THE ESTIMATION FROM ABOVE FOR THE TOPOLOGICAL ENTROPY OF A DIFFEOMORPHISM

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1. The topological entropy of a diffeomorphism of a compact manifold is always finite. It follows from the rough estimation [1] which is similar to the earlier estimation of the metric entropy of a diffeomorphism with respect to a smooth invariant measure [2]. Later an exact formula was proved in this case ([3]; the estimation from above belongs to G. A. Margulis). We prove a refined estimation from above for the topological entropy which is similar to the estimation of Margulis. Expressing the right-hand part of our estimation in terms of differential forms we get a relation between the topological entropy and spectral properties of the operator induced by a diffeomorphism in a space of differential forms.

2. Let M be a compact metric space, $T : M \to M$ a homeomorphism of M onto itself. Background material about the topological entropy can be found in [4]. We shall indicate several simple facts concerning the notion of conditional topological entropy which is introduced by M. Misiurewicz [5].

<u>Definition</u>. Let $\mathcal{A},\mathcal{B}$ be two open cover of the space M. The conditional topological entropy $h(\mathcal{A}/\mathcal{B})$ of the cover $\mathcal{A}$ relative to the cover $\mathcal{B}$ is defined by the formula

$$h(\mathcal{A}/\mathcal{B}) = \max_{B\in\mathcal{B}} \log(N_{\mathcal{A}}(B))$$

where $N_{\mathcal{A}}(B)$ is the minimal number of elements of $\mathcal{A}$ which cover the element $B \in \mathcal{B}$.

PROPOSITION 1. Suppose that $\mathcal{B} < \mathcal{C}$, i.e., each element of $\mathcal{C}$ is contained in an element of $\mathcal{B}$. Then $h(A/\mathcal{B}) \geq n(A/\mathcal{C})$.

PROPOSITION 2. $h(A \vee \mathcal{B}) \leq h(\mathcal{B}) + h(A/\mathcal{B})$.

PROOF: Let us denote the minimal number of elements of a subcover of the cover A by N_A. Further, let $\mathcal{B}'$ be a subcover of $\mathcal{B}$, which contains exactly $N_{\mathcal{B}}$ elements. Then

$$N_{A\vee\mathcal{B}} \leq \sum_{B\in\mathcal{B}'} N_A(B) \leq N_{\mathcal{B}} \max_{B\in\mathcal{B}} N_A(B) \quad \text{and}$$

$$h(A\vee\mathcal{B}) = \log N_{A\vee\mathcal{B}} \leq \log N_{\mathcal{B}} + \log \max_{B\in\mathcal{B}} N_A(B) = h(\mathcal{B}) + h(A/\mathcal{B}).$$

PROPOSITION 3. $h(T,A) \leq \lim_{n\to\infty} h(T^nA/A\vee\ldots\vee T^{n-1}A)$.

PROOF: Let us apply the previous proposition n times to the cover $A\vee\ldots\vee T^{n-1}A$. We have

$$h(A\vee TA\vee\ldots\vee T^nA) \leq h(A) + h(TA/A) + \ldots + h(T^nA/A\vee\ldots\vee T^{n-1}A).$$

By the definition of the topological entropy and proposition 1

$$h(T,A) = \overline{\lim_{n\to\infty}} \frac{h(A\vee\ldots\vee T^nA)}{n} \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} h(T^iA/A\vee\ldots\vee T^{i-1}A)}{n} =$$

$$= \lim_{n\to\infty} h(T^nA/A\vee\ldots\vee T^{n-1}A).$$

PROPOSITION 4. $h(T,A) \leq h(TA/A)$.

PROOF: $h(T^iA/A\vee\ldots\vee T^{i-1}A) = h(TA/A\vee T^{-1}A\vee\ldots\vee T^{-(i-1)}A) \leq h(TA/A)$.

(The last inequality follows from proposition 1). Combining this inequality with porposition 3 we obtain

$$h(T,A) \leq \lim_{n\to\infty} (T^nA/A\vee\ldots\vee T^{n-1}A) \leq h(TA/A).$$

3. Let M be an m-dimensional smooth compact Riemannian manifold and $T: M \to M$ be a C^1 diffeomorphism.

<u>THEOREM</u>. $h(T) \le \log \max_{x \in M} \max_{L \subset T_x M} |J(DT_x|_L)| = \alpha(T)$, where J means the Jacobian and the inner maximum is taken over the set of all linear subspaces of the tangent space $T_x M$.

