APPLICATIONS OF (a,b)-CONTINUED FRACTION TRANSFORMATIONS

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Dedicated to the memory of Dan Rudolph

Abstract. We describe a general method of arithmetic coding of geodesics on the modular surface based on the study of one-dimensional Gauss-like maps associated to a two parameter family of continued fractions introduced in [16]. The finite rectangular structure of the attractors of the natural extension maps and the corresponding "reduction theory" play an essential role. In special cases, when an (a,b)-expansion admits a so-called "dual", the coding sequences are obtained by juxtaposition of the boundary expansions of the fixed points, and the set of coding sequences is a countable sofic shift. We also prove that the natural extension maps are Bernoulli shifts and compute the density of the absolutely continuous invariant measure and the measure-theoretic entropy of the one-dimensional map.

1. Introduction and background

In [16], the authors studied a new two-parameter family of continued fraction transformations. These transformations can be defined using the standard generators $T(x)=x+1$, $S(x)=-1/x$ of the modular group $SL(2,\mathbb{Z})$ and considering $f_{a,b}:\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ given by

$$f_{a,b}(x)=\begin{cases}x+1 & \text{if } x<a\\ -\dfrac{1}{x} & \text{if } a\le x<b\\ x-1 & \text{if } x\ge b\,.\end{cases} \tag{1.1}$$

Under the assumption that the parameters (a,b) belong to the set

$$\mathcal{P}=\{(a,b)\,|\,a\le 0\le b,\ b-a\ge 1,\ -ab\le 1\}\,,$$

one can introduce corresponding continued fraction algorithms by using the first return map of $f_{a,b}$ to the interval $[a,b)$. Equivalently, these so called (a,b)*-continued fractions* can be defined using a generalized integral part function:

$$\lfloor x\rceil_{a,b}=\begin{cases}\lfloor x-a\rfloor & \text{if } x<a\\ 0 & \text{if } a\le x<b\\ \lceil x-b\rceil & \text{if } x\ge b\,,\end{cases} \tag{1.2}$$

where $\lfloor x\rfloor$ denotes the integer part of x and $\lceil x\rceil=\lfloor x\rfloor+1$.

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A starting point of the theory is the following result [16, Theorem 2.1]: if $(a,b) \in \mathcal{P}$, then any irrational number x can be expressed uniquely as an infinite continued fraction of the form

$$x = n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots}}} = \lfloor n_0, n_1, \cdots \rceil_{a,b}, \ (n_k \neq 0 \text{ for } k \geq 1),$$

where $n_0 = \lfloor x \rceil_{a,b}$, $x_1 = -\frac{1}{x-n_0}$ and $n_{k+1} = \lfloor x_{k+1} \rceil_{a,b}$, $x_{k+1} = -\frac{1}{x_k - n_k}$, i.e. the sequence of partial fractions $r_k = \lfloor n_0, n_1, \ldots, n_k \rceil_{a,b}$ converges to x.

It is possible to construct (a,b)-continued fraction expansions for rational numbers, too. However, such expansions will terminate after finitely many steps if $b \neq 0$. If $b = 0$, the expansions of rational numbers will end with a tail of 2's, since $0 = \lfloor 1, 2, 2, \ldots \rceil_{a,0}$.

The above family of continued fraction transformations contains three classical examples: the case $a = -1$, $b = 0$ described in [22, 12] gives the "minus" (backward) continued fractions, the case $a = -1/2$, $b = 1/2$ gives the "closest-integer" continued fractions considered first by Hurwitz in [9], and the case $a = -1$, $b = 1$ was presented in [19, 14] in connection with a method of coding symbolically the geodesic flow on the modular surface following Artin's pioneering work [6] and corresponds to the regular "plus" continued fractions with alternating signs of the digits.

The main object of study in [16] is a two-dimensional realization of the *natural extension map* of $f_{a,b}$, $F_{a,b} : \bar{\mathbb{R}}^2 \setminus \Delta \to \bar{\mathbb{R}}^2 \setminus \Delta$, $\Delta = \{(x,y) \in \bar{\mathbb{R}}^2 | x = y\}$, defined by

$$F_{a,b}(x,y) = \begin{cases} (x+1, y+1) & \text{if } y < a \\ \left(-\frac{1}{x}, -\frac{1}{y}\right) & \text{if } a \leq y < b \\ (x-1, y-1) & \text{if } y \geq b\,. \end{cases} \tag{1.3}$$

Here is the main result of that paper:

Theorem 1.1 ([16]). *There exists an explicit one-dimensional Lebesgue measure zero, uncountable set $\mathcal{E}$ that lies on the diagonal boundary $b = a+1$ of $\mathcal{P}$ such that:*

1. *for all $(a,b) \in \mathcal{P} \setminus \mathcal{E}$ the map $F_{a,b}$ has an attractor $D_{a,b} = \cap_{n=0}^{\infty} F_{a,b}^n(\bar{\mathbb{R}}^2 \setminus \Delta)$ on which $F_{a,b}$ is essentially bijective.*
2. *The set $D_{a,b}$ consists of two (or one, in degenerate cases) connected components each having* finite rectangular structure*, i.e. bounded by non-decreasing step-functions with a finite number of steps.*
3. *Almost every point (x,y) of the plane ($x \neq y$) is mapped to $D_{a,b}$ after finitely many iterations of $F_{a,b}$.*

An essential role in the argument is played by the forward orbits associated to a and b: to a, the *upper orbit* $\mathcal{O}_u(a)$ (i.e. the orbit of Sa) and the *lower orbit* $\mathcal{O}_\ell(a)$ (i.e. the orbit of Ta), and to b, the *upper orbit* $\mathcal{O}_u(b)$ (i.e. the orbit of $T^{-1}b$) and the *lower orbit* $\mathcal{O}_\ell(b)$ (i.e. the orbit of Sb). It was proved in [16] that if $(a,b) \in \mathcal{P} \setminus \mathcal{E}$, then $f_{a,b}$ satisfies the *finiteness condition*, i.e. for both a and b, their upper and lower orbits are either eventually periodic, or they satisfy the *cycle property*, i.e. they meet forming a cycle; more precisely, there exist $k_1, m_1, k_2, m_2 \geq 0$ s.t.

$$f_{a,b}^{m_1}(Sa) = f_{a,b}^{k_1}(Ta) = c_a, \text{ (resp., } f_{a,b}^{m_2}(T^{-1}b) = f_{a,b}^{k_2}(Sb) = c_b),$$

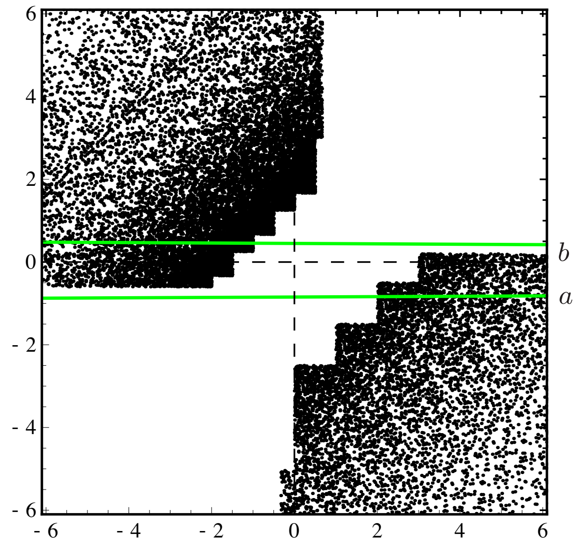


FIGURE 1. Attracting domain $D_{a,b}$ for $a = -\frac{4}{5}$, $b = \frac{2}{5}$

where c_a and c_b are the *ends of the cycles.* If the products of transformations over the upper and lower sides of the cycle are equal, the cycle property is *strong*, otherwise, it is *weak*. In both cases the set $\mathcal{L}_{a,b}$ of the corresponding values is finite; ends of the cycles belong to the set $\mathcal{L}_{a,b}$ if and only if they are equal to 0, i.e. if the cycle is weak. The structure of the attractor $D_{a,b}$ is explicitly "computed" from the finite set $\mathcal{L}_{a,b}$.

The paper is organized as follows. In Section 2 we give some background information about geodesic flows and their representations as special flows over symbolic dynamical systems, and define the coding map. In Section 3 we describe the reduction procedure for coding geodesics via (a,b)-continued fractions based on the study of the attractor of the associated natural extension map, define the corresponding cross-section set, and introduce the notion of *reduced geodesic.* In Section 4 we prove that the first return map to the cross-section corresponds to a shift of the coding sequence (Theorem 4.1) and, as a consequence, show that (a,b)-continued fractions satisfy the *Tail Property*, i.e. two $SL(2,\mathbb{Z})$-equivalent real numbers have the same tails in their (a,b)-continued fraction expansions. In Section 5 we introduce a notion of a *dual code* and show that if an (a,b)-expansion has a dual (a',b')-expansion, then the coding sequence of a reduced geodesic is obtained by juxtaposition of the (a,b)-expansion of its attracting endpoint w and the (a',b')-expansion of $1/u$, where u is its repelling endpoint. We also prove that if the (a,b)-expansion admits a dual, then the set of admissible coding sequences is a sofic shift (Theorem 5.8). In Section 6 we derive formulas for the density of the absolutely continuous invariant measure and the measure-theoretic entropy of the one-dimensional Gauss-type maps and their natural extensions. We also prove that the natural extension maps are Bernoulli shifts. And finally, in Section 7 we apply results of [16] to obtain explicit formulas for invariant measure for the one-dimensional maps for some regions of the parameter set $\mathcal{P}$.

