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Closed geodesics, periods and arithmetic of modular forms

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Introduction

In this paper we study modular forms on Fuchsian groups of the first kind. Our original motivation was to find an analog of Fourier coefficients for a modular form on a compact surface $\Gamma\backslash\mathscr{H}$, where

$$\mathscr{H}=\{z\in\mathbb{C},\ \operatorname{Im} z>0\}$$

is the upper half-plane and Γ is a discrete cocompact subgroup of $SL(2,\mathbb{R})$ acting by fractional linear transformations. We demonstrate that in this case the periods over closed geodesics play a role somewhat similar to that of Fourier coefficients of modular forms on $SL(2,\mathbb{Z})$ and its congruence subgroups. More specifically, those periods uniquely determine a modular form (Theorem 2). This result is valid for cusp forms on any Fuchsian group of the first kind with or without cusps and is closely related to the study of relative Poincaré series associated to closed geodesics. For each integer $k\geqq 2$ and each closed geodesic $[\gamma_0]$ we define special cusp forms of weight $2k$, called relative Poincaré series $\theta_{k,[\gamma_0]}$ and prove that they generate the whole space $S_{2k}(\Gamma)$ of cusp forms (Theorem 1). In § 3 we give an expression for periods of a relative Poincaré series over closed geodesics in purely geometrical terms through the intersection of the corresponding geodesics (Theorem 3). An application of Theorems 1 and 3 to arithmetic subgroups of $SL(2,\mathbb{R})$ gives two natural rational structures on $S_{2k}(\Gamma)$ (Theorem 4).

The relative Poincaré series have been studied for general Γ by Petersson [18, 19] and Hejhal [5] ($g\geqq 2$). Wolpert [23] gives a basis of $S_4(\Gamma)$ for $g\geqq 2$. For $SL(2,\mathbb{Z})$ the relative Poincaré series have been studied by Zagier [25], Kohnen [8], Kohnen and Zagier [9], and Kramer [13]. A related problem of constructing cusp forms of weight two associated to closed geodesics has been treated by Kudla and Millson in [14]. In connection with the problem of

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choosing spanning sets for $S_{2k}(\Gamma)$ from Poincaré series (not relative Poincaré series) we should mention Hejhal [5] and Kra [11].

The essential difference between all the earlier work and our approach lies in the use of qualitative (geometrical and dynamical) methods based on the representation of a cusp form as a function on the unit tangent bundle of $\Gamma\backslash\mathscr{H}$. A very general theorem of Livshitz [16] and a theorem of Guillemin and Kazhdan [4] play the key role in our considerations.

This paper contains the results of the first three chapters of my Ph.D. thesis [7]. I want to thank my adviser, Don Zagier, for all his help and advice, and J. Bernstein and S. Wolpert for discussions from which I profited a great deal and also for their enthusiasm and encouragement for my work. The results of this paper have been announced in [6].

§ 1. Relative Poincaré series

Let Γ be a Fuchsian group of the first kind acting on the upper half-plane $\mathscr{H}$, i.e. a discrete subgroup of $SL(2,\mathbb{R})$ with $\mathrm{vol}(\Gamma\backslash\mathscr{H})<\infty$; $k\geqq 2$ be an integer; $S_{2k}(\Gamma)$ be the space of all cusp forms of weight $2k$ on Γ. All relevant definitions and properties of cusp forms can be found in [20, 15, 21]. We shall use the following notations: for $z=x+iy\in\mathscr{H}$ and $\alpha=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{R})$ $j(\alpha,z)=cz+d$, $(f|_{2k}\alpha)(z)=f(\alpha z)\cdot j(\alpha,z)^{-2k}$.

The following proposition describes a general way of obtaining cusp forms on Γ.

Proposition 1. *Let Γ_0 be a subgroup of Γ and let $f(z)$ be a function, holomorphic on $\mathscr{H}$ and satisfying*

$$f|_{2k}\gamma=f \quad \text{for all } \gamma\in\Gamma_0 \tag{1.1}$$

$$\iint_{\Gamma_0\backslash\mathscr{H}} |f(z)|\,(\mathrm{Im}\,z)^k\,dV<\infty \tag{1.2}$$

where $dV=\dfrac{dx\,dy}{y^2}$. Then

i) *The series $F(z)=\sum_{\gamma\in\Gamma_0\backslash\Gamma}(f|_{2k}\gamma)(z)$, called the relative Poincaré series associated to the function f, converges absolutely on $\mathscr{H}$, uniformly on compact sets;*

ii) *$F(z)\in S_{2k}(\Gamma)$.*

Remark. It is easy to see that $F(z)$ is well defined, i.e. it does not depend on the choice of representatives of $\Gamma_0\backslash\Gamma$, and (1.2) does not depend on the choice of the fundamental domain for Γ_0.

The proof is a standard complex variable argument (cf. [12], Ch. 1, § 7).

Example. Let $\gamma_0=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be a *hyperbolic* ($|\mathrm{tr}\,\gamma_0|>2$) element in Γ. γ_0 is called *primitive* if there is no element $\gamma_1\in\Gamma$ such that $\gamma_0=\gamma_1^n$ for an integer $n>1$. If $c\neq 0$ γ_0 has two hyperbolic fixed points on the real axis, one repulsive (w_1) and the other attractive (w_2). The oriented geodesic $C(\gamma_0)$ on $\mathscr{H}$ from w_1 to w_2 is called the *axis* of γ_0 and clearly is $\langle\gamma_0\rangle$-invariant. Let $Q_{\gamma_0}(z)=cz^2+(d-a)z-b$ be a quadratic function havings its zeros at w_1 and w_2 and $D_0=\mathrm{Disc}(Q_{\gamma_0})=(\mathrm{tr}\,\gamma_0)^2-4$. If $c=0$, then $Q_{\gamma_0}(z)$ becomes a linear function and $C(\gamma_0)$ is parallel to the imaginary axis.