<u>PROOF OF THE THEOREM</u>: It is known [4] that $h(T) = \lim_{k\to\infty} (T, A_k)$ if A_k is an exhaustive sequence of covers i.e., the maximal diameter of elements of the covers A_k tends to zero as $k \to \infty$. We shall consider a special exhaustive sequence of covers having bounded multiplicities.

Let us choose a positive number ε_0 and a finite set of points $x_1,\dots,x_s$ M such that the mappings $\exp_{x_i}$, $i = 1,\dots,s$ are injective on ε_0-balls and the images of these balls cover M. Denote the maximal multiplicity of this cover by N. Let $\delta > 0$ be so small that the images of $(\varepsilon_0 - \delta)$-balls still cover M. Let us denote by $D_y(v,r)$ the ball in the tangent space $T_y M$ of radius r about $v \in T_y M$. Let us fix a sequence of positive numbers $\delta_k \to 0$, $\delta_k < \frac{\delta}{8}$ and cover each of the balls $D_{x_i}(0,\varepsilon_0 - \delta)$, $i = 1,\dots,s$ by a system of δ_k-balls with the maximal multiplicity bounded by a number R which does not depend on k and i. The images of these δ_k-balls under the action of the corresponding mappings $\exp_{x_i}$ form the cover A_k of M with the maximal multiplicity bounded by the number $C = NR$.

For every element $A \in A_k$ there exist $i \in \{1,\dots,s\}$ and a tangent vector $v \in T_{x_i} M$ such that $(\exp_{x_i})^{-1} A = D_{x_i}(v,\delta_k)$. Let $\exp_{x_i} v = x_A$. Suppose that the number $\varepsilon_0 > 0$ is chosen so small that

$$D_{x_A}\left(0,\frac{\delta_k}{2}\right) \subset (\exp_{x_A})^{-1} A \subset D_{x_A}(0,2\delta_k) \tag{1}$$

LEMMA. There exists a constant C' such that for every integer n there exists a positive integer $k(n)$ such that for $k > k(n)$

$$h(\mathcal{A}_k/T^n\mathcal{A}_k) \le \alpha(T^n) + C'$$

PROOF OF THE LEMMA: Let us fix an integer n and choose k to provide the mapping T^n be sufficiently close to a linear mapping on each element A of the cover $\mathcal{A}_k$. To be more precise, let us consider a set A and the corresponding point x_A. Condition (1) implies that:

$$(\exp_{T^n x_A})^{-1}B \subset (\exp_{T^n x_A})^{-1}T^n \exp_{x_A}(D_{x_A}(0,2\delta_k)) \quad \text{where} \quad B = T^n A.$$

We can choose the number k so large (and consequently δ_k so small) that

$$(\exp_{T^n x_A})^{-1}T^n \exp_{x_A}(D_{x_A}(0,2\delta_k)) \subset DT^n D_{x_A}(0,3\delta_k)$$

whence

$$(\exp_{T^n x_A})^{-1}B \subset DT^n D_{x_A}(0,3\delta_k) \tag{2}$$

Besides that, we shall claim the diameter of the set $DT^n D_{x_A}(0,3\delta_k)$ to be less than $\frac{\delta}{2}$. Condition (2) expresses our demand to the mapping DT^n to be close to a linear map. Let us estimate now $h(\mathcal{A}_k/T^n\mathcal{A}_k) = \max_{B\in T^n\mathcal{A}_k} \log N_{\mathcal{A}_k}(B)$. Denote by B' the set $\exp_{T^n x_A} U_{2\delta_k}((\exp_{T^n x})^{-1}B)$ where $U_\alpha(E)$ is an α-neighborhood of the set E. By (2) we have

$$(\exp_{T^n x_A})^{-1}B' = U_{2\delta_k}((\exp_{T^n x_A})^{-1}B) \subset U_{2\delta_k}(DT^n D_{x_A}(0,3\delta_k)) \tag{3}$$

and the diameter of the set at the right hand part of this formula is less than δ.