2. Geodesic flow on the modular surface and its representation as a special flow over a symbolic dynamical system

Let $\mathcal{H} = \{z = x + iy : y > 0\}$ be the upper half-plane endowed with the hyperbolic metric, $\mathcal{F} = \{z \in \mathcal{H} : |z| \geq 1,\ |\text{Re } z| \leq \frac{1}{2}\}$ be the standard fundamental region of the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$, and $M = PSL(2,\mathbb{Z})\backslash\mathcal{H}$ be the modular surface. Let $S\mathcal{H}$ denote the unit tangent bundle of $\mathcal{H}$. We will use the coordinates $v = (z,\zeta)$ on $S\mathcal{H}$, where $z \in \mathcal{H}$, $\zeta \in \mathbb{C}$, $|\zeta| = \text{Im }(z)$. The quotient space $PSL(2,\mathbb{Z})\backslash S\mathcal{H}$ can be identified with the unit tangent bundle of M, SM, although the structure of the fibered bundle has singularities at the elliptic fixed points (see [11, §3.6] for details). Recall that geodesics in this model are half-circles or vertical half-rays. The geodesic flow $\{\tilde{\varphi}^t\}$ on $\mathcal{H}$ is defined as an $\mathbb{R}$-action on the unit tangent bundle $S\mathcal{H}$ which moves a tangent vector along the geodesic defined by this vector with unit speed. The geodesic flow $\{\tilde{\varphi}^t\}$ on $\mathcal{H}$ descents to the *geodesic flow* $\{\varphi^t\}$ on the factor M via the canonical projection

$$\pi : S\mathcal{H} \to SM \tag{2.1}$$

of the unit tangent bundles. Geodesics on M are orbits of the geodesic flow $\{\varphi^t\}$.

A *cross-section* C for the geodesic flow is a subset of the unit tangent bundle SM visited by (almost) every geodesic infinitely often both in the future and in the past. In other words, every $v \in C$ defines an oriented geodesic $\gamma(v)$ on M which will return to C infinitely often. The "ceiling" function $g : C \to \mathbb{R}$ giving the *time of the first return* to C is defined as follows: if $v \in C$ and t is the time of the first return of $\gamma(v)$ to C, then $g(v) = t$. The map $R : C \to C$ defined by $R(v) = \varphi^{g(v)}(v)$ is called the *first return map*. Thus $\{\varphi^t\}$ can be represented as a *special flow* on the space

$$C^g = \{(v,s) : v \in C,\ 0 \leq s \leq g(v)\},$$

given by the formula $\varphi^t(v,s) = (v, s+t)$ with the identification $(v, g(v)) = (R(v), 0)$.

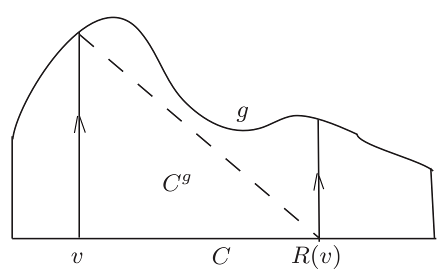


Figure 2. Geodesic flow is a special flow

Let $\mathcal{N}$ be a finite or countable alphabet, $\mathcal{N}^{\mathbb{Z}} = \{x = \{n_i\}_{i\in\mathbb{Z}} \mid n_i \in \mathcal{N}\}$ be the space of all bi-infinite sequences endowed with the Tikhonov (product) topology,

$$\sigma : \mathcal{N}^{\mathbb{Z}} \to \mathcal{N}^{\mathbb{Z}} \text{ defined by } (\sigma x)_i = n_{i+1}$$

be the left shift map, and $\Lambda \subset \mathcal{N}^{\mathbb{Z}}$ be a closed σ-invariant subset. Then (Λ, σ) is called a *symbolic dynamical system*. There are some important classes of such dynamical systems. The space $(\mathcal{N}^{\mathbb{Z}}, \sigma)$ is called the *full shift* (or the *topological Bernoulli shift*). If the space Λ is given by a set of simple transition rules which can be described with the help of a matrix consisting of zeros and ones, we say that (Λ,σ) is a *one-step topological Markov chain* or simply a *topological Markov chain* (also called a *subshift of finite type*). A factor of a topological Markov chain is called a *sofic shift*. (See [10, §1.9] for the definitions.)

In order to represent the geodesic flow as a special flow over a symbolic dynamical system, one needs to choose an appropriate cross-section C and code it, i.e. to find an appropriate symbolic dynamical system (Λ,σ) and a continuous surjective map $\mathrm{Cod}:\Lambda\to C$ (in some cases the actual domain of Cod is Λ except a finite or countable set of excluded sequences) defined such that the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\ \sigma\ } & \Lambda \\ {\scriptstyle \mathrm{Cod}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Cod}} \\ C & \xrightarrow{\ R\ } & C \end{array}$$

is commutative. We can then talk about *coding sequences* for geodesics defined up to a shift which corresponds to a return of the geodesic to the cross-section C. Notice that usually the coding map is not injective but only finite-to-one (see e.g. [2, §3.2 and §5]).

There are two essentially different methods of coding geodesics on surfaces of constant negative curvature. The geometric code, with respect to a given fundamental region, is obtained by a construction universal for all Fuchsian groups. The second method is specific for the modular group and is of arithmetic nature: it uses continued fraction expansions of the end points of the geodesic at infinity and a so-called reduction theory (see [15, 14] for the three classical cases). Here we will describe a general method of arithmetic coding via (a,b)-continued fractions that is based on study of the attractor of the associated natural extension map. This approach, coupled with ideas of Bowen and Series [7], may be useful for coding of geodesics on quotients by general Fuchsian groups.

3. The reduction procedure

In what follows we will denote the end points of geodesics on $\mathcal{H}$ by u and w, and whenever we refer to such geodesics, we use (u,w) as their coordinates on $\bar{\mathbb{R}}^2$ $(u\neq w)$.

The reduction procedure for coding symbolically the geodesic flow on the modular surface via continued fraction expansions was presented for the three classical cases in [14]; for a survey on symbolic dynamics of the geodesic flow see also [15]. Here we describe the reduction procedure for (a,b)-continued fractions and explain how it can be used for coding purposes.

Let γ be an arbitrary geodesic on $\mathcal{H}$ from u to w (irrational end points), and $w=\lfloor n_0,n_1,\dots\rceil_{a,b}$. We construct the sequence of real pairs $\{(u_k,w_k)\}$ $(k\geq 0)$ defined by

$$u_0=u,\ w_0=w \text{ and } w_{k+1}=ST^{-n_k}w_k\,,\quad u_{k+1}=ST^{-n_k}u_k\,. \tag{3.1}$$

Each geodesic γ_k from u_k to w_k is $PSL(2,\mathbb{Z})$-equivalent to γ by construction. It is convenient to describe this procedure using the *reduction map* that combines the appropriate iterate of the map $F_{a,b}$:

$$R_{a,b}:\mathbb{R}^2\setminus\Delta\to\mathbb{R}^2\setminus\Delta$$

given by the formula $R_{a,b}(u,w) = (ST^{-n}u, ST^{-n}w)$, where n is the first digit in the (a,b)-expansion of w. Notice that $(u_k, w_k) = R^k_{a,b}(u,w)$.

Definition 3.1. A geodesic in $\mathcal{H}$ from u to w is called (a,b)*-reduced* if $(u,w) \in \Lambda_{a,b}$, where

$$\Lambda_{a,b} = F_{a,b}(D_{a,b} \cap \{a \le w \le b\}) = S(D_{a,b} \cap \{a \le w \le b\}).$$

According to Theorem 1.1, for (almost) every geodesic γ from u to w in $\mathcal{H}$, the above algorithm produces in finitely many steps an (a,b)-reduced geodesic $PSL(2,\mathbb{Z})$-equivalent to γ, and an application of this algorithm to an (a,b)-reduced geodesic produces another (a,b)-reduced geodesic. In other words, there exists a positive integer ℓ such that $R^\ell_{a,b}(u,w) \in \Lambda_{a,b}$ and $R_{a,b} : \Lambda_{a,b} \to \Lambda_{a,b}$ is bijective (with the exception of some segments of the boundary of $\Lambda_{a,b}$ and their images).

Let γ be a reduced geodesic with the repelling point $u \neq 0$ and the attracting point

$$w = \lfloor n_0, n_1, \dots \rceil_{a,b}. \tag{3.2}$$

Then, by successive applications of the map $R_{a,b}$, we obtain a sequence of real pairs $\{(u_k, w_k)\}$ $(k \ge 0)$ (see (3.1) above) such that each geodesic γ_k from u_k to w_k is (a,b)-reduced. Using the bijectivity of the map $R_{a,b}$, we extend the sequence (3.2) to the past to obtain a bi-infinite sequence of integers

$$\lfloor \gamma \rceil = \lfloor \dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots \rceil, \tag{3.3}$$

called the *coding sequence* of γ, as follows. There exists an integer $n_{-1} \neq 0$ and a real pair $(u_{-1}, w_{-1}) \in \Lambda_{a,b}$ such that $ST^{-n_{-1}}w_{-1} = w = w_0$ and $ST^{-n_{-1}}u_{-1} = u = u_0$. Notice that $\lfloor w_{-1} \rceil_{a,b} = n_{-1}$. By uniqueness of the (a,b)-expansion, we conclude that $w_{-1} = \lfloor n_{-1}, n_0, n_1, \dots \rceil_{a,b}$. Continuing inductively, we define the sequence of integers n_{-k} and the real pairs $(u_{-k}, w_{-k}) \in \Lambda_{a,b}$ $(k \ge 2)$, where

$$w_{-k} = \lfloor n_{-k}, n_{-k+1}, n_{-k+2}, \dots \rceil_{a,b}$$

by $ST^{-n_{-k}}w_{-k} = w_{-(k-1)}$ and $ST^{-n_{-k}}u_{-k} = u_{-(k-1)}$. We also associate to γ a bi-infinite sequence of (a,b)-reduced geodesics

$$(\dots, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \dots), \tag{3.4}$$

where γ_k is the geodesic from u_k to w_k.

Remark 3.2. Notice that all "intermediate" geodesics $T^{-s}\gamma_k$ $(1 \le s \le n_k)$ obtained from γ_k using the map $F_{a,b}$ are not (a,b)-reduced.