Let $f(z)=Q_{\gamma_0}^{-k}(z)$ and $\Gamma_0=\langle\gamma_0\rangle$. (1.1) follows from the equality $Q_{\gamma_0}(\gamma_0 z)=Q_{\gamma_0}(z)j^{-2}(\gamma_0,z)$. For any $\alpha\in SL(2,\mathbb{R})$ and any quadratic function $Q(z)$ put $(Q\circ\alpha)(z)=Q(\alpha z)j(\alpha,z)^2$. Notice that $Q_{\gamma_0}\circ\alpha=Q_{\alpha^{-1}\cdot\gamma_0\cdot\alpha}$. Also we have $|Q_{\gamma_0}^{-k}(\alpha z)|(\mathrm{Im}\,\alpha z)^k=|(Q_{\gamma_0}\circ\alpha)^{-k}(z)|(\mathrm{Im}\,z)^k$ and $Q_{\gamma_0}^{-k}|_{2k}\,\alpha=(Q_{\gamma_0}\circ\alpha)^{-k}$. In order to verify (1.2) we make the "canonical" change of variables by means of a matrix $R\in SL(2,\mathbb{R})$ which transforms the oriented imaginary axis $[0,i\infty]$ into $C(\gamma_0)$. Then $R^{-1}\cdot\gamma_0\cdot R=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$ with $\varepsilon=\frac{1}{2}(\mathrm{tr}\,\gamma_0+\sqrt{D_0})$ if $\mathrm{tr}\,\gamma_0>0$ and $\varepsilon=\frac{1}{2}(\mathrm{tr}\,\gamma_0-\sqrt{D_0})$ if $\mathrm{tr}\,\gamma_0<0$, $(Q_{\gamma_0}\circ R)(z)=(-\mathrm{sgn}\,\mathrm{tr}\,\gamma_0)\sqrt{D_0}\,z$ and the integral over the fundamental domain for $\langle R^{-1}\cdot\gamma_0\cdot R\rangle$, $\mathfrak{F}_0=\{z\in\mathscr{H}\,|\,1\leqq|z|<\varepsilon^2\}$ converges obviously for $k\geqq 2$. Thus we obtain a cusp form for each hyperbolic element $\gamma_0\in\Gamma$, the following normalization of which we denote by

$$\theta_{k,\gamma_0}(z)=D_0^{k-\frac{1}{2}}(-\mathrm{sgn}\,\mathrm{tr}\,\gamma_0)\,\pi^{-1}\binom{2k-2}{k-1}^{-1}\cdot 2^{k-2}\cdot\sum_{\gamma\in\langle\gamma_0\rangle\backslash\Gamma}(Q_{\gamma_0}\circ\gamma)^{-k}(z). \tag{1.3}$$

The following properties of the relative Poincaré series are proved by a straightforward calculation.

Proposition 2.

i) *If* γ_0 *and* γ_1 *are conjugate in* Γ *(we shall denote this by* $\gamma_0\sim\gamma_1$*), then* $\theta_{k,\gamma_0}(z)=\theta_{k,\gamma_1}(z)$;

ii) *For any integer* n, $\theta_{k,\gamma_0^n}=n\cdot\alpha_n^{k-1}\,\theta_{k,\gamma_0}(z)$, *where* $\alpha_n=Q_{\gamma_0^n}(z)/Q_{\gamma_0}(z)$.

Thus, the relative Poincaré series (1.3) are actually defined on conjugacy classes of hyperbolic elements and it is sufficient to consider them only for primitive hyperbolic elements. We denote by $[\gamma_0]$ the conjugacy class of the hyperbolic element γ_0 in Γ. Since conjugacy classes of primitive hyperbolic elements of Γ are in one-to-one correspondence with oriented (primitive) closed geodesics on $\Gamma\backslash\mathscr{H}$, we shall usually denote closed geodesics also by $[\gamma_0]$ and the relative Poincaré series associated to them by $\theta_{k,[\gamma_0]}(z)$. The finiteness of the number of conjugacy classes of hyperbolic elements with a given trace ("class number") interpreted as the number of closed geodesics of given length (see [17]), follows immediately from this representation.

The space $S_{2k}(\Gamma)$ is a finite dimensional complex Hilbert space with respect to the Petersson scalar product

$$\langle f,g\rangle=\int_{\Gamma\backslash\mathscr{H}} f(z)\overline{g(z)}\,y^{2k}\,dV.$$

Definition. For each hyperbolic element $\gamma_0\in\Gamma$ and each $g\in S_{2k}(\Gamma)$, the integral

$$r_k(g,\gamma_0)=\int_{z_0}^{\gamma_0 z_0} g(z)\,Q_{\gamma_0}^{k-1}(z)\,dz \tag{1.4}$$

is called the *period of* g *over the closed geodesic associated to* γ_0.

The integral $\int_{z_0}^{\gamma_0 z_0} g(z) Q_{\gamma_0}^{k-1}(z)\,dz$ does not depend on the choice of the point z_0 and the path from z_0 to $\gamma_0 z_0$. This follows from the fact that the differential form $g(z) Q_{\gamma_0}^{k-1}(z)\,dz$ is holomorphic on $\mathscr{H}$ and $\langle\gamma_0\rangle$-invariant. Thus $r_k(g,\gamma_0)$ is well-defined. It is convenient, however, to choose $z_0 \in C(\gamma_0)$ and to integrate along the geodesic $C(\gamma_0)$. The following result first appeared in [5] §7.

Proposition 3. *For any $g(z) \in S_{2k}(\Gamma)$ and any hyperbolic element $\gamma_0 \in \Gamma$,*

$$\langle g, \theta_{k,\gamma_0}\rangle = r_k(g,\gamma_0).$$

A proof can be found in [8] and is based on the following observation. For $F(z) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} (f|\gamma)(z)$, the relative Poincaré series associated to f satisfying the hypotheses of Proposition 1 and any $g \in S_{2k}(\Gamma)$ we have $\langle g, F\rangle = \iint_{\Gamma_0 \backslash \mathscr{H}} g(z)\overline{f(z)}\, y^{2k}\, dV$. Then one uses the same change of variables as in the Example and reduces the proof to a routine computation.

