λ: $A'v = \lambda v$ with all its entries in $\mathcal{O}_K$, which can be always done, and to this eigenvector, an ideal in $\mathcal{O}_K$ with the $\mathbb{Z}$-basis $v_1, \ldots, v_n$. The described map is a bijection between the $GL(n, \mathbb{Z})$-conjugacy classes of matrices in $SL(n, \mathbb{Z})$ which have the same characteristic polynomial f and the set of ideal classes in $\mathcal{O}_K$. Moreover, it allows us to reach conclusions about centralizers as well.

Theorem 4.5. *Let $A \in SL(n, \mathbb{Z})$ be a hyperbolic matrix with irreducible characteristic polynomial f and distinct real eigenvalues, $K = \mathbb{Q}(\lambda)$ where λ is an eigenvalue of A, and $\mathcal{O}_K = \mathbb{Z}[\lambda]$. Suppose the number of eigenvalues among $\lambda_1, \ldots, \lambda_n$ that belong to K is equal to r. If the class number $h(K) > r$, then there exists a matrix $A' \in SL(n, \mathbb{Z})$ having the same eigenvalues as A whose centralizer $Z(A')$ is not conjugate in $GL(n, \mathbb{Z})$ to $Z(A)$. Furthermore, the number of matrices in $SL(n, \mathbb{Z})$ having the same eigenvalues as A with pairwise nonconjugate (in $GL(n, \mathbb{Z})$) centralizers is at least $[\frac{h(K)}{r}] + 1$, where $[x]$ is the largest integer $< x$.*

Proof. Suppose the matrix A corresponds to the ideal class I_1 with the $\mathbb{Z}$-basis

$v^{(1)}$. Then

$$Av^{(1)} = \lambda v^{(1)}.$$

Since $h(K) > 1$, there exists a matrix A_2 having the same eigenvalues which corresponds to a different ideal class I_2 with the basis $v^{(2)}$, and we have

$$A_2 v^{(2)} = \lambda v^{(2)}.$$

The eigenvectors $v^{(1)}$ and $v^{(2)}$ are chosen with all their entries in $\mathcal{O}_K$. Now assume that $Z(A_2)$ is conjugate to $Z(A)$. Then $Z(A_2)$ contains a matrix B_2 conjugate to A. Since B_2 commutes with A_2 we have $B_2 v^{(2)} = \mu_2 v^{(2)}$, and since B_2 is conjugate to A, μ_2 is one of the roots of f. Moreover, since $B_2 \in SL(n, \mathbb{Z})$ and all entries of $v^{(2)}$ are in K, $\mu_2 \in K$. Thus μ_2 is one of r roots of f which belongs to K.

From $B_2 = S^{-1}AS$ ($S \in GL(n, \mathbb{Z})$) we deduce that $\mu_2(Sv^{(2)}) = A(Sv^{(2)})$. Since I_1 and I_2 belong to different ideal classes, $Sv^{(2)} \neq kv^{(1)}$ for any k in the quotient field of $\mathcal{O}_K$, and since λ is a simple eigenvalue for A, we deduce that $\mu_2 \neq \lambda$, and thus μ_2 can take one of the $r - 1$ remaining values.

Now assume that A_3 corresponds to the third ideal class, i.e

$$A_3 v^{(3)} = \lambda v^{(3)},$$

and B_3 commutes with A_3 and is conjugate to A, and hence to B_2. Then $B_3 v^{(3)} = \mu_3 v^{(3)}$ where μ_3 is a root of f belonging to the field K. By the previous considerations, $\mu_3 \neq \lambda$ and $\mu_3 \neq \mu_2$. An induction argument shows that if the class number of K is greater than r, there exists a matrix A' such that no matrix in $Z(A')$ is conjugate to A, i.e. $Z(A')$ and $Z(A)$ are not conjugate in $GL(n, \mathbb{Z})$.

Since A' has the same characteristic polynomial as A, continuing the same process, we can find not more than r matrices representing different ideal classes having centralizers conjugate to $Z(A')$, and the required estimate follows.

5. Measure-theoretic rigidity of conjugacies, centralizers, and factors

5.1. Conjugacies

Suppose α and α' are measurable actions of the same group G by measure-preserving transformations of the spaces (X, μ) and (Y, ν), respectively. If $H : (X, \mu) \to (Y, \nu)$ is a metric isomorphism (conjugacy) between the actions then the lift of the measure μ onto the graph $H \subset X \times Y$ coincides with the lift of ν to graph H^{-1}. The resulting measure η is a very special case of a *joining* of α and α': it is invariant under the diagonal (product) action $\alpha \times \alpha'$ and its projections to X and Y coincide with μ and ν, respectively. Obviously the projections establish metric isomorphism of the action $\alpha \times \alpha'$ on $(X \times Y, \eta)$ with α on (X, μ) and α' on (Y, ν) correspondingly.

Similarly, if an automorphism $H : (X, \mu) \to (X, \mu)$ commutes with the action α, the lift of μ to graph $H \subset X \times X$ is a self-joining of α, i.e. it is $\alpha \times \alpha$-invariant and both of its projections coincide with μ. Thus an information about invariant measures of the products of different actions as well as the product of an action with itself may give an information about isomorphisms and centralizers.