Proposition 3.3. *A formal minus continued fraction comprised from the digits of the "past" of (3.3),*

$$n_{-1} - \cfrac{1}{n_{-2} - \cfrac{1}{n_{-3} - \cfrac{1}{\ddots}}} = (n_{-1}, n_{-2}, n_{-3}, \dots)$$

converges to $1/u$.

Proof. By [13, Lemma 1.1], it will be sufficient to check that $|n_{-k}| = 1$ implies $n_{-k} \cdot n_{-(k+1)} < 0$, i.e. the digit 1 must be followed by a negative integer and the digit -1 must be followed by a positive integer. We use the following properties of the set $\Lambda_{a,b}$ that can be derived from our knowledge of the shape of the set $D_{a,b}$ determined in [16, Lemmas 5.6, 5.10, 5.11]. The upper part of $\Lambda_{a,b}$ is contained in the region

$$\begin{aligned} &[-1,0]\times\left[-\frac{1}{a},+\infty\right]\cup[0,1]\times\left[-\frac{1}{b-1},+\infty\right] && \text{if } b<1\\ &[-1,0]\times\left[-\frac{1}{a},+\infty\right] && \text{if } b\geq 1. \end{aligned} \tag{3.5}$$

The lower part of $\Lambda_{a,b}$ is contained in the region

$$\begin{aligned} &[-1,0]\times\left[-\infty,-\frac{1}{a+1}\right]\cup[0,1]\times\left[-\infty,-\frac{1}{b}\right] && \text{if } a>-1\\ &[0,1]\times\left[-\infty,-\frac{1}{b}\right] && \text{if } a\leq -1. \end{aligned} \tag{3.6}$$

Recall that $(u_{-(k+1)}, w_{-(k+1)}) = (T^{n_{-(k+1)}}Su_{-k}, T^{n_{-(k+1)}}Sw_{-k})$ for an appropriate integer $n_{-(k+1)} \neq 0$. Suppose $n_{-k} = 1$. Then $w_{-k} > 0$. If $u_{-k} < 0$, then $Su_{-k} > 0$ and $Sw_{-k} < 0$, and it takes a negative power of T to bring it back to (the lower component of) $\Lambda_{a,b}$, i.e. $n_{-(k+1)} < 0$. The case $u_{-k} > 0$, according to (3.5), can only take place if $b \leq 1$. In this case, $-1/(b-1) \leq w_{-k} < b+1$, which is equivalent to $b > 1$, a contradiction. Therefore $n_{-k} = 1$ implies $n_{-(k+1)} < 0$. A similar argument shows that $n_{-k} = -1$ implies $n_{-(k+1)} > 0$. We conclude that the formal minus continued fraction converges. In order to prove that the limit is equal to $1/u$ we use the recursive definition of the digits $n_{-1}, n_{-2}, \dots$, to write

$$\frac{1}{u} = n_{-1} - u_{-1} = n_{-1} - \frac{1}{n_{-2} - u_{-2}} = \cdots = (n_{-1}, n_{-2}, \dots, n_{-k} - u_{-k}) = \cdots,$$

and the conclusion follows since the formal minus continued fraction converges. □

Let

$$C = \{z \in \mathcal{H} \mid |z| = 1,\ -1 \leq \mathrm{Re}\ z \leq 1\}$$

be the upper-half of the unit circle, and

$$C^- = \{z \in \mathcal{H} \mid |z+1| = 1,\ -\frac{1}{2} \leq \mathrm{Re}\ z \leq 0\}$$

and

$$C^+ = \{z \in \mathcal{H} \mid |z-1| = 1,\ 0 \leq \mathrm{Re}\ z \leq \frac{1}{2}\}$$

be the images of the two vertical boundary components of the fundamental region $\mathcal{F}$ under S (see Figure 3).

Proposition 3.4. *Every (a,b)-reduced geodesic either intersects C or both curves C^- and C^+.*

Proof. If a, b are such that $-1 \leq a \leq 0$ and $0 \leq b \leq 1$, then by properties (3.5) and (3.6) of the set $\Lambda_{a,b}$, if $(u,w) \in \Lambda_{a,b}$, then $-1 \leq u \leq 1$ and $w \geq -\frac{1}{a}$ or $w \leq -\frac{1}{b}$, and hence all (a,b)-reduced geodesics intersect C. For the case $b > 1$ we have: if $-1 < u < 0$, then either $w > -\frac{1}{a} > b > 1$ or $w < -\frac{1}{a+1} < -1$, i.e. the geodesic intersects C; if $0 < u < 1$, then (3.5) implies that $w < -\frac{1}{b} < a < 0$, thus the corresponding geodesic intersects C if $w < -1$, and it intersects first C^+ and then C^-, if $-1 < w < 0$. Similarly, for the case $a < -1$ we have: if $0 < u < 1$, then either $w < -\frac{1}{b} < a < -1$ or $w > -\frac{1}{b-1} > 1$, i.e. the geodesic intersects C; if $-1 < u < 0$, then (3.6) implies that $w > -\frac{1}{a} > b > 0$, therefore the corresponding geodesic intersects C if $w > 1$, and it intersects first C^- and then C^+ if $0 < w < 1$. □

Based on Proposition 3.4 we introduce the notion of the *cross-section point*. It is either the intersection of a reduced geodesic γ with C, or, if γ does not intersect C, its first intersection with $C^- \cup C^+$.

Now we can define a map

$$\varphi : \Lambda_{a,b} \to S\mathcal{H}\,, \varphi(u,w) = (z,\zeta)$$

where $z \in \mathcal{H}$ is the cross-section point on the geodesic γ from u to w, and ζ is the unit vector tangent to γ at z. The map is clearly injective. Composed with the canonical projection π introduced in (2.1) we obtain a map

$$\pi \circ \varphi : \Lambda_{a,b} \to SM.$$

Let $C_{a,b} = \pi \circ \varphi(\Lambda_{a,b}) \subset SM$. This set can be described as follows: $C_{a,b} = P \cup Q_1 \cup Q_2$, where P consists of the unit vectors based on the circular boundary of the fundamental region $\mathcal{F}$ pointing inward such that the corresponding geodesic γ on the upper half-plane $\mathcal{H}$ is (a,b)-reduced, Q_1 consists of the unit vectors based on the right vertical boundary of $\mathcal{F}$ pointing inward such that either $S\gamma$ or $TS\gamma$ is (a,b)-reduced (notice that they cannot both be reduced), and Q_2 consists of the unit vectors based on the left vertical boundary of $\mathcal{F}$ pointing inward such that either $S\gamma$ or $T^{-1}S\gamma$ is (a,b)-reduced (see Figure 3). Then a.e. orbit of $\{\varphi^t\}$ returns to $C_{a,b}$, i.e. $C_{a,b}$ is a *cross-section* for $\{\varphi^t\}$, and $\Lambda_{a,b}$ is a parametrization of $C_{a,b}$. The map $\pi \circ \varphi$ is injective, as follows from Remark 3.2: only one of the geodesics γ, $S\gamma$, $T^{-1}S\gamma$, and $TS\gamma$ can be reduced.

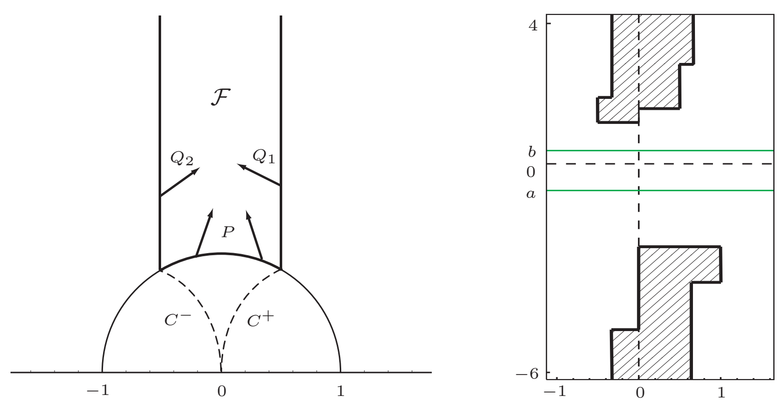


FIGURE 3. The cross-section (left) and its $\Lambda_{a,b}$ parametrization (right)

4. Symbolic coding of the geodesic flow via (a,b)-continued fractions.

If γ is a geodesic on $\mathcal{H}$, we denote by $\bar{\gamma}$ the canonical projection of γ on M. For a given geodesic on M that can be reduced in finitely many steps, we can always choose its lift γ to $\mathcal{H}$ to be (a,b)-reduced.

The following theorem provides the basis for coding geodesics on the modular surface using (a,b)-coding sequences.

Theorem 4.1. *Let γ be an (a,b)-reduced geodesic on $\mathcal{H}$ and $\bar\gamma$ its projection to M. Then*

1. *each geodesic segment of $\bar\gamma$ between successive returns to the cross-section $C_{a,b}$ produces an (a,b)-reduced geodesic on $\mathcal{H}$, and each reduced geodesic $SL(2,\mathbb{Z})$-equivalent to γ is obtained this way;*
2. *the first return of $\bar\gamma$ to the cross-section $C_{a,b}$ corresponds to a left shift of the coding sequence of γ.*

Proof. (1) By lifting a geodesic segment on M starting on $C_{a,b}$ to $\mathcal{H}$, we obtain a segment of a geodesic γ on $\mathcal{H}$ that is reduced by the definition of the cross-section $C_{a,b}$. A coding sequence of $\gamma=\gamma_0$ that connects u_0 to $w_0=\lfloor n_0,n_1,\dots\rceil_{a,b}$,

$$\lfloor\gamma_0\rceil=\lfloor\dots,n_{-2},n_{-1},n_0,n_1,n_2,\dots\rceil,$$

is obtained by extending the sequence of digits of w_0 to the past as explained in the previous section.