According to Borel [1] (see also [2]), cusp forms may be interpreted as functions on $\Gamma \backslash SL(2,\mathbb{R})$, which can be identified with the unit tangent bundle SM to the surface[1] $M = \Gamma \backslash \mathscr{H}$. We shall describe this interpretation in a convenient coordinate form. The unit tangent bundle to $\mathscr{H}$, $S\mathscr{H}$, may be parametrized by local coordinates (z,ζ), where $z \in \mathscr{H}$, $\zeta \in \mathbb{C}$, $|\zeta| = \operatorname{Im} z$ since the metric on each fiber is induced by the hyperbolic metric on $\mathscr{H}$. The second coordinate ζ may be regarded as a complex-valued function on $S\mathscr{H}$. For any $\gamma \in \Gamma$, $(d\gamma)\zeta = j^{-2}(\gamma, z)\zeta$ ($d\gamma$ is the differential of γ) and thus for any $f(z) \in S_{2k}(\Gamma)$ the function $f(z)\zeta^k$ is invariant under Γ, and therefore is a well-defined function on SM. For the same reason $f(z)\zeta^{k-1}\,dz$ is a well-defined differential form on SM. Now we can give a geometrical interpretation of the periods of $f(z)$ over closed geodesics (1.4). The arc $(z_0, \gamma_0 z_0)$ of the geodesic $C(\gamma_0)$ in $\mathscr{H}$ becomes the closed geodesic $[\gamma_0]$ in M. We can lift $[\gamma_0]$ into SM by $z \to (z,\zeta)$, where ζ is the tangent vector to $C(\gamma_0)$ at the point z of hyperbolic length 1. We shall use the same notation $[\gamma_0]$ for the geodesic lifted to SM. Consider the geodesic flow $\{\psi^t\}$ on SM. More precisely, each point $v = (z,\zeta) \in SM$ defines a geodesic on M, closed or not, which is lifted to SM by the standard lifting $z \to (z,\zeta)$ described above. $\psi^t v = (z_t, \zeta_t)$ is the point on the same geodesic such that the hyperbolic length of the arc(z, z_t) equals t. $\{\psi^t\}$ can be regarded as a motion with unit speed along geodesics. The closed geodesics become periodic orbits of $\{\psi^t\}$.

Proposition 4. *Let $f(z) \in S_{2k}(\Gamma)$ and $\gamma_0 \in \Gamma$ be a primitive hyperbolic element, $[\gamma_0]$ the closed geodesic on SM corresponding to the conjugacy class of γ_0. Then*

$$\int_{[\gamma_0]} f(z)\zeta^k\,dt = \int_{[\gamma_0]} f(z)\zeta^{k-1}\,dz = (-\operatorname{sgn}(\operatorname{tr}\gamma_0))^{k-1} D_0^{-\frac{k-1}{2}}\, r_k(f,\gamma_0),$$

where t is the "time", or the parameter of the geodesic flow and $dt = d\ell = \dfrac{\sqrt{dx^2+dy^2}}{y}$.

[1] If Γ has elliptic elements, M is not a smooth surface, but a so-called orbifold

Proof. Let $\gamma_0=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$ and $(z,\zeta)\in C(\gamma_0)$. Then $Q_{\gamma_0}(z)=c z^2+(d-a)z-b=c(z-w_1)(z-w_2)$. An elementary geometrical argument shows that $\operatorname{Arg}\zeta=\operatorname{Arg}Q_{\gamma_0}(z)$ if $\operatorname{tr}\gamma_0<0$ and $\operatorname{Arg}\zeta=\operatorname{Arg}Q_{\gamma_0}(z)\pm\pi$ if $\operatorname{tr}\gamma_0>0$. Let $|\ |$ denote the usual Euclidean norm on $\mathbb{C}$. Then $|\zeta|=y$ and $Q_{\gamma_0}(z)=|c|\cdot|z-w_1|\cdot|z-w_2|=|c|\cdot y\cdot|w_2-w_1|=\sqrt{D_0}\,|\zeta|$. Finally, we obtain $Q_{\gamma_0}(z)=-\operatorname{sgn}(\operatorname{tr}\gamma_0)\sqrt{D_0}\,\zeta$. Taking as a lift of $[\gamma_0]$ to $\mathscr{H}$ the segment between z_0 and $\gamma_0 z_0$ we obtain the right equality. The left equality follows from the formula $dz=\zeta\, d\ell=\zeta\, dt$. □

§ 2. Periods of cusp forms and the geodesic flow

In this section we shall prove the following theorem.

Theorem 1.

i) *The relative Poincaré series $\theta_{k,[\gamma_0]}$ for conjugacy classes of primitive hyperbolic elements $[\gamma_0]$ of Γ generate the whole space $S_{2k}(\Gamma)$.*

ii) *Suppose, in addition, that Γ is symmetric,* i.e.

$$\gamma\in\Gamma \Rightarrow \gamma'=\varepsilon\gamma\varepsilon\in\Gamma, \qquad \left(\varepsilon=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and let

$$\theta^+_{k,[\gamma_0]}=\frac{\theta_{k,[\gamma_0]}+\theta_{k,[\gamma_0']}}{2} \quad \text{and} \quad \theta^-_{k,[\gamma_0]}=\frac{\theta_{k,[\gamma_0]}-\theta_{k,[\gamma_0']}}{2i}.$$

Then each of the families of the cusp forms $\{\theta^+_{k,[\gamma_0]}\}$ and $\{\theta^-_{k,[\gamma_0]}\}$ generates $S_{2k}(\Gamma)$.