The use of this joining construction in order to deduce rigidity of isomorphisms and centralizers from properties of invariant measures of the product was first suggested in this context to the authors by J.-P. Thouvenot.

In both cases the ergodic properties of the joining would be known because of the isomorphism with the original actions. Very similar considerations apply to the actions of semi-groups by noninvertible measure-preserving transformations. We will use the following corollary of the results of [11].

Theorem 5.1. *Let α be an action of $\mathbb{Z}^2$ by ergodic toral automorphisms and let μ be a weakly mixing α-invariant measure such that for some $\mathbf{m} \in \mathbb{Z}^2$, $\alpha^{\mathbf{m}}$ is a K-automorphism. Then μ is a translate of Haar measure on an α-invariant rational subtorus.*

Proof. We refer to Corollary 5.2' from ([11], "Corrections..."). According to this corollary the measure μ is an extension of a zero entropy measure for an algebraic factor of smaller dimension with Haar conditional measures in the fiber. But since α contains a K-automorphism it does not have non-trivial zero entropy factors. Hence the factor in question is the action on a single point and μ itself is a Haar measure on a rational subtorus.

Conclusion of Theorem 5.1 obviously holds for any action of $\mathbb{Z}^d$, $d \geq 2$ which contains a subgroup $\mathbb{Z}^2$ satisfying assumptions of Theorem 5.1. Thus we can deduce the following result which is central for our constructions.

Theorem 5.2. *Let α and α' be two actions of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ correspondingly and assume that α satisfies condition $(\mathcal{R})$. Suppose that $H : \mathbb{T}^n \to \mathbb{T}^{n'}$ is a measure-preserving isomorphism between (α, λ) and (α', λ), where λ is Haar measure. Then $n = n'$ and H coincides* (mod 0) *with an affine automorphism on the torus $\mathbb{T}^n$, and hence α and α' are algebraically isomorphic.*

Proof. First of all, condition $(\mathcal{R})$ is invariant under metric isomorphism, hence α' also satisfies this condition. But ergodicity with respect to Haar measure can also be expressed in terms of the eigenvalues; hence $\alpha \times \alpha'$ also satisfies $(\mathcal{R})$. Now consider the joining measure η on graph $H \subset \mathbb{T}^{n+n'}$. The conditions of Theorem 5.1 are satisfied for the invariant measure η of the action $\alpha \times \alpha'$. Thus η is a translate of Haar measure on a rational $\alpha \times \alpha'$-invariant subtorus $\mathbb{T}' \subset \mathbb{T}^{n+n'} = \mathbb{T}^n \times \mathbb{T}^{n'}$. On the other hand we know that projections of $\mathbb{T}'$ to both $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ preserve Haar measure and are one-to-one. The partitions of $\mathbb{T}'$ into pre-images of

points for each of the projections are measurable partitions and Haar measures on elements are conditional measures. This implies that both projections are onto, both partitions are partitions into points, and hence $n = n'$ and $\mathbb{T}' = \operatorname{graph} I$, where $I : \mathbb{T}^n \to \mathbb{T}^n$ is an affine automorphism which has to coincide (mod 0) with the measure-preserving isomorphism H.

Since a time change is in a sense a trivial modification of an action we are primarily interested in distinguishing actions up to a time change. The corresponding rigidity criterion follows immediately from Theorem 5.2.

Corollary 5.3. *Let α and α' be two actions of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$, respectively, and assume that α satisfies condition $(\mathcal{R})$. If α and α' are measurably isomorphic up to a time change then they are algebraically isomorphic up to a time change.*

5.2. Centralizers

Applying Theorem 5.2 to the case $\alpha = \alpha'$ we immediately obtain rigidity of the centralizers.

Corollary 5.4. *Let α be an action of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$. Any invertible Lebesgue measure-preserving transformation commuting with α coincides (mod 0) with an affine automorphism of $\mathbb{T}^n$.*

Any affine transformation commuting with α preserves the finite set of fixed points of the action. Hence the centralizer of α in affine automorphisms has a finite index subgroups which consist of automorphisms and which corresponds to the centralizer of $\rho_\alpha(\mathbb{Z}^d)$ in $GL(n, \mathbb{Z})$.

Thus, in contrast with the case of a single automorphism, the centralizer of such an action α is not more than countable, and can be identified with a finite extension of a certain subgroup of $GL(n, \mathbb{Z})$. As an immediate consequence we obtain the following result.

Proposition 5.5. *For any d and k, $2 \le d \le k$, there exists a $\mathbb{Z}^d$-action by hyperbolic toral automorphisms such that its centralizer in the group of Lebesgue measure-preserving transformations is isomorphic to $\{\pm 1\} \times \mathbb{Z}^k$.*

Proof. Consider a hyperbolic matrix $A \in SL(k+1, \mathbb{Z})$ with irreducible characteristic polynomial and real eigenvalues such that the origin is the only fixed point of F_A. Consider a subgroup of $Z(A)$ isomorphic to $\mathbb{Z}^d$ and containing A as one of its generators. This subgroup determines an embedding $\rho : \mathbb{Z}^d \to SL(k+1, \mathbb{Z})$. Since $d \ge 2$ and by Proposition 4.2, all matrices in $\rho(\mathbb{Z}^d)$ are hyperbolic and hence ergodic, condition $(\mathcal{R})$ is satisfied. Hence by Corollary 5.4, the measure-theoretic

centralizer of the action α_ρ coincides with its algebraic centralizer, which, in turn, and obviously, coincides with centralizer of the single automorphism F_A isomorphic to $\{\pm 1\} \times \mathbb{Z}^k$.