Let us assume that $w_0>0$, i.e. $n_0\ge 1$. The case $w_0<0$ can be treated similarly. The geodesic $ST^{-n_0}\gamma_0=\gamma_1$ is reduced by Theorem 1.1. Let z_0 and z_1 be the cross-section points on γ_0 and γ_1, respectively. Then $z_1'=T^{n_0}Sz_1\in\gamma_0$; it is the intersection point of γ_0 with the circle $|z-n_0|=1$. We will show that the geodesic segment of γ_0, $[z_0,z_1']$ projected to M is the segment between two successive returns to the cross-section $C_{a,b}$. Since $ST^{-n_0}(z_1')=z_1$ is the cross-section point on γ_1, the geodesic segment $[z_0,z_1']$ projected to M is between two returns to $C_{a,b}$. Recall that a geodesic in $\mathcal{F}$ consists of countably many oriented geodesic segments between consecutive crossings of the boundary of $\mathcal{F}$ that are obtained by the canonical projection of γ_0 to $\mathcal{F}$.

If z_0 is the intersection of γ_0 with C, there are two possibilities. First, when γ_0 intersects $\mathcal{F}$ or γ_0 does not intersect $\mathcal{F}$ and $ST^{-1}\gamma_0$ exists $\mathcal{F}$ through it circular boundary, and, second, when γ_0 does not intersect $\mathcal{F}$ and $ST^{-1}\gamma_0$ exists $\mathcal{F}$ through it left vertical boundary. In the first case the segments in $\mathcal{F}$ are represented by the intersection with $\mathcal{F}$ of the following geodesics in $\mathcal{H}$: $T^{-1}\gamma_0$, $T^{-2}\gamma_0$, …, $T^{-n_0+1}\gamma_0$, either $ST^{-n_0+1}\gamma_0$ or $T^{-n_0}\gamma_0$, and either γ_0, or $ST^{-1}\gamma_0$.

Suppose that for some intermediate point $z\in\gamma_0$, $z\in[z_0,z_1']$ the unit vector tangent to γ_0 at z, (z,ζ) is projected to $C_{a,b}$. By tracing the geodesic γ_0 inside $\mathcal{F}$, we see that (z,ζ) must be projected to $(\bar z,\bar\zeta)$ with $\bar z$ on the boundary of $\mathcal{F}$ and $\bar\zeta$ directed inward. Then the geodesic through $(\bar z,\bar\zeta)$

(a) enters $\mathcal{F}$ through its vertical boundary and exits it also through the vertical boundary,
(b) enters $\mathcal{F}$ through its vertical boundary and exits through its circular boundary, or
(c) enters $\mathcal{F}$ through its circular boundary and exits through its vertical boundary.

The following assertions are implied by the analysis of the attractor $D_{a,b}$. In case (a), $T^{-1}ST^{-s}\gamma_0$ is not reduced for $1\le s<n_0$ since $s<n_0$, $T^{-s}w_0>b$, hence $ST^{-s}w_0>-\frac{1}{b}$, i.e. $(ST^{-s}u_0,ST^{-s}w_0)\notin D_{a,b}$, therefore

$$(T^{-1}ST^{-s}u_0,T^{-1}ST^{-s}w_0)\notin\Lambda_{a,b}.$$

In case (b), either the segment $T^{-n_0}\gamma_0$ exits through the circular boundary of $\mathcal{F}$, $ST^{-n_0}\gamma_0=\gamma_1$ is reduced, and we reached the point z_1 on the cross-section. If the segment $T^{-n_0+1}\gamma_0$ intersects the circular boundary of $\mathcal{F}$, $ST^{-n_0+1}\gamma_0$ is not reduced. In case (c), ST^{-n_0+1} is not reduced.

In the second case the first digit of w_0, $n_0 = 2$. This is because $n_0 = 1$ would imply $b+1 < w < -\frac{1}{b-1}$ which is impossible. Thus $ST^{-2}\gamma_0 = \gamma_1$ is reduced. In this case the geodesic in $\mathcal{F}$ consists of the intersection with $\mathcal{F}$ of a single geodesic $ST^{-1}\gamma_0$ that enters $\mathcal{F}$ through its right vertical and leave it through its left vertical boundary, since $(TS)T(ST^{-1}\gamma_0) = ST^{-2}\gamma_0 = \gamma_1$ is reduced. In all cases the geodesic segment $[z_0, z_1']$ projected to M is between two consecutive returns to $C_{a,b}$.

If $z_0 \notin C$, by Proposition 3.4, since $w_0 > 0$, $z_0 \in C^-$. Notice that this implies that $a < -1$ and $n_0 = 1$, and $\gamma_1 = ST^{-1}\gamma_0$ is reduced. In this case the geodesic in $\mathcal{F}$ also consists of the intersection with $\mathcal{F}$ of a single geodesic $S\gamma_0$ that enters $\mathcal{F}$ through its right vertical and leave it through its left vertical boundary, since $(TS)T(S\gamma_0) = ST^{-1}\gamma_0 = \gamma_1$ is reduced, and hence the geodesic segment $[z_0, z_1']$ projected to M is between two consecutive returns to $C_{a,b}$. Continuing this argument by induction in both positive and negative direction, we obtain a bi-infinite sequence of points

$$(\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots),$$

where z_k is the cross-section point of the reduced geodesic γ_k in the sequence of γ_0, that represents the sequence of all successive returns of the geodesic γ_0 in M to the cross-section $C_{a,b}$.

If $\tilde{\gamma}_0$ is a reduced geodesic in $\mathcal{H}$, $SL(2,\mathbb{Z})$-equivalent to γ_0, then both project to the same geodesic on M. Therefore, the cross-section point $\tilde{z}_0$ of $\bar{\gamma}_0$ projects on $C_{a,b}$ to a cross-section point z_k of γ_k for some k. This completes the proof of (1).

(2) Since $\gamma_1 = ST^{-n_0}\gamma_0$, $w_1 = ST^{-n_0}w_0 = \lfloor n_1, n_2, \dots \rceil_{a,b}$. The first digit of the past is evidently n_0, and the remaining digits are the same as for γ_0. Thus (2) follows. □

The following corollary is immediate.

Corollary 4.2. *If γ' is $SL(2,\mathbb{Z})$-equivalent to γ, and both geodesics can be reduced in finitely many steps, then the coding sequences of γ and γ' differ by a shift.*

It implies a very important property of (a,b)-continued fractions that escapes a direct proof.

Corollary 4.3. (The Tail Property) *For almost every pair of real numbers that are $SL(2,\mathbb{Z})$-equivalent, the "tails" of their (a,b)-continued fraction expansions coincide.*

Remark 4.4. The set of exceptions in Corollary 4.3 is the same as the one described in Theorem 1.1(3).

Thus we can talk about *coding sequences of geodesics on M*. To any geodesic γ that can be reduced in finitely many steps we associate the coding sequence (3.3) of a reduced geodesic $SL(2,\mathbb{Z})$-equivalent to it. Corollary 4.2 implies that this definition does not depend on the choice of a particular representative: sequences for equivalent reduced geodesics differ by a shift.

Let $X_{a,b}$ be the closure of the set of admissible sequences and σ be the left shift map. The coding map $\mathrm{Cod} : X_{a,b} \to C_{a,b}$ is defined by

$$\mathrm{Cod}(\lfloor \dots, n_{-2}, n_{-1}, n_0, n_1, \dots \rceil) = (1/(n_{-1}, n_{-2}, \dots), \lfloor n_0, n_1, \dots \rceil_{a,b}). \tag{4.1}$$

This map is essentially bijective.

The symbolic system $(X_{a,b},\sigma) \subset (\mathcal{N}^{\mathbb{Z}},\sigma)$ is defined on the infinite alphabet $\mathcal{N} \subset \mathbb{Z}\setminus\{0\}$. The product topology on $\mathcal{N}^{\mathbb{Z}}$ is induced by the distance function

$$d(x,x') = \frac{1}{m}\,,$$

where $x=(n_i), x'=(n'_i) \in \mathcal{N}^{\mathbb{Z}}$, and $m = \max\{k \mid n_i = n'_i \text{ for } |i| \le k\}$.

Proposition 4.5. *The map* Cod *is continuous.*

Proof. If $d(x,x') < \frac{1}{m}$, then the (a,b)-expansions of the attracting end points $w(x)$ and $w(x')$ of the corresponding geodesics given by (3.2) have the same first m digits. Hence the first m convergents of their (a,b)-expansions are the same, and using the properties of (a,b) continued fraction and the rate of convergence of[16, Theorem 2.1] we obtain $|w(x) - w(x')| < \frac{2}{m}$. Similarly, the first m digits in the convergent formal minus continued fraction of $\frac{1}{u(x)}$ and $\frac{1}{u(x')}$ are the same, and hence $|u(x)-u(x')| < \frac{2|u(x)u'(x)|}{m} < \frac{2}{m}$. Therefore the geodesics are uniformly $\frac{2}{m}$-close. But the tangent vectors $v(x), v(x') \in C_{a,b}$ are determined by the intersection of the corresponding geodesic with the unit circle or the curves C^+ and C^-. Hence, by making m large enough we can make $v(x')$ as close to $v(x)$ as we wish. □

In conclusion, the geodesic flow becomes a special flow over a symbolic dynamical system $(X_{a,b},\sigma)$ on the infinite alphabet $\mathcal{N}\subset\mathbb{Z}\setminus\{0\}$. The ceiling function $g_{a,b}(x)$ on $X_{a,b}$ coincides with the time of the first return of the associated geodesic $\gamma(x)$ to the cross-section $C_{a,b}$. One can establish an explicit formula for $g_{a,b}(x)$ as the function of the end points of the corresponding geodesic $\gamma(x)$, $u(x)$, $w(x)$, following the ideas explained in [8]. If $-1\le a\le 0$ and $0\le b\le 1$, then $g_{a,b}(x)$ is cohomologous to $2\log|w(x)|$; more precisely,

$$g_{a,b}(x) = 2\log|w(x)| + \log h(x) - \log h(\sigma x) \text{ where } h(x) = \frac{|w(x)-u(x)|\sqrt{w(x)^2-1}}{w(x)^2\sqrt{1-u(x)^2}}\,.$$