By Proposition 3, Theorem 1 follows immediately from the following statement.

Theorem 2.

i) *If the periods of a cusp form $f\in S_{2k}(\Gamma)$ over all closed geodesics are equal to zero, then $f(z)=0$.*

ii) *If Γ is symmetric and $r_k(f,\gamma_0)+r_k(f,\gamma_0')=0$ (or $r_k(f,\gamma_0)-r_k(f,\gamma_0')=0$) for all primitive hyperbolic elements $\gamma_0\in\Gamma$, then $f(z)=0$.*

Proof of Theorem 2. (i) Let $f(z)\in S_{2k}(\Gamma)$ such that $r_k(f,\gamma_0)=0$ for all primitive hyperbolic elements $\gamma_0\in\Gamma$. The function $\phi=f(z)\zeta^k$ on SM is smooth and, according to Proposition 4, has zero integrals over all closed geodesics. We shall use now the result of A. Livshitz[2] [16] which we formulate in a slightly modified form to include the possible presence of cusps.

Theorem. *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ with* $\operatorname{vol}(\Gamma\backslash\mathscr{H})<\infty$, *$M=\Gamma\backslash\mathscr{H}$, and let $\{\psi^t\}$ be the geodesic flow acting on SM. Let f be a smooth function on SM whose integrals over all periodic orbits of $\{\psi^t\}$ are equal to zero. Then there exists a function F on SM satisfying a Lipschitz condition and differ-*

[2] We use the natural English spelling of his name instead of a phoenetic transliteration, Livčic, which appeared in the translations of his papers from Russian into English

entiable in the direction of the geodesic flow and such that $\frac{dF}{dt}=f$, *where* $\mathscr{D}=\frac{d}{dt}$ *is the operator of differentiation along the orbits of the flow* $\{\psi^t\}$.

The original proof for geodesic flows on compact manifolds works with minor alterations for manifolds with cusps (see [7] Appendix).

By Livshitz's theorem there exists a function F satisfying a Lipschitz condition and such that

$$\frac{d}{dt}F=f(z)\zeta^k. \tag{2.1}$$

Let us consider $L^2(SM)$ with the scalar product

$$\langle F,G\rangle=\int_{SM} F\cdot\bar{G}\,dV\,d\theta,$$

where $dV=\frac{dx\,dy}{y^2}$ and $dV d\theta$ is the $SL(2,\mathbb{R})$-invariant volume on $SM(\theta=\operatorname{Arg}\zeta)$.

Lemma. *Let F be the function obtained from Livshitz's theorem for $\phi=f(z)\zeta^k$, where $f(z)\in S_{2k}(\Gamma)$. Then $F\in L^2(SM)$.*

Proof. If we prove that F is uniformly bounded on SM, then, using the assumption $\operatorname{vol}(\Gamma\backslash\mathscr{H})<\infty$ which implies that $\operatorname{vol}(SM)<\infty$, we get $F\in L^2(SM)$. If $\Gamma\backslash\mathscr{H}$ is compact, then the boundedness follows from a Lipschitz condition. Suppose that $\Gamma\backslash\mathscr{H}$ has cusps. Since the number of cusps is finite, it is enough to prove that F is bounded at each cusp σ. Let $R\in SL(2,\mathbb{R})$ be such that $R(\infty)=\sigma$, and let $\Phi(z)=(f|_{2k}R)(z)=f(Rz)j(R,z)^{-2k}\cdot\Phi(z)$ is invariant under $\left\langle\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\right\rangle=R^{-1}\Gamma_\sigma R$, where $\Gamma_\sigma=\{\gamma\in\Gamma,\ \gamma\sigma=\sigma\}$, $h>0$, and therefore has a Fourier development at ∞: $\Phi(z)=\sum_{n=1}^{\infty}a_n e^{2\pi niz/h}$; $a_0=0$ since f is a cusp form. Let I be the imaginary axis, then $R(I)$ is the geodesic, going to the cusp σ. Fix $v\in R(I)$, then

$$F(\psi^s v)-F(v)=\int_0^s \phi(\psi^t v)\,dt \qquad \text{since } \phi=\frac{dF}{dt}.$$

Notice that for some constant C_1

$$\left|\int_0^s \phi(\psi^t v)\,dt\right|=\left|\int_0^s f(w_t)\,\eta_t^k\,dt\right|=\left|\int_0^s \Phi(z_t)\,\zeta_t^k\,dt\right|$$
$$\leqq\int_{y_0}^{\infty}|\Phi(z)|\,y^{k-1}\,dy<C_1,$$

since on I, $\zeta_t=iy$, $dt=\frac{dy}{y}$, $|\Phi(z)|=0(e^{-\frac{2\pi y}{h}})$ and therefore the last integral converges. Therefore, there exists a constant C_2 such that for $0<s<\infty$, $|F(\psi^s v)|<C_2$.

The finiteness of the volume implies that for any $\varepsilon>0$ there exists a neighborhood of σ in M, $U(\sigma)\not\ni v$ such that $\ell(z,R(I))<\varepsilon$ for all $z\in U(\sigma)$, where ℓ is the hyperbolic metric on M. Let $\overline{U(\sigma)}=S(U(\sigma))\subset SM$, $\overline{R(I)}$ be the lift of $R(I)$

onto SM, and s be the metric in SM induced by the hyperbolic metric on $M(ds^2=dl^2+d\theta^2)$. Then for all $u=(z,\zeta)\in\overline{U(\sigma)}$ we have $s(u,\overline{R(I)})<\varepsilon+\pi$ and since F satisfies a Lipschitz condition, we have $|F(u)|<C$ for some constant C. $\square$

A standard Fourier analysis argument shows that we can decompose F in $L^2(SM)$ as follows: $F(z,\zeta)=\sum_{-\infty}^{\infty} F_m$, $F_m\in H_m$, where $H_m\subset L^2(SM)$ are defined as follows: $H_m=\{G(z,\zeta)\in L^2(SM)\,|\,G(z,\zeta)=g(z)\zeta^m\}$. Notice that $g(z)$ is not supposed to be holomorphic on $\mathscr{H}$.