5.3. Factors, noninvertible centralizers and weak isomorphism

A small modification of the proof of Theorem 5.2 produces a result about rigidity of factors.

Theorem 5.6. *Let α and α' be two actions of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ and $\mathbb{T}^{n'}$ respectively, and assume that α satisfies condition $(\mathcal{R})$. Suppose that $H : \mathbb{T}^n \to \mathbb{T}^{n'}$ is a Lebesgue measure-preserving transformation such that $H \circ \alpha = \alpha' \circ H$. Then α' also satisfies $(\mathcal{R})$ and H coincides (mod 0) with an epimorphism $h : \mathbb{T}^n \to \mathbb{T}^{n'}$ followed by translation. In particular, α' is an algebraic factor of α.*

Proof. Since α' is a measurable factor of α, every element which is ergodic for α is also ergodic for α'. Hence α' also satisfies condition $(\mathcal{R})$. As before consider the product action $\alpha \times \alpha'$ which now by the same argument also satisfies $(\mathcal{R})$. Take the $\alpha \times \alpha'$ invariant measure $\eta = (\mathrm{Id} \times H)_* \lambda$ on graph H. This measure provides a joining of α and α'. Since $(\alpha \times \alpha', (\mathrm{Id} \times H)_* \lambda)$ is isomorphic to (α, λ) the conditions of Corollary 5.1 are satisfied and η is a translate of Haar measure on an invariant rational subtorus $\mathbb{T}'$. Since $\mathbb{T}'$ projects to the first coordinate one-to-one we deduce that H is an algebraic epimorphism (mod 0) followed by a translation.

Similarly to the previous section the application of Theorem 5.6 to the case $\alpha = \alpha'$ gives a description of the centralizer of α in the group of all measure-preserving transformations.

Corollary 5.7. *Let α be an action of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$. Any Lebesgue measure-preserving transformation commuting with α coincides (mod 0) with an affine map on $\mathbb{T}^n$.*

Now we can obtain the following strengthening of Proposition 2.1 for actions satisfying condition $(\mathcal{R})$ which is one of the central conclusions of this paper.

Theorem 5.8. *Let α be an action of $\mathbb{Z}^d$ by automorphisms of $\mathbb{T}^n$ satisfying condition $(\mathcal{R})$ and α' another $\mathbb{Z}^d$-action by toral automorphisms. Then (α, λ) is weakly isomorphic to (α', λ') if and only if ρ_α and $\rho_{\alpha'}$ are isomorphic over $\mathbb{Q}$, i.e. if α and α' are finite algebraic factors of each other.*

Proof. By Theorem 5.6, α and α' are algebraic factors of each other. This implies that α' acts on the torus of the same dimension n and hence both algebraic factor-maps have finite fibres. Now the statement follows from Proposition 2.1.

5.4. Distinguishing weakly isomorphic actions

Similarly we can translate criteria for algebraic conjugacy of weakly algebraically conjugate actions to the measurable setting.

Theorem 5.9. *If α is an irreducible cyclic action of $\mathbb{Z}^d$, $d \geq 2$, on $\mathbb{T}^n$ and α' is a non-cyclic $\mathbb{Z}^d$-action by toral automorphisms. Then α and α' are not measurably isomorphic up to a time change.*

Proof. Since action α satisfies condition $(\mathcal{R})$ (Corollary 3.2) we can apply Theorem 5.8 and conclude that we only need to consider the case when ρ_α and $\rho_{\alpha'}$ are isomorphic over $\mathbb{Q}$ up to a time change. But then, by Proposition 3.3, α and α' are not algebraically isomorphic up to a time change and hence, by Corollary 5.3, they are not measurably isomorphic up to a time change.

Combining Proposition 3.9 and Corollary 5.3 we immediately obtain rigidity for the minimal irreducible models.

Corollary 5.10. *Assume that $\mathcal{O} \supsetneq \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. Then the action $\alpha_{\bar{\lambda}, \mathcal{O}}$ is not measurably isomorphic up to a time change to $\alpha_{\bar{\lambda}}^{\min}$. In particular, if $\mathcal{O}_K \supsetneq \mathbb{Z}[\lambda_1, \ldots, \lambda_d]$, then the actions $\alpha_{\bar{\lambda}}^{\max}$ and $\alpha_{\bar{\lambda}}^{\min}$ are not measurably isomorphic up to a time change.*

6. Examples

Now we proceed to produce examples of actions for which the entropy data coincide but which are not algebraically isomorphic, and hence by Theorem 5.2 not measure-theoretically isomorphic.

6.1. Weakly nonisomorphic actions

In this section we consider actions which are not algebraically isomorphic over $\mathbb{Q}$ (or, equivalently, over $\mathbb{R}$) and hence by Theorem 5.8 are not even weakly isomorphic. The easiest way is as follows.

