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Livshitz theorem for the unitary frame flow

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With an Appendix

Spanning sets for cusp forms on complex hyperbolic spaces

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Abstract. Let Γ be a lattice in $SU(n, 1)$. For each loxodromic element $\gamma_0 \in \Gamma$ we define a closed curve $\{\gamma_0\}$ on $\Gamma\backslash SU(n, 1)$ that projects to the closed geodesic on the factor of the complex hyperbolic space $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ associated with γ_0. We prove that the cohomological equation $\mathfrak{D}F = f$ has a solution if f is the lift of a holomorphic cusp form to $SU(n, 1)$ under the following condition: for each restriction of f to $\{\gamma_0\}$ a finite number of Fourier coefficients vanish, and this finite number grows linearly with the length of the curve. This is a generalization of the classical Livshitz theorem for $SU(1, 1)$ (A. Livshitz. *Mat. Zametki* **10** (1971), 555–564) where the curves are the closed geodesics themselves and the vanishing of the integrals of f over them, i.e. the zeroth Fourier coefficients, is both necessary and sufficient. An application of our result to the construction of spanning sets for spaces of holomorphic cusp forms on complex hyperbolic spaces is given in Appendix A.

0. *Introduction*

One form of the celebrated Livshitz theorem [**7**] states that given a compact manifold X, a topologically transitive Anosov flow $\varphi : \mathbb{R} \times X \to X$ and a Lipschitz function $f : X \to \mathbb{R}^m$, the *cohomological equation*

$$\mathfrak{D}F = f, \tag{1}$$

where $\mathfrak{D}$ is the differentiation operator in the direction of the flow, has a solution $F : X \to \mathbb{R}^m$ which is Lipschitz and differentiable in the direction of the flow if

$$f \text{ has zero integrals over all closed orbits of } \varphi. \tag{2}$$

It was proved later [**6**] that if the function f is C^∞, the solution F is also C^∞. The leading example of an Anosov flow is the geodesic flow on the unit tangent bundle to a negatively curved manifold. The original proof works with minor alterations for some non-compact manifolds of finite volume, in which case it is natural to formulate it for the class of bounded Lipschitz functions on X (the proof for geodesic flows corresponding to Fuchsian groups of the first kind is given in [**3**, Appendix]). The hyperbolic properties of the geodesic flow imply the Anosov Closing Lemma and its strengthening (see [**4**, Theorem 6.4.15 and Proposition 6.4.16]). As a consequence, the Livshitz theorem holds. In particular, it can be applied to the geodesic flow on factors of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}} = G/K$, where $G = SU(n,1)$ and K is a maximal compact subgroup of G, by lattices $\Gamma \subset G$, i.e. on the unit tangent bundle to $M = \Gamma\backslash G/K$ denoted by *SM*. Closed geodesics on M are exactly the axes of loxodromic elements in Γ. We denote a closed geodesic corresponding to the axis of a loxodromic element $\gamma_0 \in \Gamma$ in M as well as its lift to *SM* by $[\gamma_0]$.

In this paper we study the *unitary frame flow* on $\Gamma\backslash G$. It projects to the geodesic flow on *SM* and is partially hyperbolic but not Anosov. In fact, it is an isometric extension of the geodesic flow on *SM*. By Moore's ergodicity theorem [**8**, Theorem 2.2.6], the unitary frame flow is ergodic and, hence, topologically transitive but the Closing Lemma does not hold. The reason for this is that the lifts of the closed geodesics to $\Gamma\backslash G$ are usually not closed. However, each lift of the closed geodesic $[\gamma_0]$ belongs to a torus in $\Gamma\backslash G$ which projects to $[\gamma_0]$. Its generators can be chosen in such a way that one of them is a closed curve in $\Gamma\backslash G$ which projects to $[\gamma_0]$, and the others are curves in $\Gamma\backslash G$ which project to a point. We denote the generator of the torus containing the standard lift of $[\gamma_0]$ which projects to $[\gamma_0]$, by $\{\gamma_0\}$. This construction is described in §2.

A loxodromic element in $\gamma_0 \in G$ has two eigenvalues of the form

$$\lambda e^{-i\psi_0} \quad \text{and} \quad \lambda^{-1}e^{-i\psi_0}, \tag{3}$$

where $\lambda > 1$ is real, and all other eigenvalues are of absolute value one. The number ψ_0 is called the *coefficient of loxodromy* of γ_0. If $\gamma_0 \in \Gamma$ is loxodromic, $\ln\lambda = T$ is half the length of the closed geodesic $[\gamma_0]$.

Let $\kappa \geq 1$ be a half-integer such that $(n+1)\kappa$ is an integer, $p_0 = 2(n+1)\kappa$ and $S_{p_0}(\Gamma)$ be the space of holomorphic cusp forms of weight p_0. The cusp forms and their lifts to G are described in §3.

For any loxodromic $\gamma_0 \in \Gamma$ and any $c > 0$, we define a finite set of integers

$$\Phi_{\gamma_0,c} = \left([-cT,\, cT] \cup \frac{p_0\psi_0}{2\pi}\right) \cap \mathbb{Z}, \tag{4}$$

where T and ψ_0 are as before. The main result of this paper is the following generalization of the Livshitz theorem.