Since the volume form $dVd\phi$ is invariant for the geodesic flow $\{\psi^t\}$, the differential operator $\mathscr{D}=\frac{d}{dt}$, defined on a dense set of functions differentiable along the orbits of the geodesic flow $\{\psi^t\}$, is skew-self-adjoint: $\mathscr{D}^*=-\mathscr{D}$, or equivalently, $\langle\mathscr{D}f,g\rangle=\langle f,-\mathscr{D}g\rangle$. A direct calculation shows that $\mathscr{D}=\mathscr{D}^+ +\mathscr{D}^-$ and for $G=g(z)\zeta^m\in H_m$, $\mathscr{D}^+G=\left(\frac{\partial g}{\partial z}-\mathrm{i}m\,y^{-1}g(z)\right)\zeta^{m+1}\in H_{m+1}$ and $\mathscr{D}^-G=y^2\frac{\partial g}{\partial\bar z}\zeta^{m-1}\in H_{m-1}$, and also $(\mathscr{D}^+)^*=-\mathscr{D}^-$; $(\mathscr{D}^-)^*=-\mathscr{D}^+$. Thus we can rewrite (2.1) as follows

$$\begin{aligned}&\mathscr{D}^-F_{k+1}+\mathscr{D}^+F_{k-1}=f(z)\zeta^k\\&\mathscr{D}^-F_{i+1}+\mathscr{D}^+F_{i-1}=0\qquad\text{for all } i\neq k.\end{aligned}\tag{2.2}$$

A theorem of Guillemin and Kazhdan ([4], Theorem 3.6) implies that $F_i=0$ for $i\geqq k$ and $i\leqq -k$, so the first equation has the form $\mathscr{D}^+F_{k-1}=f(z)\zeta^k$. Then $\mathscr{D}^-\mathscr{D}^+F_{k-1}=\mathscr{D}^-(f(z)\zeta^k)=y^2\left(\frac{\partial}{\partial\bar z}f(z)\right)\zeta^{k-1}=0$ since $f(z)$ is holomorphic. We have $0=\langle F_{k-1},\mathscr{D}^-\mathscr{D}^+F_{k-1}\rangle=-\langle\mathscr{D}^+F_{k-1},\mathscr{D}^+F_{k-1}\rangle=-\|\mathscr{D}^+F_{k-1}\|^2$. Therefore $\mathscr{D}^+F_{k-1}=0$ and $f(z)=0$. This completes the proof of statement (i).

(ii) Let $f(z)\in S_{2k}(\Gamma)$ be such that $r_k(f,\gamma_0)+r_k(f,\gamma_0')=0$ for all primitive hyperbolic elements $\gamma_0\in\Gamma$. Then we have

$$\int_{[\gamma_0]} f(z)\zeta^k\,dt+\int_{[\gamma_0']} f(z)\zeta^k\,dt=0.$$

Notice that $C(\gamma_0)$ and $C(\gamma_0')$ on $\mathscr{H}$ are symmetric with respect to the imaginary axis and have the opposite orientation. Therefore after the change of variables $z=-\bar w$, we have $\zeta=-\bar\eta=-y^2\eta^{-1}$ and

$$\int_{[\gamma_0']} f(z)\zeta^k\,dt=(-1)^k\int_{[\gamma_0]} f(-\bar w)(\operatorname{Im}w)^{2k}\eta^{-k}\,dt.$$

Therefore for all closed geodesics $[\gamma_0]$ on SM we have

$$\int_{[\gamma_0]} f(z)\zeta^k\mp f(-\bar z)\,y^{2k}\zeta^{-k}\,dt=0$$

where $f(z)\zeta^k\in H_k$ and $f(-\bar z)\,y^{2k}\zeta^{-k}\in H_{-k}$.

By Livshitz's Theorem there exists a function F on SM such that

$$\frac{d}{dt}F=f(z)\zeta^k\mp f(-\bar{z})\,y^{2k}\zeta^{-k}. \tag{2.3}$$

The same argument as in Lemma shows that $F\in L^2(SM)$, and we can rewrite (2.3) as follows:

$$\begin{aligned}
\mathscr{D}^- F_{k+1}+\mathscr{D}^+ F_{k-1}&=f(z)\zeta^k\\
\mathscr{D}^- F_{-k+1}+\mathscr{D}^+ F_{-k-1}&=\mp f(-\bar{z})\,y^{2k}\zeta^{-k}\\
\mathscr{D}^- F_{i+1}+\mathscr{D}^+ F_{i-1}&=0 \qquad \text{for all } i\neq k,\ -k.
\end{aligned}$$

Theorem 3.6 ([4]) implies again that $F_i=0$ for $i\geqq k$ and $i\leqq -k$. Therefore, the same arguments as in (i) can be carried out, and we get $f(z)=0$. □

§ 3. Periods of relative Poincaré series

Throughout this section Γ is supposed to be symmetric (see Theorem 1 (ii)).

Theorem 3. *Let γ_0 and γ_1 be two primitive hyperbolic elements and $\chi_i=-\operatorname{sgn}\operatorname{tr}\gamma_i$ for $i=0,1$. Then*

$$\begin{aligned}
&r_k(\theta_{k,[\gamma_0]},\gamma_1)-r_k(\theta_{k,[\gamma_0']},\gamma_1')\\
&\quad=D_0^{\frac{k-1}{2}}D_1^{\frac{k-1}{2}}(\chi_0\chi_1)^{k-1}\binom{2k-2}{k-1}^{-1}2^{2k-1}\cdot i\sum_{p\in[\gamma_0]\cap[\gamma_1]}\mu_p P_{k-1}(\cos\theta_p),
\end{aligned}\tag{3.1}$$

where the summation is taken over all intersection points p of $[\gamma_0]$ and $[\gamma_1]$ on $\Gamma\backslash\mathscr{H}$ (counted with multiplicities); if $[\gamma_0]=[\gamma_1]$ the summation is taken over all points of the self-intersection of the geodesic $[\gamma_0]$. Here $\theta_p=\theta_p(\gamma_0,\gamma_1)$ is the angle between the oriented geodesics $[\gamma_0]$ and $[\gamma_1]$ at their intersection point p; $\mu_p=\operatorname{sgn}(\sin\theta_p)$; P_{k-1} is the $(k-1)^{\text{st}}$ Legendre polynomial.