Example 1a. Start with any action α of $\mathbb{Z}^d$, $d \geq 2$, by ergodic automorphisms of $\mathbb{T}^n$. We may double the entropies of all its elements in two different ways: by considering the Cartesian square $\alpha \times \alpha$ acting on $\mathbb{T}^{2n}$, and by taking second powers of all elements: $\alpha_2^{\mathbf{n}} = \alpha^{2\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$. Obviously $\alpha \times \alpha$ is not algebraically isomorphic to α_2, since, for example, they act on tori of different dimension. Hence by Theorem 5.2 $(\alpha \times \alpha, \lambda)$ is not metrically isomorphic to (α_2, λ) either.

Now we assume that α contains an automorphism F_A where A is hyperbolic

with an irreducible characteristic polynomial and distinct positive real eigenvalues. In this case it is easy to find an invariant distinguishing the two actions, namely, the algebraic type of the centralizer of the action in the group of measure-preserving transformations. By Corollary 5.4, the centralizer of α in the group of measure-preserving transformations coincides with the centralizer in the group of affine maps, which is a finite extension of the centralizer in the group of automorphisms. By the Dirichlet Unit Theorem, the centralizer of $Z(\alpha_2)$ in the group of automorphisms of the torus is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{n-1}$, whereas the centralizer of $\alpha \times \alpha$ contains the $\mathbb{Z}^{2(n-1)}$-action by product transformations $\alpha^{\mathbf{n}_1} \times \alpha^{\mathbf{n}_2}$, $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{Z}^{n-1}$. In fact, the centralizer of $\alpha \times \alpha$ can be calculated explicitly:

Proposition 6.1. *Let λ be an eigenvalue of A. Then $K = \mathbb{Q}(\lambda)$ is a totally real algebraic field. If its ring of integers $\mathcal{O}_K$ is equal to $\mathbb{Z}[\lambda]$ then the centralizer of $\alpha \times \alpha$ in $GL(2n, \mathbb{Z})$ is isomorphic to the group $GL(2, \mathcal{O}_K)$, i.e. the group of 2×2 matrices with entries in $\mathcal{O}_K$ whose determinant is a unit in $\mathcal{O}_K$.*

Proof. First we notice that a matrix in block form $B = \left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right)$ with $X, Y, Z, T \in M(n, \mathbb{Z})$ commutes with $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ if an only if X, Y, Z, T commute with A and can thus be identified with elements of $\mathcal{O}_K$. In this case B can be identified with a matrix in $M(2, \mathcal{O}_K)$. Since $\det\left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right) = \det(XT - YZ) = \pm 1$ (cf. [5]), the norm of the determinant of the 2×2 matrix corresponding to B is equal ± 1. Hence this determinant is a unit in $\mathcal{O}_K$, and we obtain the desired isomorphism.

It is not difficult to modify Example 1a to obtain weakly nonisomorphic actions with the same entropy on the torus of the same dimension.

Example 1b. For a natural number k define the action α_k similarly to α_2: $\alpha_k^{\mathbf{n}} = \alpha^{k\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$.

The actions $\alpha_3 \times \alpha$ and $\alpha_2 \times \alpha_2$ act on $\mathbb{T}^{2n}$, have the same entropies for all elements and are not isomorphic.

As before, we can see that centralizers of these two actions are not isomorphic. In particular, the centralizer of $\alpha_3 \times \alpha$ is abelian since it has simple eigenvalues, while the centralizer of $\alpha_2 \times \alpha_2$ is not.

6.2. Cartan actions distinguished by cyclicity or maximality

We give two examples which illustrate the method of Section 3.3. They provide weakly algebraically isomorphic Cartan actions of $\mathbb{Z}^2$ on $\mathbb{T}^3$ which are not algebraically isomorphic even up to a time change (i.e. a linear change of coordinates in $\mathbb{Z}^2$) by Proposition 3.9. These examples utilize the existence of number fields $K = \mathbb{Q}(\lambda)$ and units $\bar{\lambda} = (\lambda_1, \lambda_2)$ in them for which $\mathcal{O}_K \neq \mathbb{Z}[\lambda_1, \lambda_2]$. In each example one action has a form $\alpha_{\bar{\lambda}}^{\min}$ and the other $\alpha_{\bar{\lambda}}^{\max}$. Hence by Corollary 5.10 they are not measurably isomorphic up to a time change

In other words, in each example one action, namely, $\alpha_{\bar\lambda}^{\min}$, is a cyclic Cartan action, and the other is not.

We will also show that in these examples the conjugacy type of the pair $(Z(\alpha), \alpha)$ distinguishes weakly isomorphic actions. Let us point out that a non-cyclic action for example $\alpha_{\bar\lambda}^{\max}$ may be maximal, for example when fundamental units lie in a proper subring of $\mathcal{O}_K$. However in our examples centralizers for the cyclic actions will be different and thus will serve as a distinguishing invariant.

The information about cubic fields is either taken from [4] or obtained with the help of the computer package Pari-GP. Some calculations were made by Arsen Elkin during the REU program at Penn State in summer of 1999.