5. Dual codes

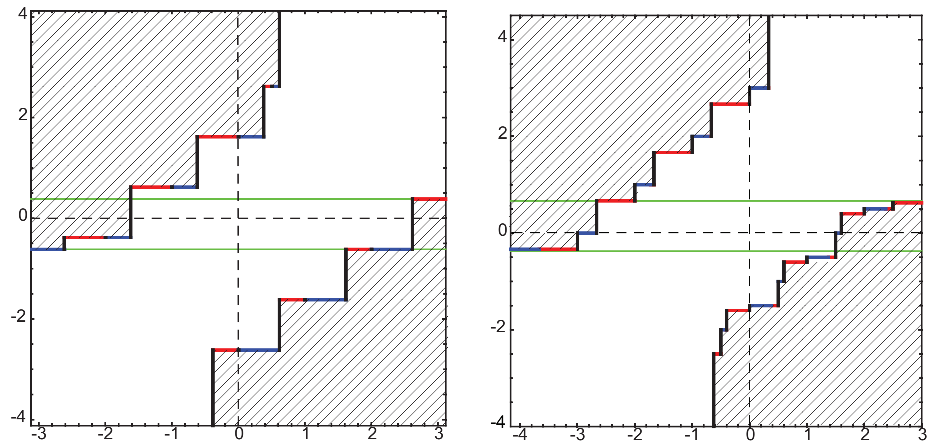
We have seen that a coding sequence for a reduced geodesic from u to w (3.3) is comprised from the sequence of digits in (a,b)-expansion of w and the "past", an infinite sequence of non-zero integers, each digit of which depends on w and u. In some special cases the "past" only depends on u, and, in fact, it will coincide with the sequence of digits of $1/u$ by using a so-called *dual expansion* to (a,b).

Let $\psi(x,y) = (-y,-x)$ be the reflection of the plane about the line $y=-x$.

Definition 5.1. If $\psi(D_{a,b})$ coincides with the attractor set $D_{a',b'}$ for some $(a',b')\in\mathcal{P}$, then the (a',b')-expansion is called the *dual* expansion to (a,b). If $(a',b')=(a,b)$, then the (a,b)-expansion is called *self-dual.*

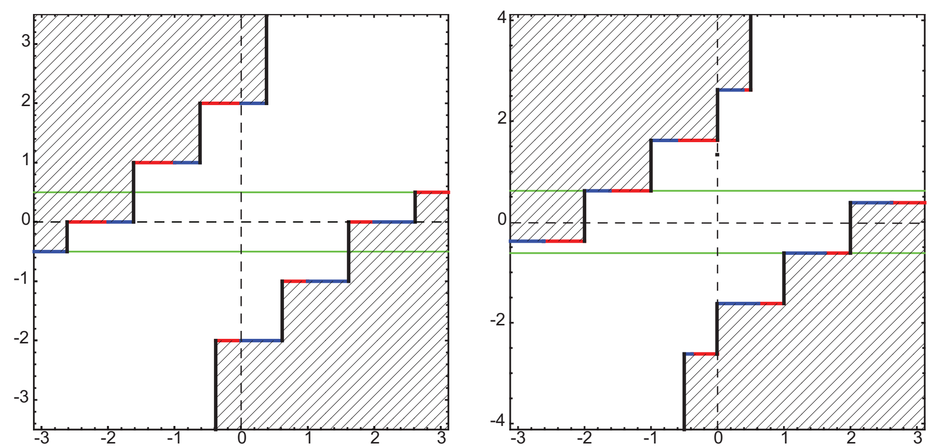
Example 5.2. The classical situations of $(-1,0)$- and $(-1,1)$-expansions are self-dual. Two more sophisticated examples $(\frac{1-\sqrt{5}}{2},\frac{3-\sqrt{5}}{2})$ and $(-\frac{3}{8},\frac{2}{3})$, respectively, are shown in Figure 4.

Example 5.3. The expansions $(-\frac{1}{n},1-\frac{1}{n})$, $n\ge 1$, satisfy a weak cycle property and have dual expansions that are periodic. A classical example in this series is the Hurwitz case $(-\frac{1}{2},\frac{1}{2})$ whose dual is $(\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2})$ (see [9, 14]). Their domains are shown in Figure 5.



$(a,b) = (\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$ $(a,b) = (-\frac{3}{8}, \frac{2}{3})$

FIGURE 4. Domains of self-dual expansions



$(a,b) = (-\frac{1}{2}, \frac{1}{2})$ $(a',b') = (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$

FIGURE 5. Dual expansions

The following result gives equivalent characterizations for an expansion to admit a dual.

Proposition 5.4. *The following are equivalent:*

(i) *the (a,b)-expansion has a dual;*
(ii) *the boundary of the lower part of the set $D_{a,b}$ does not have y-levels with $a < y < 0$, and the boundary of the upper part of the set $D_{a,b}$ does not have y-levels with $0 < y < b$;*
(iii) *a and b do not have the strong cycle property.*

Proof. If the (a,b)-expansion has a dual (a',b')-expansion, then the parameters a', b' are obtained from the boundary of $D_{a,b}$ as follows: the right vertical boundary of the upper part of $D_{a,b}$ is the ray $x = 1 - b'$, and the left vertical boundary of the lower part of $D_{a,b}$ is the ray $x = -1 - a'$. Now assume that (ii) does not hold. Then at least one of the parameters a, b has the strong cycle property, and either the left boundary of the upper part of $\Lambda_{a,b}$ or the right boundary of the lower part of $\Lambda_{a,b}$ is not a straight line. Assume the former. Then the reflection of $D_{a,b}$ with respect to the line $y = -x$ is not $D_{a',b'}$ since the map $F_{a',b'}$ is not bijective on it: the black rectangle in Figure 6 belongs to it, but its image under T^{-1}, colored in grey, does not. Thus (i)⇒(ii).

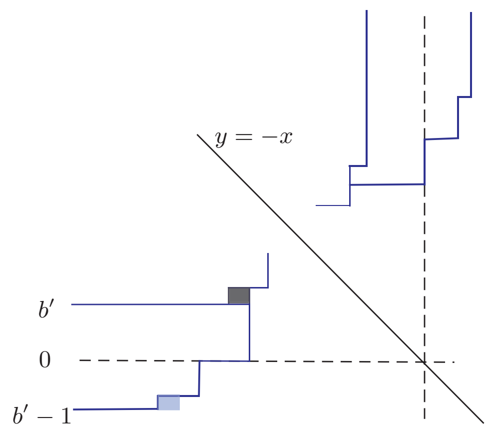


FIGURE 6. Dual expansions and $D_{a,b}$

Conversely, let the vertical line $x = 1 - b'$ be the right boundary of the upper part of $D_{a,b}$ and the vertical line $x = -1 - a'$ be the left boundary of the lower part of $D_{a,b}$. Let $[x_a, \infty] \times \{a\}$ be the intersection of $D_{a,b}$ with the horizontal line at the level a, and $[-\infty, x_b] \times \{b\}$ be the intersection of $D_{a,b}$ with the horizontal line at the level b. Then $a' = \frac{1}{x_b}$ and $b' = \frac{1}{x_a}$. We also see that $1 - b' = -\frac{1}{t}$, where $t = x_b$ or $t < x_b$ if $[t, x_b] \times \{0\}$ is a segment of the boundary of $D_{a,b}$. Then $-b' + 1 = -\frac{1}{t} \le a'$, which implies $b' - a' \ge 1$. By Lemma 5.6 of [16] $x_b \le -1$ and $x_a \ge 1$, therefore

$$-1 \le a' \le 0 \le b' \le 1, \tag{5.1}$$

and

$$\Lambda_{a,b} = D_{a,b} \cap \{(u,w) \in \bar{\mathbb{R}}^2 : -b' \le u \le -a'\}. \tag{5.2}$$

We now show that $\psi(D_{a,b}) = D_{a',b'}$ is the attractor for $F_{a',b'}$, where

$$F_{a',b'} = \psi \circ F_{a,b}^{-1} \circ \psi^{-1}. \tag{5.3}$$

For $(u,w) \in D_{a',b'}$ with $a' < w < b'$ we have $\psi^{-1}(u,w) = (-w,-u)$ with $-b' < u < -a'$, so $\psi^{-1}(u,w) \in \Lambda_{a,b}$ by (5.2), hence $F_{a,b}^{-1}(-w,-u) = (1/w, 1/u)$, and $F_{a',b'}(u,w) = (-1/u, -1/w)$. For $(u,w) \in D_{a',b'}$ with $w > b'$ we have $\psi^{-1}(u,w) = (-w,-u)$ with $u < -b'$, so $F_{a,b}^{-1}(-w,-u) = (-w+1, -u+1)$, and $F_{a',b'}(u,w) = (u-1, w-1)$. Similarly, for $(u,w) \in D_{a',b'}$ with $w < a'$ we have $\psi^{-1}(u,w) = (-w,-u)$ with $u > -a'$, so $F_{a,b}^{-1}(-w,-u) = (-w-1, -u-1)$, and $F_{a',b'}(u,w) = (u+1, w+1)$. This proves that (ii)⇒(i).

Notice that (ii) and (iii) are equivalent by Theorems 4.2 and 4.5 of [16]. □

Remark 5.5. Notice that if an (a,b)-expansion has a dual, then $-1 \le a \le 0 \le b \le 1$. This follows from (5.1) and the fact that the relation of duality is symmetric.

Theorem 5.6. *If an (a,b)-expansion admits a dual expansion (a',b'), and γ_0 is an (a,b)-reduced geodesic, then its coding sequence*

$$\lfloor \gamma_0 \rceil = \lfloor \dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots \rceil, \tag{5.4}$$

is obtained by juxtaposing the (a,b)-expansion of $w_0 = \lfloor n_0, n_1, n_2, \dots \rceil_{a,b}$ and the (a',b')-expansion of $1/u_0 = \lfloor n_{-1}, n_{-2}, \dots \rceil_{a',b'}$. This property is preserved under the left shift of the sequence.