Remarks. 1. For $k=2$, formula (3.1) agrees with the cosine formula of [24], (cf. also [5], Theorem 5).

2. D. Kazhdan suggested that formula (3.1) may possibly be obtained by using Eichler cohomology.

Proof. Let us assume first that $\gamma_1\nsim\gamma_0$, $\gamma_1\nsim\gamma_0^{-1}$. Let $\Gamma_0=\langle\gamma_0\rangle$, $\Gamma_1=\langle\gamma_1\rangle$, $K_1=D_0^{k-\frac{1}{2}}(-\operatorname{sgn}\operatorname{tr}\gamma_0)\,\pi^{-1}\binom{2k-2}{k-1}^{-1}\cdot 2^{2k-2}$, then $\theta_{k,[\gamma_0]}=K_1\cdot\sum_{\gamma\in\Gamma_0\backslash\Gamma}(Q_{\gamma_0}\circ\gamma)^{-k}(z)$.

Lemma 1. *Let $\gamma_0,\gamma_1\in\Gamma$ be two primitive hyperbolic elements, such that $\gamma_1\nsim\gamma_0$, $\gamma_1\nsim\gamma_0^{-1}$ in Γ; let $\{\gamma_\alpha\in\Gamma\}$ be a complete set of representatives of $\Gamma_0\backslash\Gamma/\Gamma_1$. Then $\{\gamma_\alpha\gamma_1^m,\ m\in\mathbb{Z}\}$ forms a complete set of representatives of $\Gamma_0\backslash\Gamma$.*

The proof is straightforward and we shall omit it.

Using Lemma 1 and interchanging the summation and the integration (which is legitimate since the relative Poincaré series converges absolutely) we have

$$\begin{aligned} r_k(\theta_{k,[\gamma_0]},\gamma_1) &= \int_{z_0}^{\gamma_1 z_0} (K_1 \cdot \sum_{\gamma\in\Gamma_0\backslash\Gamma} (Q_{\gamma_0}^{-k}|\gamma)(z)\, Q_{\gamma_1}^{k-1}(z))\,dz \\ &= K_1 \cdot \int_{z_0}^{\gamma_1 z_0} \sum_{\gamma\in\Gamma_0\backslash\Gamma/\Gamma_1} \sum_{n\in\mathbb{Z}} (Q_{\gamma_0}^{-k}|\gamma|\gamma^n)(z)\, Q_{\gamma_1}^{k-1}(z)\,dz \\ &= K_1 \cdot \sum_{\gamma\in\Gamma_0\backslash\Gamma/\Gamma_1} \int_{C(\gamma_1)} (Q_{\gamma_0}\circ\gamma)^{-k}(z)\, Q_{\gamma_1}^{k-1}(z)\,dz. \end{aligned}$$

We obtain the corresponding formula for $r_k(\theta_{k,[\gamma_0']},\gamma_1')$. Making a change of variables $z\to -z$ and using that $(Q_{\gamma_0'}\circ\gamma')(z) = -(Q_{\gamma_0}\circ\gamma)(-z)$ and $Q_{\gamma_1'}(z) = -Q_{\gamma_1}(-z)$, we obtain the formula

$$r_k(\theta_{k,[\gamma_0]},\gamma_1) - r_k(\theta_{k,[\gamma_0']},\gamma_1') = K_1 \cdot \sum_{\gamma\in\Gamma_0\backslash\Gamma/\Gamma_1} \oint_{C(\gamma_1)} \frac{Q_{\gamma_1}^{k-1}(z)\,dz}{(Q_{\gamma_0}\circ\gamma)^k(z)}, \tag{3.2}$$

where the integration is over a circle $C(\gamma_1)\cup(-C(\gamma_1'))$.

It follows from the Residue Theorem that the double cosets $\Gamma_0\gamma\Gamma_1 \in \Gamma_0\backslash\Gamma/\Gamma_1$ which contribute non-zero terms to the sum (3.2) have the property that $\gamma^{-1}C(\gamma_0)$ intersects $C(\gamma_1)$ on $\mathscr{H}$, and those are in $1-1$ correspondence with intersection points of $[\gamma_0]$ and $[\gamma_1]$ on $\Gamma\backslash\mathscr{H}$ counted with multiplicities. Since $[\gamma_0]$ and $[\gamma_1]$ may have only finitely many intersection points, the sum in (3.2) is in fact finite.

After the "canonical" change of variables R (cf. Example, § 1), transforming the imaginary axis into $C(\gamma_1)$, each term in (3.2) can be rewritten in the form

$$\oint_{C(\gamma_1)} \frac{Q_{\gamma_1}^{k-1}(z)\,dz}{(Q_{\gamma_0}\circ\gamma)^k(z)} = \chi_1^{k-1} D_1^{\frac{k-1}{2}} \int_{-i\infty}^{i\infty} \frac{z^{k-1}\,dz}{(Q_{\gamma_0}\circ\gamma\circ R)^k(z)}.$$

The next lemma gives an expression for such an integral.