We construct two $\mathbb{Z}^2$-actions, α, generated by commuting matrices A and B, and α', generated by commuting matrices A' and B' in $GL(3, \mathbb{Z})$. These actions are weakly algebraically isomorphic by Proposition 3.8 since they are produced with the same set of units on two different orders, $\mathbb{Z}[\lambda]$ and $\mathcal{O}_K$, but not algebraically isomorphic by Proposition 3.9. In these examples the action α is cyclic by Corollary 3.10 and will be shown to be a maximal Cartan action. Thus $Z(\alpha) = \alpha \times \{\pm\mathrm{Id}\}$. The action α' is not maximal, specifically, $Z(\alpha')/\{\pm\mathrm{Id}\}$ is a nontrivial finite extension of α'.

Example 2a. Let K be a totally real cubic field given by the irreducible polynomial $f(x) = x^3 + 3x^2 - 6x + 1$, i.e. $K = \mathbb{Q}(\lambda)$ where λ is one of its roots. The discriminant of K is equal to 81, hence its Galois group is cyclic, and $[\mathcal{O}_K : \mathbb{Z}[\lambda]] = 3$. The algebraic integers $\lambda_1 = \lambda$ and $\lambda_2 = 2 - 4\lambda - \lambda^2$ are units with $f(\lambda_1) = f(\lambda_2) = 0$. The minimal order in K containing λ_1 and λ_2 is $\mathbb{Z}[\lambda_1, \lambda_2] = \mathbb{Z}[\lambda]$, and the maximal order is $\mathcal{O}_K$. A basis in fundamental units is $\epsilon = \frac{\lambda^2 + 5\lambda + 1}{3}$ and $\epsilon - 1$, hence $\mathcal{U}_K$ is not contained in $\mathbb{Z}[\lambda]$.

With respect to the basis $\{1, \lambda, \lambda^2\}$ in $\mathbb{Z}[\lambda]$, multiplications by λ_1 and λ_2 are given by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 6 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -4 & -1 \\ 1 & -4 & -1 \\ 1 & -5 & -1 \end{pmatrix},$$

respectively (if acting from the right on row-vectors). A direct calculation shows that this action is maximal.

With respect to the basis $\{-\frac{2}{3} + \frac{5}{3}\lambda + \frac{1}{3}\lambda^2, -\frac{1}{3} + \frac{7}{3}\lambda + \frac{2}{3}\lambda^2\}$ in $\mathcal{O}_K$, multiplications by λ_1 and λ_2 are given by the matrices

$$A' = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 2 \\ 2 & 5 & -2 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -4 & -2 \end{pmatrix}.$$

We have $A' = VAV^{-1}$, $B' = VBV^{-1}$ for $V = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -3 & 0 \\ 1 & -4 & -2 \end{pmatrix}$. Since A is a companion matrix of f, $\alpha = \langle A, B \rangle$ has a cyclic element in $\mathbb{Z}^3$. If A' also had a cyclic element $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$, then the vectors

$$\mathbf{m} = (m_1, m_2, m_3), \quad \mathbf{m}A' = (m_1 - m_2 + 2m_3, 2m_1 - 2m_2 + 5m_3, -m_1 + 2m_2 - 2m_3)$$
$$\mathbf{m}(A')^2 = (-3m_1 + 5m_2 - 7m_3, -7m_1 + 12m_2 - 16m_3, 5m_1 - 7m_2 + 12m_3),$$

would have to generate $\mathbb{Z}^3$ or, equivalently

$$\det \begin{pmatrix} m_1 & m_2 & m_3 \\ m_1-m_2+2m_3 & 2m_1-2m_2+5m_3 & -m_1+2m_2-2m_3 \\ -3m_1+5m_2-7m_3 & -7m_1+12m_2-16m_3 & 5m_1-7m_2+12m_3 \end{pmatrix}$$
$$= 3m_1^3 + 18m_1^2m_3 - 9m_1m_2^2 - 9m_1m_2m_3$$
$$+ 27m_1m_3^2 + 3m_2^3 - 9m_2m_3^2 + 3m_3^3 = 1.$$

This contradiction shows that A' has no cyclic vector, and since $B' = 2 - 4A' - A'^2$, the action α' is not cyclic. In this example both actions α and α' have a single fixed point $(0,0,0)$, hence their linear and affine centralizers coincide, and by Corollary 5.3 α and α' are not measurably isomorphic up to a time change.

The action α' is not maximal because $Z(\alpha')$ contains fundamental units.

Example 2b. Let us consider a totally real cubic field K given by the irreducible polynomial $f(x) = x^3 - 7x^2 + 11x - 1$. Thus $K = \mathbb{Q}(\lambda)$ where λ is one of its roots. In this field the ring of integers $\mathcal{O}_K$ has basis $\{1, \lambda, \frac{1}{2}\lambda^2 + \frac{1}{2}\}$ and hence $[\mathcal{O}_K : \mathbb{Z}[\lambda]] = 2$. The fundamental units in $\mathcal{O}_K$ are $\{\frac{1}{2}\lambda^2 - 2\lambda + \frac{1}{2}, \lambda - 2\}$. We choose the units $\lambda = \lambda_1 = (\frac{1}{2}\lambda^2 - 2\lambda + \frac{1}{2})^2$ and $\lambda_2 = \lambda - 2$ which are contained in both orders, $\mathcal{O}_K$ and $\mathbb{Z}[\lambda]$.