Proof. We will show that the digits in the (a',b')-expansion of $1/u_0$ coincide with the digits of the "past" of (5.4). By (5.3), the following diagram

$$\begin{array}{ccc} \Lambda_{a,b} & \xrightarrow{S\psi} & \Lambda_{a',b'} \\ {\scriptstyle R^{-1}_{a,b}}\downarrow & & \downarrow{\scriptstyle R_{a',b'}} \\ \Lambda_{a,b} & \xrightarrow{S\psi} & \Lambda_{a',b'} \end{array}$$

is commutative. The pair $(u_0, w_0) \in \Lambda_{a,b}$, therefore $(Su_0, Sw_0) \in S\Lambda_{a,b} \subset D_{a,b}$, and $(1/w_0, 1/u_0) \in \Lambda_{a',b'}$. The first digit of the (a',b')-expansion of $1/u_0$ is n_{-1}, so

$$R_{a',b'}(1/w_0, 1/u_0) = (ST^{-n_{-1}}(1/w_0), ST^{-n_{-1}}(1/u_0))$$

maps $\Lambda_{a',b'}$ to itself. Then

$$(u_{-1}, w_{-1}) := R^{-1}_{a,b}(u_0, w_0) = (T^{n_{-1}}Su_0, T^{n_{-1}}Sw_0) \in \Lambda_{a,b}$$

and $(ST^{-n_{-1}}u_{-1}, ST^{-n_{-1}}w_{-1}) = (u_0, w_0)$. Also $w_{-1} = \lfloor n_{-1}, n_0, n_1, \dots \rceil_{a,b}$, and $ST^{-n_{-1}}(1/u_0) = 1/u_{-1} = \lfloor n_{-2}, \dots \rceil_{a',b'}$.

Continuing by induction, one proves that all digits of the "past" of the sequence (5.4) are the digits of the (a',b')-expansion of $1/u_0$.

In order to see what happens under a left shift, we reverse the diagram to obtain:

$$\begin{array}{ccc} \Lambda_{a,b} & \xrightarrow{S\psi} & \Lambda_{a',b'} \\ {\scriptstyle R_{a,b}}\downarrow & & \downarrow{\scriptstyle R^{-1}_{a',b'}} \\ \Lambda_{a,b} & \xrightarrow{S\psi} & \Lambda_{a',b'} \end{array}$$

Since the first digit of (a,b)-expansion of w_0 is n_0,

$$R_{a,b}(u_0, w_0) = (ST^{-n_0}u_0, ST^{-n_0}w_0)$$

maps $\Lambda_{a,b}$ to itself. Then $(u_1, w_1) := (ST^{-n_0}u_0, ST^{-n_0}w_0)$ and $w_1 = \lfloor n_1, n_2, \dots \rceil_{a,b}$. Also

$$(1/w_1, 1/u_1) = R^{-1}_{a',b'}(1/w_0, 1/u_0) = (T^n_0 S(1/w_0), T^n_0 S(1/u_0)),$$

hence $1/u_1 = \lfloor n_0, n_{-1}, n_{-2}, \dots \rceil_{a',b'}$. □

Remark 5.7. Under conditions of Theorem 5.6, if γ_0 projects to a closed geodesic on M, then its coding sequence is periodic, and $w_0 = \lfloor \overline{n_0, n_1, \dots, n_m} \rceil_{a,b}$, $1/u_0 = \lfloor \overline{n_m, \dots, n_1, n_0} \rceil_{a',b'}$.

Theorem 5.8. *If an (a,b)-expansion admits a dual expansion, then the symbolic space $(X_{a,b}, \sigma)$ is a sofic shift.*

Proof. The "natural" (topological) partition of the set $\Lambda_{a,b}$ related to the alphabet $\mathcal{N}$ is $\Lambda_{a,b} = \cup_{n\in\mathcal{N}}\Lambda_n$, where Λ_n are labeled by the symbols of the alphabet $\mathcal{N}$ and are defined by the following condition: $\Lambda_n = \{(u,w) \in \Lambda_{a,b} \mid n_0(u,w) = n_0(w) = n\}$. In order to prove that the space $(X_{a,b},\sigma)$ is sofic one needs to find a topological Markov chain $(M_{a.b},\tau)$ and a surjective continuous map $h : M_{a,b} \to X_{a,b}$ such that $h\circ\tau = \sigma\circ h$.

Notice that the elements Λ_n are rectangles for large n; in fact, at most two elements in the upper part and at most two elements in the lower part of $\Lambda_{a,b}$ are incomplete rectangles (see Figure 7).

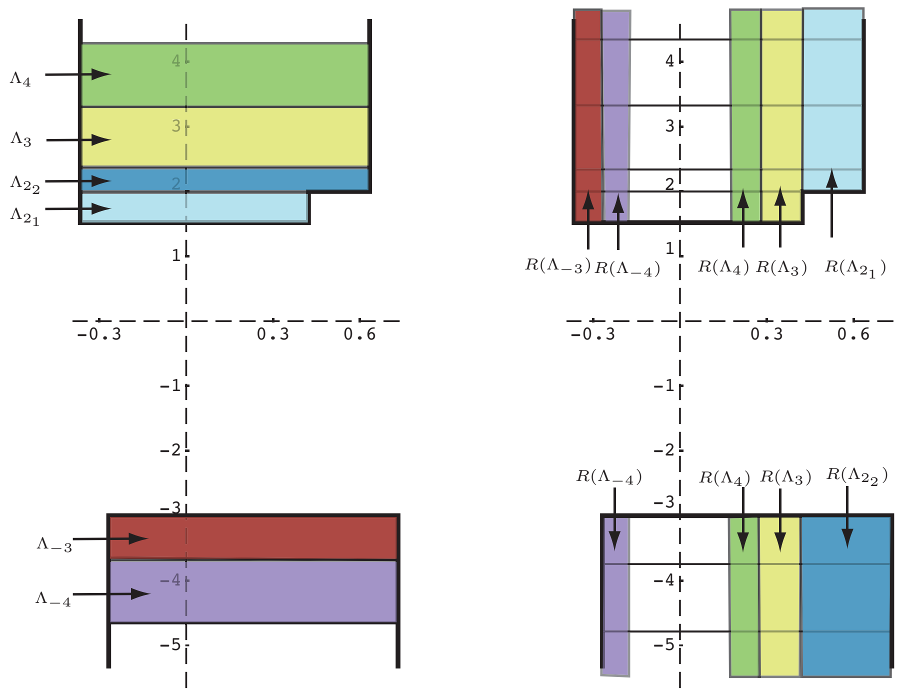


FIGURE 7. The partition of $\Lambda_{a,b}$ and its image through $R_{a,b}$.

Since $\Lambda_{a,b}$ has finite rectangular structure, we can sub-divide horizontally these incomplete rectangles into rectangles, and extend the alphabet $\mathcal{N}$ by adding subscripts to the corresponding elements of $\mathcal{N}$. For example, if Λ_2 is subdivided into two rectangles, $\Lambda_2 = \cup_{i=1}^2\Lambda_{2_i}$, the "digit" 2 will give rise to two digits, $2_1, 2_2$ in the extended alphabet $\mathcal{N}'$ (see Figure 7). We denote the new partition of $\Lambda_{a,b}$ by $\cup_{n\in\mathcal{N}'}M_n$. Notice that it consists of rectangles with horizontal and vertical sides. Since the first return R to $\Lambda_{a,b}$ corresponds to the left shift of the coding sequence x associated to the geodesic (u,w), we see that $x = \{n_k\}_{-\infty}^{\infty}$, where n_k is defined by $R^k(u,w) \in \Lambda_{n_k}$. Now we define the symbolic space $M_{a,b}$ as follows: to each sequence $x \in X_{a,b}$ we associate a geodesic (u,w) by (4.1), and define a new coding sequence $y = \{m_k\}_{-\infty}^{\infty}$, where m_k is defined by $R^k(u,w) \in M_{m_k}$, and τ is the left shift.

We will prove that $(M_{a,b},\tau)$ is a topological Markov chain. For this, in accordance to [2, Theorem 7.9], it is sufficient to prove that for any pair of distinct symbol $n,m\in\mathcal{N}'$, $R(M_n)$ and M_m either do not intersect, or intersect "transversally" i.e. their intersection is a rectangle with two horizontal sides belonging to the horizontal boundary of M_m and two vertical sides belonging to the vertical boundary of $R(M_n)$. Let us recall that $-1\le a\le 0\le b\le 1$ (see Remark 5.5). Therefore, if $M_n=\Lambda_n$ is a complete rectangle, it is, in fact, a 1×1 square, and its image under R is an infinite vertical rectangle intersecting all M_m transversally. If M_n is obtained by subdivision of some Λ_k and belongs to the lower part of $\Lambda_{a,b}$, its horizontal boundaries are the levels of the step-function defining the lower component of $D_{a,b}$, and by Proposition 5.4, since the lower boundary of $D_{a,b}$ does not have y-levels with $a<y<0$, its image is a vertical rectangle intersecting only the lower component of $D_{a,b}$ whose horizontal boundaries are the levels of the step-function defining the lower component of $D_{a,b}$. Therefore, all possible intersections with M_m are transversal. A similar argument applies to the case when M_n belongs to the upper part of $\Lambda_{a,b}$. The map $h:M_{a,b}\to X_{a,b}$ is obviously continuous, surjective, and, in addition, $h\circ\tau=\sigma\circ h$. □

6. Invariant measures and ergodic properties

Based on the finite rectangular geometric structure of the domain $D_{a,b}$ and the connections with the geodesic flow on the modular surface, we study some of the measure-theoretic properties of the Gauss-type map $\hat f_{a,b}:[a,b)\to[a,b)$,

$$\hat f_{a,b}(x)=-\frac{1}{x}-\left\lfloor -\frac{1}{x}\right\rceil_{a,b},\quad \hat f_{a,b}(0)=0\,. \tag{6.1}$$

Notice that the associated natural extension map $\hat F_{a,b}$

$$\hat F_{a,b}(x,y)=\left(\hat f_{a,b}(x),-\frac{1}{y-\lfloor -1/x\rceil_{a,b}}\right) \tag{6.2}$$

is obtained from the map $F_{a,b}$ induced on the set $\Lambda_{a,b}$ by the change of coordinates

$$x=-1/w,\ y=u \tag{6.3}$$

(or, equivalently, on the set $D_{a,b}\cap\{(u,w)|a\le w<b\}$ by the change of coordinates $x=w,y=-1/u$). Therefore the domain $\hat\Lambda_{a,b}$ of $\hat F_{a,b}$ is easily identified knowing $\Lambda_{a,b}$ and may be considered as its "compactification".