Lemma 2. *Suppose $k\geqq 1$ and $A,B,C\in\mathbb{R}$ with $D=B^2-4AC>0$. Then*

$$\int_{-i\infty}^{i\infty} \frac{z^{k-1}\,dz}{(Az^2+Bz+C)^k} = \begin{cases} (-\operatorname{sgn} A)\,\dfrac{2\pi i^{2k-1}}{(k-1)!}\,\dfrac{\partial^{k-1}}{\partial B^{k-1}}(B-4AC)^{-\frac{1}{2}} & \text{if } AC<0 \\ 0 & \text{if } AC>0, \end{cases} \tag{3.3}$$

The proof is a simple exercise and we shall omit it.

We define $P_{k-1}\left(\frac{B}{\sqrt{D}}\right)$ as follows

$$\frac{\partial^{k-1}}{\partial B^{k-1}}(B^2-4AC)^{-\frac{1}{2}} = D^{-\frac{k}{2}}(-1)^{k-1}(k-1)!\,P_{k-1}\left(\frac{B}{\sqrt{D}}\right)$$

and prove by induction that $P_{k-1}(t)$ satisfies the following recurrent formula for $k=1,2,\ldots$

$$P_k(t) = t\,P_{k-1}(t) + \frac{t^2-1}{k}\,P_k'(t) \qquad \text{with } P_0(t)=1$$

which identifies $P_{k-1}(t)$ as the $(k-1)^{\text{st}}$ Legendre polynomial (see [10], p. 724).

To each of the double cosets $\Gamma_0 \gamma \Gamma_1 \in \Gamma_0 \backslash \Gamma / \Gamma_1$ which contribute non-zero terms to the sum (3.2) we shall apply Lemma 2 with $Az^2+Bz+C=Q_{\tilde{\gamma}_0}(z)$, where $\tilde{\gamma}_0=R^{-1}\cdot\gamma^{-1}\cdot\gamma_0\cdot\gamma\cdot R$. An orientation on $C(\tilde{\gamma}_0)$ is given by $\mu(\tilde{\gamma}_0)=\operatorname{sgn}(\operatorname{tr}\tilde{\gamma}_0\cdot A)$ and is clockwise if $\mu(\tilde{\gamma}_0)>0$ and counterclockwise if $\mu(\tilde{\gamma}_0)<0$. Let θ_p be the angle between the oriented geodesic $C(\tilde{\gamma}_0)$ and the positive direction of the imaginary axis (i.e. $C(R^{-1}\cdot\gamma_1\cdot R)$. Then $\mu(\tilde{\gamma}_0)=\operatorname{sgn}(\sin\theta_p)=\mu_p$. An easy geometric argument shows that $\cos\theta_p=(-\operatorname{sgn}\operatorname{tr}\gamma_0)\frac{B}{\sqrt{D}}$. Since P_k is even for even k and odd for odd k, we have $P_{k-1}\left(\frac{B}{\sqrt{D}}\right)=(-\operatorname{sgn}\operatorname{tr}\gamma_0)^{k-1}P_{k-1}(\cos\theta_p)$. Thus, since the linear fractional transformation R is conformal, each non-zero term in (3.2) can be rewritten in purely geometrical terms (note that $D=D_0$):

$$\oint_{C(\gamma_1)}\frac{Q^{k-1}(z)\,dz}{(Q_{\gamma_0}\circ\gamma)^k(z)}=2\pi i\,D_0^{-\frac{k}{2}}D_1^{\frac{k-1}{2}}(\chi_0\chi_1)^{k-1}\cdot\chi_0^{-1}\mu_p P_{k-1}(\cos\theta_p),$$

which implies formula (3.1).

Let us now consider the case when γ_0 is conjugate in Γ to γ_1 or to γ_1^{-1}. Since $\theta_{k,[\gamma_0]}$ depends only on the conjugacy class of γ_0 and since $\theta_{k,[\gamma_1^{-1}]}=(-1)^k\theta_{k,[\gamma_1]}$, it is sufficient to consider the case $\gamma_1=\gamma_0$. In this case we have $\Gamma_0=\Gamma_1$ and therefore instead of Lemma 1 we need the following lemma:

Lemma 3.

i) *Let γ_0 be a primitive hyperbolic element, and let $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Suppose $\gamma_0\sim\gamma_0^{-1}$*[3] *and let $\sigma\in\Gamma$ be such that $\sigma^{-1}\cdot\gamma\cdot\sigma=\gamma_0^{-1}$; let $\{\gamma_\alpha\in\Gamma\}$ be a complete set of representatives of $\Gamma_0\backslash\Gamma/\Gamma_0$ containing e and σ. Then $\{\gamma_\alpha\cdot\gamma_0^m,\ m\in\mathbb{Z},\ \gamma_\alpha\neq e,\sigma\}\cup e\cup\sigma$ forms a complete set of representatives of $\Gamma_0\backslash\Gamma$.*

ii) *Let γ_0 be a primitive hyperbolic element such that $\gamma_0\nsim\gamma_0^{-1}$, $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and let $\{\gamma_\alpha\in\Gamma\}$ be a complete set of representatives of $\Gamma_0\backslash\Gamma/\Gamma_0$, containing e. Then $\{\gamma_\alpha\cdot\gamma_0^m,\ m\in\mathbb{Z},\ \gamma_\alpha\neq e\}\cup e$ forms a complete set of representatives of $\Gamma_0\backslash\Gamma$.*

The only other fact we need to get through all the calculations is that $\int_{z_0}^{\gamma_0 z_0}\frac{dz}{Q_{\gamma_0}(z)}=\int_{z_0'}^{\gamma_0' z_0'}\frac{dz}{Q_{\gamma_0'}(z)}$, which becomes obvious after our "canonical" change of variables (see Example, § 1). □

§ 4. Rational structures on $S_{2k}(\Gamma)$ in the arithmetic case

Let Γ be an arithmetic subgroup of $SL(2,\mathbb{R})$, i.e. a group obtained from a quaternion algebra over $\mathbb{Q}$ which splits over $\mathbb{R}$ (see [3], pp. 116–119, and [22]). It is known that Γ can be embedded in $SL(2,\mathbb{R})$ in a symmetric way so that it satisfies the assumption of Theorem 1 (ii). The only other fact about arithmetic subgroups we need is that traces of all elements are integers.