In $\mathbb{Z}[\lambda]$ we consider the basis $\{1, \lambda, \lambda^2\}$ relative to which the multiplication by λ_1 is represented by the companion matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -11 & 7 \end{pmatrix}$ and multiplication by λ_2 is represented by the matrix $B = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & -11 & 5 \end{pmatrix}$.

For $\mathcal{O}_K$ with the basis $\{1, \lambda, \frac{1}{2}\lambda^2 + \frac{1}{2}\}$ multiplications by λ_1 and λ_2 are represented by the matrices $A' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ -3 & -5 & 7 \end{pmatrix}$ and $B' = \begin{pmatrix} -2 & 1 & 0 \\ -1 & -2 & 2 \\ -3 & -5 & 5 \end{pmatrix}$.

It can be seen directly that α and α' are not algebraically conjugate up to a time change since A' is a square of a matrix from $SL(3, \mathbb{Z})$: $A' = \begin{pmatrix} 0 & -2 & 1 \\ -1 & -5 & 3 \\ -2 & -9 & 6 \end{pmatrix}^2$, while A is not a square of a matrix in $GL(3, \mathbb{Z})$, which is checked by reducing modulo 2. In this case it is also easily seen that the action α' is not cyclic since the corresponding determinant is divisible by 2. The action α has 2 fixed points on $\mathbb{T}^3$: $(0,0,0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, while the action α' has 4 fixed points: $(0,0,0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(0, 0, \frac{1}{2})$. Hence the affine centralizer of α is $Z(\alpha) \times \mathbb{Z}/2\mathbb{Z}$, and the affine centralizer of α' is $Z(\alpha') \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

By Lemma 4.2, the group of elements of finite order in $Z_{Aff}(\alpha)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in $Z_{Aff}(\alpha')$ it is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The indices of each action in its affine centralizer are $[Z_{Aff}(\alpha) : \alpha] = 4$ and $[Z_{Aff}(\alpha') : \alpha'] = 16$.

This gives two alternative arguments that the actions are not measurably isomorphic up to a time change.

6.3. Nonisomorphic maximal Cartan actions

We find examples of weakly algebraically isomorphic maximal Cartan actions which are not algebraically isomorphic up to time change. For such an action α the structure of the pair $(Z(\alpha), \alpha)$ is always the same: $Z(\alpha)$ is isomorphic as a group to $\alpha \times \{\pm \text{Id}\}$. The algebraic tool which allows to distinguish the actions is described in Section 4.2. In particular due to Theorem 4.5 we may conclude existence of such actions from certain information about the class number and the Galois group.

Let A a hyperbolic matrix $A \in SL(n, \mathbb{Z})$ with irreducible characteristic polynomial f, and distinct real eigenvalues, $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A. Nontrivial time changes in a Cartan action which includes A exist only if another root belongs to the field $\mathbb{Q}(\lambda)$ (Proposition 3.8). For, the image B of A under such a time change must have the same characteristic polynomial as A and hence $\gamma(B) \in \mathbb{Q}(\lambda)$ is the root in question. For $n = 3$ this situation correspond to the Galois group of the field being cyclic.

Example 3a. An example for $n = 3$ can be obtained from a totally real cubic field with class number 2 and the Galois group S_3. The class number 2 guarantees that the actions obtained from two different ideal classes are not isomorphic and the Galois group S_3 guarantees that there are no nontrivial time changes.

The smallest discriminant for such a field is 1957 ([4], Table B4), and it can be represented as $K = \mathbb{Q}(\lambda)$ where λ is a unit in K with minimal polynomial $f(x) = x^3 - 2x^2 - 8x - 1$. In this field the ring of integers $\mathcal{O}_K = \mathbb{Z}[\lambda]$ and the fundamental units are $\lambda_1 = \lambda$ and $\lambda_2 = \lambda + 2$. Two actions are constructed with this set of units (fundamental, hence multiplicatively independent) on two different lattices, $\mathcal{O}_K$ with the basis $\{1, \lambda, \lambda^2\}$, representing the principal ideal class, and $\mathcal{L}$ with the basis $\{2, 1 + \lambda, 1 + \lambda^2\}$ representing to the second ideal class. Notice that the units λ_1 and λ_2 do not belong to $\mathcal{L}$, but $\mathcal{L}$ is a $\mathbb{Z}[\lambda]$-module. The first action α is generated by the matrices $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 8 & 4 \end{pmatrix}$ which represent multiplication by λ_1 and λ_2, respectively, on $\mathcal{O}_K$. The second action α' is generated by matrices $A' = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -5 & 9 & 2 \end{pmatrix}$ and $B' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -5 & 9 & 5 \end{pmatrix}$ which represent multiplication by λ_1 and λ_2, respectively, on $\mathcal{L}$ in the given basis. By Proposition 3.8 these actions are weakly algebraically isomorphic. By Theorem 4.5 they are not algebraically isomorphic. Since the Galois group is S_3 there are no nontrivial time changes which produce conjugacy over $\mathbb{Q}$. Therefore, by Theorem 5.2 the actions are not measurably isomorphic.