Let $\mathcal{A}' = \{A' \in \mathcal{A}_k, \quad A' \cap B \neq \emptyset\}$. Obviously the inclusion $A' \in \mathcal{A}'$ implies that $A' \subset B'$. Consequently,

$$\sum_{A' \in A'} v(A') \leq C\, v(B')$$

where v is a Riemannian volume on M and C is the maximal multiplicity of the cover A_k. Thus

$$N_{A_k}(B) \leq |A'| \leq C \frac{v(B')}{\min\limits_{A' \in A'} v(A')} .$$

Compactness of M and our choice of the number ε_0 guarantee that the ratio $\frac{v(A')}{v(A'')}$ for every two elements $A', A'' \in A_k$ bounded from positive constant. In particular $v(A') > C_1 v(A)$ so that

$$N_{A_k}(B) \leq \frac{C}{C_1} \cdot \frac{v(B')}{v(A)} . \tag{4}$$

Now we shall estimate the volume $v(B')$. Let $\sigma : T_{x_A} M \to T_{T^n x_A} M$ be an isometry. Then $v = DT^n \cdot \sigma^{-1} : T_{x_A} M \to T_{x_A} M$ is a linear operator in $T_{x_A} M$. It can be represented in the form $V = U \cdot S$ where U is an isometric operator and S is a positively definite symmetric operator with eigenvalues $\lambda_1, \dots, \lambda_m$, where $\lambda_1 \geq \lambda_2 \dots \lambda_l > 1 \geq$ $\geq \lambda_{l+1} \geq \dots \geq \lambda_m$. Condition (3) shows that there exist constants C_2, C_3 such that

$$v(B') \leq C_2 \bar{v}((\exp_{T^n x_A})^{-1} B') \leq C_3 (3\delta_k)^m \prod_{i=1}^{m} \left(\frac{2}{3} + \lambda_i\right) \tag{5}$$

where $\bar{v}$ is a volume in a tangent space. On the other hand

$$v(A) \geq C_4 \bar{v}((\exp_{x_A})^{-1} A) \geq C_5 \delta_k^m \tag{6}$$

Combining inequalities (4), (5), (6) we get an estimation for $N_{A_k}(B)$:

$$N_{A_k}(B) \leq C_6 \prod_{i=1}^{m} \left(\frac{2}{3} + \lambda_i\right) \leq C_7 \prod_{i=1}^{m} \lambda_i .$$

The value $\prod_{i=1}^{l} \lambda_i$ is equal to $|J(DT^n x|_L)|$ where L is a subspace of $T_n x$ generated by the eigenvectors of S with eigenvalues $\lambda_1, \ldots, \lambda_l$. Thus we obtain the estimation

$$h(A_k/T^n A_k) \le \log \max_{x \in M} \max_{L \subset T_x M} |J(DT^n_x|_L)| + \log C_7 .$$

Lemma is proved.

Now we can finish the proof of the theorem. Let us fix a positive integer n and $\varepsilon > 0$ and choose k according to the lemma. Moreover, k can be chosen so large that

$$nh(T) = h(T^n) < h(T^n, A_k) + \varepsilon .$$

Proposition 4.1 implies that
$h(T^n, A_k) = h(T^{-n}, A_k) \le h(T^{-n}A_k/A_k) = h(A_k/T^n A_k)$, so that $nh(T) < h(A_k/T^n A_k) + \varepsilon$. By the lemma we have $h(A_k/T^n A_k) \le \alpha(T^n) + C'$. But $\alpha(T^n) \le n\alpha(T)$, that $h(T) \le \alpha(T) + \frac{C'+\varepsilon}{n}$. Since n can be chosen arbitrary large, $h(T) \le \alpha(T)$. The theorem is proved.

4. Let us denote by $\Omega^k(M)$ the space of all continuous real-valued differential antisymmetric k-forms on M with the norm

$$\|\omega\| = \max_{x \in M} \max_{\substack{v_1, \ldots, v_k \in T_x M \\ \det(v_1, \ldots, v_k) = 1}} |\omega(v_1 \ldots v_k)|$$

The diffeomorphism T induces a linear operator $T^{\#}_k : \Omega^k(M) \to \Omega^k(M)$. Let us denote the direct sum $\overset{m}{\underset{0}{\oplus}} \Omega^k(M)$ by $\Omega(M)$ and $\overset{m}{\underset{0}{\oplus}} T^{\#}_k$ by $T^{\#}$. The proof of the following proposition is routine.

PROPOSITION 5. $\lim_{n \to \infty} \frac{\alpha(T^n)}{n} = \log s(T^{\#})$

where $s(T^{\#})$ is a spectral radius of the operator $T^{\#}$. The next fact follows immediately from our Theorem and proposition 5.

COROLLARY. $h(T) \le \log s(T^{\#})$.

<u>Remark</u>. K. Krzyzewski has generalized our result from diffeomorphisms to arbitrary C^1 mappings of smooth manifolds. His proof used the definition of the topological entropy through ε-separated sets (see [4]).

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