[3] The existence of such a $\gamma_0\in\Gamma$ with $\gamma_0\sim\gamma_0^{-1}$ is a very special property of Γ. This may be possible only if Γ contains an elliptic element of order 2

Theorem 4. *Let Γ be an arithmetic subgroup of $SL(2,\mathbb{R})$, and let $\gamma_0,\gamma_1\in\Gamma$ be two primitive hyperbolic elements. Then $\langle\theta^+_{k,[\gamma_0]},\theta^-_{k,[\gamma_1]}\rangle\in\mathbb{Q}$.*

Proof. We have

$$\langle\theta^+_{k,[\gamma_0]},\theta^-_{k,[\gamma_1]}\rangle=\tfrac{1}{4}i(\langle\theta_{k,[\gamma_0]},\theta_{k,[\gamma_1]}\rangle-\langle\theta_{k,[\gamma_0']},\theta_{k,[\gamma_1']}\rangle)$$
$$+\tfrac{1}{4}i(\langle\theta_{k,[\gamma_0']},\theta_{k,[\gamma_1]}\rangle-\langle\theta_{k,[\gamma_0]},\theta_{k,[\gamma_1']}\rangle). \tag{4.1}$$

According to Proposition 3 (§ 1) and Theorem 3, we have

$$\tfrac{1}{4}i(\langle\theta_{k,[\gamma_0]},\theta_{k,[\gamma_1]}\rangle-\langle\theta_{k,[\gamma_0']},\theta_{k,[\gamma_1']}\rangle)=q\cdot D_0^{\frac{k-1}{2}}D_1^{\frac{k-1}{2}}\sum_{p\in[\gamma_0]\cap[\gamma_1]}\mu_p P_{k-1}(\cos\theta_p)$$
$$=-\tfrac{1}{2}\operatorname{Im}\langle\theta_{k,[\gamma_0]},\theta_{k,[\gamma_1]}\rangle, \tag{4.2}$$

where $q\in\mathbb{Q}$.

Lemma (Wolpert [24]). *The following formula gives the expression for the cosine of the angle between the axes of two hyperbolic transformations S ans T at their intersection point p:*

$$\cos\theta_p=\frac{-\operatorname{sgn}(\operatorname{tr}S\cdot\operatorname{tr}T)(\operatorname{tr}S\cdot\operatorname{tr}T-2\operatorname{tr}ST)}{\sqrt{(\operatorname{tr}S)^2-4}\cdot\sqrt{(\operatorname{tr}T)^2-4}}.$$

This formula shows that for each $p\in[\gamma_0]\cap[\gamma_1]$ the numerator of the expression for $\cos\theta_p$ is an integer, while the denominator is $\sqrt{D_0D_1}$. If k is odd, P_{k-1} has only even powers of $\cos\theta_p$, if k is even, p_{k-1} has only odd powers of $\cos\theta_p$ and it is clear from (4.2) that we get a rational number in both cases.

The same argument gives us the rationality of the second term in (4.1), which completes the proof. □

Formula (4.2) allows us to find bases in $S_{2k}(\Gamma)$ for particular arithmetic groups Γ and small k (cf. Chapter IV of [7]).

Now we can define two rational subspaces of $S_{2k}(\Gamma)$:

$$S^+_{2k}(\Gamma)=\{f\in S_{2k}(\Gamma)|\langle f,\theta^-_{k,[\gamma_0]}\rangle\in\mathbb{Q}\text{ for all hyperbolic }\gamma_0\in\Gamma\},$$
$$S^-_{2k}(\Gamma)=\{f\in S_{2k}(\Gamma)|\langle f,\theta^+_{k,[\gamma_0]}\rangle\in\mathbb{Q}\text{ for all hyperbolic }\gamma_0\in\Gamma\}.$$

Theorems 1(ii) and 4 now imply that $S^\pm_{2k}(\Gamma)$ is the $\mathbb{Q}$-span of $\{\theta^\pm_{k,[\gamma_0]}\}$ and that $S^+_{2k}(\Gamma)$ and $S^-_{2k}(\Gamma)$ are rational structures on $S_{2k}(\Gamma)$ (i.e. $S^\pm_{2k}(\Gamma)\otimes_{\mathbb{Q}}\mathbb{C}\xrightarrow{\sim}S_{2k}(\Gamma)$) which are dual to one another with respect to the Petersson scalar product.

Let $S^0_{2k}(\Gamma)$ be the $\mathbb{Q}$-span of $\{\theta_{k,[\gamma_0]}\}$. Then $S^0_{2k}(\Gamma)$ is a rational structure on $S_{2k}(\Gamma)$ considered as a *real* vector space. This also follows from Theorems 1(ii) and 4 and from the fact that $\operatorname{Im}\langle\theta^-_{\gamma_0},\theta^-_{\gamma_1}\rangle=0$ for any two hyperbolic $\gamma_0,\gamma_1\in\Gamma$ (see (4.2)) which implies $S^+_{2k}\cap iS^-_{2k}=\{0\}$ and $S^0_{2k}=S^+_{2k}\oplus iS^-_{2k}$.

Remark. As has been pointed out to the author by John Millson, one can deduce the existence of a rational structure on $S_{2k}(\Gamma)$ considered as a real vector space from Shimura isomorphism between $S_{2k}(\Gamma)$ and $H^1(\Gamma,S^{2k-2}\mathbb{R}^2)$ (see [21] Chapter 8), and of two rational structures on $S_{2k}(\Gamma)$ considered as a complex vector space from Shimura map $S_{2k}(\Gamma)\to H^1(\Gamma,S^{2k-2}\mathbb{C}^2)$, which supposedly coincide with three rational structures described in this paper.

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