Many of the measure-theoretic properties of $\hat f_{a,b}$ and $\hat F_{a,b}$ (existence of an absolutely continuous invariant measure, ergodicity) follow from the fact that the geodesic flow φ^t on the modular surface M can be represented as a special flow $(R_{a,b},\Lambda_{a,b},g_{a,b})$ on the space

$$\Lambda_{a,b}^{g_{a,b}}=\{(u,w,t):(u,w)\in\Lambda_{a,b},\ 0\le t\le g_{a,b}(u,w)\}$$

(see Section 2). We recall that $R_{a,b}=F_{a,b}|_{\Lambda_{a,b}}$ and $g_{a,b}$ is the ceiling function (the time of the first return to the cross-section $C_{a,b}$) parametrized by $(u,w)\in\Lambda_{a,b}$.

We start with the fact that the geodesic flow $\{\varphi^t\}$ preserves the smooth (Liouville) measure $dm=\dfrac{dudwdt}{(w-u)^2}$ (see, e.g., [3]), hence $R_{a,b}$ preserves the absolutely continuous measure $d\rho = \dfrac{dudw}{(w-u)^2}$. Using the change of coordinates (6.3), the map $\hat{F}_{a,b}$ preserves the absolutely continuous measure $d\nu = \dfrac{dxdy}{(1+xy)^2}$.

The set $\Lambda_{a,b}$ has finite measure $d\rho$ if $a \neq 0$ and $b \neq 0$, since it is uniformly bounded away from the line $\Delta = \{(u,w) \ : u = w\} \subset \mathbb{R}^2$ (see relations (3.5) and (3.6)). In this situation, we can normalize the measure $d\rho$ to obtain the smooth probability measure

$$d\rho_{a,b} = \frac{d\rho}{K_{a,b}} = \frac{dudw}{K_{a,b}(w-u)^2} \tag{6.4}$$

where $K_{a,b} = \rho(\Lambda_{a,b})$. Similarly, if $a \neq 0$ and $b \neq 0$, the map $\hat{F}_{a,b}$ preserves the smooth probability measure

$$d\nu_{a,b} = \frac{dxdy}{K_{a,b}(1+xy)^2} \tag{6.5}$$

and $K_{a,b} = \rho(\Lambda_{a,b}) = \nu(\hat{\Lambda}_{a,b})$.

Returning to the Gauss-type map, $\hat{f}_{a,b}$, one can obtain explicitly a Lebesgue equivalent invariant probability measure $\mu_{a,b}$ by projecting the measure $\nu_{a,b}$ onto the x-coordinate (push-forward); this is equivalent to integrating $\nu_{a,b}$ over $\hat{\Lambda}_{a,b}$ with respect to the y-coordinate as explained in [4].

We can immediately conclude that the systems $(\hat{F}_{a,b}, \nu_{a,b})$ and $(\hat{f}_{a,b}, \mu_{a,b})$ are ergodic from the fact that the geodesic flow $\{\varphi^t\}$ is ergodic with respect to dm. By using some well-known results about one dimensional maps that are piecewise monotone and expanding, and the implications for their natural extension maps, we can establish stronger measure-theoretic properties: $(\hat{f}_{a,b}, \mu_{a,b})$ is exact, and $(\hat{F}_{a,b}, \nu_{a,b})$ is a Bernoulli shift. Here we follow the presentation from [23] based on [21, 18].

Theorem 6.1. *For any $a \neq 0$ and $b \neq 0$, the system $(\hat{f}_{a,b}, \mu_{a,b})$ is exact and its natural extension $(\hat{F}_{a,b}, \nu_{a,b})$ is a Bernoulli shift.*

Proof. Let us consider first the case $-1 < a < 0 < b < 1$. The interval (a,b) admits a countable partition $\xi = \{X_i\}_{i \in \mathbb{Z}\setminus\{0\}}$ of open intervals and the map $\hat{f}_{a,b}$ satisfies conditions (A), (F), (U) listed in [23]. Condition (A) is Adler's distortion estimate:

$$(A): \quad \hat{f}''_{a,b}/(\hat{f}'_{a,b})^2 \text{ is bounded on } X = \cup_{i\in\mathbb{Z}\setminus\{0\}} X_i,$$

condition (F) requires the finite image property of the partition ξ,

$$(F): \quad \hat{f}_{a,b}(\xi) = \{\hat{f}_{a,b}(X_i)\}_{i\in\mathbb{Z}\setminus\{0\}} \text{ is finite,}$$

while condition (U) is a uniformly expanding condition

$$(U): \quad |\hat{f}'_{a,b}| \geq \tau > 1 \text{ on } X.$$

Let $m \geq 0$ and $n \geq 0$ be such that $a - m \leq -1/b < a - m - 1$ and $b + n \leq -1/a < b + n + 1$. Consider the open intervals

$$X_1 = \left(-\frac{1}{a-m-1}\, ,\, b\right),\ X_i = \left(-\frac{1}{a-m-i}\, ,\, -\frac{1}{a-m-i+1}\right) \text{ for } i \geq 2$$

and

$$X_{-1} = \left(a\,, -\frac{1}{b+n+1}\right),\ X_{-i} = \left(-\frac{1}{b+n+i-1}\,, -\frac{1}{b+n+i}\right) \text{ for } i \geq 2.$$

The map $\hat{f}_{a,b}$ satisfies conditions (A), (F), (U) with respect to the partition $\xi = \{X_i\}_{i\in\mathbb{Z}\setminus\{0\}}$. Indeed, $|\hat{f}''_{a,b}/(\hat{f}'_{a,b})^2| \leq 2$ on X, the collection of images $\hat{f}_{a,b}(\xi)$ consists of four sets $\hat{f}_{a,b}(X_1)$, $\hat{f}_{a,b}(X_{-1})$, $(b-1,b)$, $(a,a+1)$, and $|\hat{f}'_{a,b}| \geq \min\{\frac{1}{a^2}, \frac{1}{b^2}\} > 1$ on X. Zweimüller [23] showed that any one-dimensional map for which conditions (A), (F), (U) hold is exact and satisfies Rychlik's conditions described in [18], hence its natural extension map is Bernoulli.

We analyze now the case $b \geq 1$. Let $K > 0$ be the smallest integer such that $b(a+1)^K < 1$. We will show that there exists $\gamma > 1$ such that, for every $x \in \bigcap_{i=0}^{K} \hat{f}_{a,b}^{-i}(X)$, some iterate $\hat{f}_{a,b}^n(x)$ with $n \leq K+1$ is expanding, i.e. $|(\hat{f}_{a,b}^n)'(x)| \geq \gamma$. (For the rest of the proof, we simplify the notations and let $\hat{f}$ denote the map $\hat{f}_{a,b}$.) Notice that if $x \in \bigcap_{i=0}^{n-1} \hat{f}^{-i}(X)$, then $\hat{f}^n$ is differentiable at x and

$$\frac{d}{dx}\hat{f}^n(x) = \frac{1}{(x\hat{f}(x)\cdots\hat{f}^{n-1}(x))^2}\,.$$

Assume that $ab > -1$. We look at the following cases:

(i) If $a < x < 0$, then $b-1 \leq \hat{f}(x) \leq b$, and $|x\hat{f}(x)| \leq |ab| < 1$.

(ii) If $0 < x < b$, then $a \leq \hat{f}(x) \leq a+1$. Let K be such that $b(a+1)^K < 1$. Then either there exists $1 \leq n \leq K$ such that $0 < \hat{f}^i(x) < a+1$ for $i = 1, 2, \ldots, n-1$ and $a < \hat{f}^n(x) < 0$, or $0 < \hat{f}^i(x) < a+1$ for $i = 1, 2, \ldots, K$. In the former case we have that

$$|x\hat{f}(x)\cdots\hat{f}^n(x)| \leq |ab(a+1)^{n-1}| < 1\,, \tag{6.6}$$

while in the latter case

$$|x\hat{f}(x)\cdots\hat{f}^K(x)| \leq |b(a+1)^K| < 1\,. \tag{6.7}$$

In the case $ab = -1$, let $\tau, \epsilon > 0$ be sufficiently small such that

$$b < -1/(a+\tau) < b+1 \text{ and } a-1 < -1/(b-\epsilon) < a\,.$$

We have:

(i) If $a < x < a+\tau$, then $b-1 < \hat{f}(x) < -1/(a+\tau)$, and $|x\hat{f}(x)| \leq |a/(a+\tau)| < 1$. If $a+\tau \leq x < 0$, then $|x\hat{f}(x)| \leq |b(a+\tau)| < 1$.

(ii) If $b-\epsilon < x < b$, then $0 < \hat{f}(x) < a+1$ and one has either (6.6) with $n \geq 2$ or (6.7). If $0 < x \leq b-\epsilon$, then one has (6.6) or (6.7) where b is replaced by $b-\epsilon$.