It is interesting to point out that for actions α and α' the affine centralizers $Z_{Aff}(\alpha)$ and $Z_{Aff}(\alpha')$ are not isomorphic as abstract groups. The action α has 2 fixed points on $\mathbb{T}^3$: $(0, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, while the action α' has a single fixed point $(0, 0, 0)$. Hence $Z_{Aff}(\alpha)$ is isomorphic to $Z(\alpha) \times \mathbb{Z}/2\mathbb{Z}$, $Z_{Aff}(\alpha')$ is isomorphic to $Z(\alpha')$. As abstract groups, $Z_{Aff}(\alpha) \approx \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $Z_{Aff}(\alpha') \approx \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$.

Hence by Corollary 5.4 the measurable centralizers of α and α' are not conjugate in the group of measure-preserving transformation providing a distinguishing invariant of measurable isomorphism.

Example 3b. This example is obtained from a totally real cubic field with class number 3, Galois group S_3, and discriminant 2597. It can be represented as $K = \mathbb{Q}(\lambda)$ where λ is a unit in K with minimal polynomial $f(x) = x^3 - 2x^2 - 8x + 1$. In this field the ring of integers $\mathcal{O}_K = \mathbb{Z}[\lambda]$ and the fundamental units are $\lambda_1 = \lambda$ and $\lambda_2 = \lambda + 2$. Three actions are constructed with this set of units on three different lattices, $\mathcal{O}_K$ with the basis $\{1, \lambda, \lambda^2\}$, representing the principal ideal class, $\mathcal{L}$ with the basis $\{2, 1 + \lambda, 1 + \lambda^2\}$ representing the second ideal class, and $\mathcal{L}^2$ with the basis $\{4, 3 + \lambda, 3 + \lambda^2\}$ representing the third ideal class.

Multiplications by λ_1 and λ_2 generate the following three weakly algebraically isomorphic actions which are not algebraically isomorphic by Theorem 4.5 even up to a time change, and therefore not measurably isomorphic:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 8 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 8 & 4 \end{pmatrix};$$

$$A' = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -6 & 9 & 2 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -6 & 9 & 4 \end{pmatrix};$$

$$A'' = \begin{pmatrix} -3 & 4 & 0 \\ -3 & 3 & 1 \\ -10 & 11 & 2 \end{pmatrix} \quad \text{and} \quad B'' = \begin{pmatrix} -1 & 4 & 0 \\ -3 & 5 & 1 \\ -10 & 11 & 4 \end{pmatrix}.$$

Each action has 2 fixed point in $\mathbb{T}^3$, $(0,0,0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence all affine centralizers are isomorphic as abstract groups to $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 3c Finally we give an example of two nonisomorphic maximal Cartan actions which come from the vector of fundamental units $\bar{\lambda} = (\lambda_1, \lambda_2)$ in a totally real cubic field K such that $\mathbb{Z}(\lambda_1, \lambda_2) \neq \mathcal{O}_K$. Thus the whole group of units does not generate the ring $\mathcal{O}_K$. Both actions $\alpha_{\bar{\lambda}}^{\min}$ and $\alpha_{\bar{\lambda}}^{\max}$ of the group $\mathbb{Z}^2$ are maximal Cartan actions by Lemma 3.6. However by Corollary 3.10 the former is cyclic and the latter is not and hence they are not measurably isomorphic up to a time change by Corollary 5.10.

For a specific example we pick the totally real cubic field $K = \mathbb{Q}(\alpha)$ with class number 1 discriminant 1304 given by the polynomial $x^3 - x^2 - 11x - 1$. For this field we have $[\mathcal{O}_K : \mathbb{Z}(\alpha)] = 2$. Generators in $\mathcal{O}_K$ can be taken to be $\{1, \alpha, \beta = \frac{\alpha^2 + 1}{2}\}$. Fundamental units are $\lambda_1 = -\alpha$, $\lambda_2 = -5 + 14\alpha + 10\beta = 14\alpha + 5\alpha^2 \in \mathbb{Z}[\alpha]$. Thus the whole group of units lies in $\mathbb{Z}[\lambda]$. To construct the generators for two non-isomorphic action $\alpha_{\bar{\lambda}}^{\min}$ and $\alpha_{\bar{\lambda}}^{\max}$ we write multiplications by λ_1 and λ_2 in bases $\{1, \alpha, \alpha^2\}$ and $\{1, \alpha, \beta\}$, correspondingly. The resulting matrices are:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 11 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 14 & 5 \\ 5 & 55 & 19 \\ 19 & 214 & 74 \end{pmatrix},$$

$$A' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -6 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -5 & 14 & 10 \\ -14 & 55 & 38 \\ -30 & 114 & 79 \end{pmatrix}.$$

The first action has only one fixed point, the origin; the second has four fixed points $(0,0,0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(0, 0, \frac{1}{2})$. Thus we have an example of

two maximal Cartan actions of $\mathbb{Z}^2$ which have nonisomorphic affine and hence measurable centralizers.