In conclusion, there exists a constant $\gamma > 1$ such that for every $x \in \bigcap_{i=0}^{K} \hat{f}_{a,b}^{-i}(X)$ some iterate $\hat{f}_{a,b}^n(x)$ with $n \leq K+1$ satisfies the condition $|(\hat{f}_{a,b}^n)'(x)| \geq \gamma$. This implies that the iterate $\hat{f}_{a,b}^N$, with $N = (K+1)!$, is uniformly expanding, i.e. it satisfies property (U). Since properties (A) and (F) are automatically satisfied by any iterate of $\hat{f}_{a,b}$ (see [23]), we have that $\hat{F}_{a,b}^N$ is Bernoulli. Using one of Ornstein's results [17, Theorem 4, p. 39], it follows that $\hat{F}_{a,b}$ is Bernoulli. □

The next result gives a formula of the measure theoretic entropy of $(\hat{F}_{a,b}, \nu_{a,b})$.

Theorem 6.2. *The measure-theoretic entropy of* $(\hat{F}_{a,b}, \nu_{a,b})$ *is given by*

$$h_{\nu_{a,b}}(\hat{F}_{a,b}) = \frac{1}{K_{a,b}} \frac{\pi^2}{3} \tag{6.8}$$

Proof. To compute the entropy of this two-dimensional map, we use Abramov's formula [1]:

$$h_{\tilde{m}}(\{\phi^t\}) = \frac{h_{\rho_{a,b}}(R_{a,b})}{\int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b}},$$

where $\tilde{m}$ is the normalized Liouville measure $d\tilde{m} = \frac{dm}{m(SM)}$. It is well-known that $m(SM) = \pi^2/3$ (see [3]) and $h_{\tilde{m}}(\{\phi^t\}) = 1$ (see, e.g., [20]). The measure $d\tilde{m}$ can be represented by the Ambrose-Kakutani theorem [5] as a smooth probability measure on the space $\Lambda_{a,b}^{g_{a,b}}$

$$d\tilde{m} = \frac{d\rho_{a,b} dt}{\int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b}} \tag{6.9}$$

where $d\rho_{a,b}$ is the probability measure on the cross-section $\Lambda_{a,b}$ given by (6.4). This implies that

$$d\tilde{m} = \frac{d\rho dt}{K_{a,b} \int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b}} = \frac{dm}{K_{a,b} \int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b}}.$$

Therefore $K_{a,b} \int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b} = m(SM) = \pi^2/3$ and

$$h_{\nu_{a,b}}(\hat{F}_{a,b}) = h_{\rho_{a,b}}(R_{a,b}) = \int_{\Lambda_{a,b}} g_{a,b} d\rho_{a,b} = \frac{1}{K_{a,b}} \frac{\pi^2}{3}.$$

□

Since $(\hat{F}_{a,b}, \nu_{a,b})$ is the natural extension of $(\hat{f}_{a,b}, \mu_{a,b})$, the measure-theoretic entropies of the two systems coincide, hence

$$h_{\mu_{a,b}}(\hat{f}_{a,b}) = \frac{1}{K_{a,b}} \frac{\pi^2}{3}. \tag{6.10}$$

As an immediate consequence of the above entropy formula we derive a growth rate relation for the denominators of the partial quotients p_n/q_n of (a, b)-continued fraction expansions, similar to the classical cases.

Proposition 6.3. *Let* $\{q_n(x)\}$ *be the sequence of the denominators of the partial quotients* p_n/q_n *associated to the* (a, b)*-continued fraction expansion of* $x \in [a, b)$*. Then*

$$\lim_{n\to\infty} \frac{\log q_n(x)}{n} = \frac{1}{2} h_{\mu_{a,b}}(\hat{f}_{a,b}) = \frac{1}{K_{a,b}} \frac{\pi^2}{6} \quad \text{for a.e. } x. \tag{6.11}$$

Proof. The proof is similar to the classical case: using the Birkhoff's ergodic theorem one has

$$\lim_{n\to\infty} \frac{\log q_n(x)}{n} = -\int_a^b \log|x| d\mu_{a,b}.$$

At the same time, Rokhlin's formula tells us that

$$h_{\mu_{a,b}}(\hat{f}_{a,b}) = \int_a^b \log|\hat{f}'_{a,b}| d\mu_{a,b} = -2 \int_a^b \log|x| d\mu_{a,b},$$

hence the conclusion. □

7. Some explicit formulas for the invariant measure $\mu_{a,b}$

In order to obtain explicit formulas for $\mu_{a,b}$ and $h_{\mu_{a,b}}(\hat{f}_{a,b})$, one obviously needs an explicit description of the domain $D_{a,b}$. In [16] we describe an algorithmic approach for finding the boundaries of $D_{a,b}$ for all parameter pairs (a,b) outside of a negligible exceptional parameter set $\mathcal{E}$. Let us point out that the set $D_{a,b}$ may have an arbitrary large number of horizontal boundary segments. The qualitative structure of $D_{a,b}$ is given by the cycle properties of a and b. This structure remains unchanged for all pairs (a,b) having cycles with similar combinatorial complexity. For a large part of the parameter set the cycle descriptions are relatively simple (see [16, Section 4]) and we discuss it herein.

In what follows, we focus our attention on the situation $-1 \le a \le 0 \le b \le 1$, and due to the symmetry of the parameter set with respect to the parameter line $a=-b$ we assume that $a \le -b$.

We treat the case $1 \le -\frac{1}{a} \le b+1$ and $a \le -\frac{1}{b}+m \le a+1$ (for some $m \ge 1$). The coordinates of the corners of the boundary segments in the upper region $D_{a,b} \cap \{(u,w)|u<0, a\le w\le b\}$ are given by

$$(-2,b-1),\Big(-\frac{3}{2},T^{-2}S(b-1)\Big),\dots,\Big(-\frac{m+1}{m},(T^{-2}S)^{(m-1)}(b-1)\Big),\Big(-1,-\frac{1}{a}-1\Big)$$

while the corners of the boundary segments in the lower region $D_{a,b}\cap\{(u,w)|u>0, a\le w\le b\}$ are given by

$$\Big(m,-\frac{1}{b}+m\Big),(m+1,a+1)\,.$$

Therefore the set $\hat{\Lambda}_{a,b}$ is given by

$$\begin{aligned}\hat{\Lambda}_{a,b} &= \bigcup_{p=1}^{m-1}[(T^{-2}S)^{p-1}(b-1),(T^{-2}S)^{p}(b-1)]\times[0,\frac{p}{p+1}]\\ &\cup[(T^{-2}S)^{m-1}(b-1),-\frac{1}{a}-1]\times[0,\frac{m}{m+1}]\cup[-\frac{1}{a}-1,b]\times[0,1]\\ &\cup[a,-\frac{1}{b}+m]\times[-\frac{1}{m},0]\cup[-\frac{1}{b}+m,a+1]\times[-\frac{1}{m+1},0]\end{aligned} \tag{7.1}$$

Theorem 7.1. *If $1 \le -\frac{1}{a} \le b+1$ and $a \le -\frac{1}{b}+m\le a+1$, then*

$$\mu_{a,b}=\frac{1}{K_{a,b}}h_{a,b}(x)dx\,,$$

where $K_{a,b}=\log[(m-a)(1+b)^{2-m}]$ and $h_{a,b}(x)=h^+_{a,b}(x)+h^-_{a,b}(x)$ with

$$h^+_{a,b}(x)=\begin{cases}\dfrac{1}{x+\frac{p+1}{p}} & \text{if } (T^{-2}S)^{p-1}(b-1)\le x<(T^{-2}S)^{p}(b-1), p=1,\dots,m-1\\[2ex] \dfrac{1}{x+\frac{m+1}{m}} & \text{if } (T^{-2}S)^{m-1}(b-1)\le x<-\frac{1}{a}-1\\[2ex] \dfrac{1}{x+1} & \text{if } -\frac{1}{a}-1\le x<b\end{cases}$$

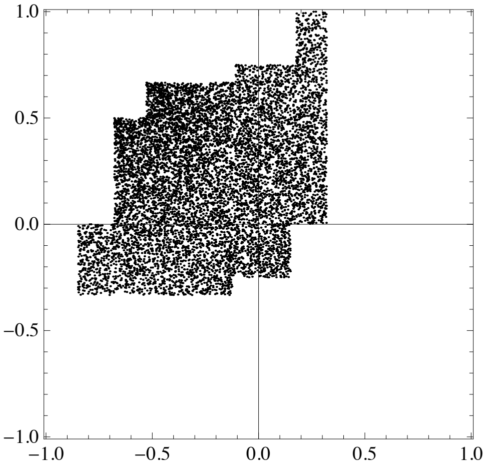


FIGURE 8. Typical domain $\hat{\Lambda}_{a,b}$ for the case studied

and

$$h^-_{a,b}(x) = \begin{cases} \dfrac{1}{m-x} & \text{if } a \le x < -\frac{1}{b} + m \\[2ex] \dfrac{1}{m+1-x} & \text{if } -\frac{1}{b} + m \le x < a+1\,. \end{cases}$$

Proof. The density formulas are obtained from the simple integration result

$$\int_c^d \frac{1}{(1+xy)^2}dy = -\frac{1}{x}\left(\frac{1}{1+dx} - \frac{1}{1+cx}\right) = \frac{d}{1+dx} - \frac{c}{1+cx}\,. \tag{7.2}$$

For the density in the upper part of $\hat{\Lambda}_{a,b}$, $y \ge 0$, all integrals have the lower boundary $c = 0$, hence the result of (7.2) becomes $1/(x + 1/d)$. This gives the description of $h^+_{a,b}(x)$. For the density in the lower part of $\hat{\Lambda}_{a,b}$, $y \le 0$, all integrals have the upper boundary $d = 0$, hence the result $-1/(-1/c-x)$ and the description of $h^-_{a,b}(x)$. By a somewhat tedious computation, we get

$$K_{a,b} = \int_{\Lambda_{a,b}} h_{a,b}(x)dx = \log[(m-a)(1+b)^{2-m}]\,,$$

and this completes the proof. □

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