References

[1] R. L. Adler and B. Weiss, Entropy, a complete metric invariant for automorphisms of the torus, *Proc. Nat. Acad. Sci.* **57** (1967), 1573–1576.

[2] D. Berend, Multi-invariant sets on tori, *Trans. Amer. Math. Soc.* **280** (1983), 509–532.

[3] Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, New York, 1966.

[4] H. Cohen, *A course in computational algebraic number theory*, Springer, Berlin–Heidelberg–New York, 1996.

[5] F. R. Gantmacher, *Theory of matrices*, vol. 1, Chelsea, New York, 1959; vol. 2, Chelsea, New York, 1960.

[6] P. R. Halmos, On automorphisms of compact groups, *Bull. Amer. Math. Soc.* **49** (1943), 619–624.

[7] B. Kalinin and A. Katok, Measurable rigidity and disjointness for $\mathbb{Z}^k$ actions by toral automorphisms, *Ergodic Theory & Dynam Sys.* **22** (2002), no. 2, 507–523.

[8] A. Katok and K. Schmidt, The cohomology of expansive Z^d actions by automorphisms of compact abelian groups, *Pacific Jour. Math.* **170** (1995), 105–142.

[9] A. Katok and R. J. Spatzier, First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, *Publ. Math. IHES* **79** (1994), 131–156.

[10] A. Katok and R. J. Spatzier, Subelliptic estimates of polynomial differential operators and applications to rigidity of abelian actions, *Math. Res. Lett.* **1** (1994), 193–202.

[11] A. Katok and R. J. Spatzier, Invariant measures for higher rank hyperbolic abelian actions, *Ergod. Th. & Dynam. Sys.* **16** (1996), 751–778; Corrections: *Ergod. Th. & Dynam. Sys.* **18** (1998), 503–507.

[12] A. Katok and R. J. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions*, *Proc. Steklov Inst. Math.* **216** (1997), 287–314.

[13] S. Katok, Coding of closed geodesics after Gauss and Morse, *Geom. Dedicata* **63** (1996), 123–145.

[14] I. Katznelson, Ergodic automorphisms of T^n are Bernoulli shifts, *Israel J. Math.* **10** (1971), 186–195.

[15] B. Kitchens and K. Schmidt, Automorphisms of compact groups, *Ergod. Th. & Dynam. Sys.* **9** (1989), 691–735.

[16] B. Kitchens and K. Schmidt, Isomorphism rigidity of irreducible algebraic $\mathbf{Z}^d$-actions, *Invent. Math.* **142** (2000), 559–577.

[17] Latimer and C. C. MacDuffee, A correspondence between classes of ideals and classes of matrices, *Ann. Math.* **74** (1933), 313–316.

[18] D. Lind, K. Schmidt and T. Ward, Mahler measure and entropy for commuting automorphisms of compact groups, *Invent. Math.* **101** (1990), 593–629.

[19] D. Ornstein, Bernoulli shifts with the same entropy are isomorphic, *Adv. Math.* **4** (1970), 337–352.

[20] M. Ratner, Rigidity of horocycle flows, *Ann. Math.* **115**, (1982), 597–614.

[21] M. Ratner, Factors of horocycle flows, *Ergod. Th. & Dynam. Sys.* **2** (1982), 465–489.

[22] M. Ratner, Horocycle flows, joinings and rigidity of products, *Ann. Math.* **118** (1983), 277–313.

[23] P. Sarnak, *Prime geodesic theorems*, Ph.D. thesis, Stanford, 1980.

[24] K. Schmidt, Automorphisms of compact abelian groups and affine varieties, *Proc. London Math. Soc.* **61** (1990), 480–496.

[25] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel–Berlin–Boston, 1995.

[26] K. Schmidt, Measurable rigidity of algebraic $\mathbf{Z}^d$-actions, *Smooth ergodic theory and its applications* (Seattle, WA, 1999), 661–676, Proc. Sympos. Pure Math. **69**, Amer. Math. Soc., Providence, RI, 2001.

[27] A. Starkov, First cohomology group, mixing and minimal sets of commutative group of algebraic action on torus, Dynamical systems, 7, *J. Math. Sci. (New York)* **95** (1999), no. 5, 2576–2582.

[28] O. Taussky, Introduction into connections between algebraic number theory and integral matrices, Appendix to: H. Cohn, *A classical invitation to algebraic numbers and class field*, Springer, New York, 1978.

[29] D. I. Wallace, Conjugacy classes of hyperbolic matrices in $SL(n, \mathbb{Z})$ and ideal classes in an order, *Trans. Amer. Math. Soc.* **283** (1984), 177–184.

Anatole Katok   Svetlana Katok
The Pennsylvania State University   The Pennsylvania State University
Department of Mathematics   Department of Mathematics
University Park   University Park
PA 16802   PA 16802
USA   USA
e-mail: katok_a@math.psu.edu   e-mail: katok_s@math.psu.edu

Klaus Schmidt
University of Vienna
Mathematics Institute
Strudlhofgasse 4
A–1090 Vienna
Austria

and

Erwin Schrödinger Institute for Mathematical Physics
Boltzmanngasse 9
A–1090 Vienna
Austria
e-mail: klaus.schmidt@univie.ac.at