THEOREM 1. *Let $c > 0$ and $\tilde{f}$ be the lift of a cusp form $f \in S_{p_0}(\Gamma)$ to G such that for every loxodromic $\gamma_0 \in \Gamma$*

$$\int_{\{\gamma_0\}} \tilde{f}(g_t)e^{-2\pi ikt/T}\,dt = 0 \tag{5}$$

for all $k \in \Phi_{\gamma_0,c}$, where $\Phi_{\gamma_0,c}$ was defined in (4). Then there exists a Lipschitz function F on $\Gamma\backslash G$ differentiable in the direction of the frame flow, such that

$$\mathfrak{D}F = \tilde{f}, \tag{6}$$

where $\mathfrak{D}$ is the operator of differentiation in the direction of the frame flow.

Remark 1. The sufficient condition (5) for solution of the cohomological equation (6) is the vanishing of $|\Phi_{\gamma_0,c}|$ Fourier coefficients of a restriction of f to each $\{\gamma_0\}$, where the number $|\Phi_{\gamma_0,c}|$ grows linearly with T. This is a weaker result than the Livshitz Theorem for the geodesic flow where the vanishing of the zeroth Fourier coefficients (2) is both necessary and sufficient for the solution of (1).

Remark 2. This result was inspired by a question from the theory of holomorphic automorphic forms and provides a valuable application given in Appendix A. It will probably hold for a wider class of functions whose restrictions to fibers belong to certain compact families.

1. *The frame flow on the complex hyperbolic space*

We give here only the information necessary for further exposition: all details can be found in [**1**].

For any $n \geq 1$, $G = SU(n,1)$ is the group of $(n+1) \times (n+1)$ complex unimodular matrices preserving the Hermitian form

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1}$$

on $\mathbb{C}^{n,1}$. The maximal compact subgroup of G

$$K = S(U(n) \times U(1)) = \left\{ \begin{pmatrix} u_n & 0 \\ 0 & a \end{pmatrix} \middle| \, u_n \in U(n),\ a = (\det u_n)^{-1} \right\}. \tag{7}$$

The complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}} = G/K$ can be identified with the unit ball in $\mathbb{C}^n$:

$$B^n = \{z \in \mathbb{C}^n \mid z_1\bar{z}_1 + \cdots + z_n\bar{z}_n < 1\}$$

endowed with a G-invariant Riemannian metric of negative sectional curvature pinched between -1 and $-\frac{1}{4}$ for $n > 1$, and equal to -1 for $n = 1$, called the *Bergman metric*. The last identification is obtained by the biholomorphic embedding of $\mathbb{C}^n$ onto the affine chart of $\mathbb{P}(\mathbb{C}^{n,1})$ defined by $z_{n+1} \neq 0$ and given by

$$z \mapsto \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{8}$$

We shall always assume that *negative vectors* (such that $\langle z, z\rangle < 0$) and *null vectors* (such that $\langle z, z\rangle = 0$) are represented in the form (8) and keep the notation

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - 1 \tag{9}$$

for them.

The group G acts on B^n by biholomorphic transformations (automorphisms): for

$$g = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & b_n \\ c_1 & \dots & c_n & d \end{pmatrix},$$

$$g(z) = \left(\frac{a_{11}z_1 + \dots + a_{1n}z_n + b_1}{c_1 z_1 + \dots + c_n z_n + d}, \dots, \frac{a_{n1}z_1 + \dots + a_{nn}z_n + b_n}{c_1 z_1 + \dots + c_n z_n + d} \right) \tag{10}$$

(sometimes we will also use gz for this action), which is an isometry of B^n with respect to the Bergman metric. The point $0 \in B^n$ is the fixed point of the subgroup K (7) in $G/K = B^n$ and the natural projection

$$\pi : G \to B^n$$

is given by

$$\pi : g \mapsto g(0). \tag{11}$$

The real rank of G is equal to 1 and one can choose

$$A = \left\{ a_t = \begin{pmatrix} 1_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}$$

as a maximal split Abelian subgroup of G. The action of A by right multiplication on G defines a standard flow $\tilde{\alpha}_t$ on G,

$$\tilde{\alpha}_t(g) = g a_t. \tag{12}$$

An orbit of $g \in G$, $\{g a_t \mid t \in \mathbb{R}\}$ projects by (11) to the geodesic $\{g a_t(0) = g a_t g^{-1}(x_g) \mid t \in \mathbb{R}\}$ on B^n passing through $x_g = g(0)$ [**2**, p. 208]. The subgroup A itself projects on B^n to the 'standard geodesic'

$$\mathfrak{I} = \{a_t(0) \mid t \in \mathbb{R}\}$$

passing through 0. There is a subgroup

$$W = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & e^{-i\psi} & 0 \\ 0 & 0 & e^{-i\psi} \end{pmatrix} \middle| \; u \in U(n-1),\ \det u = e^{2i\psi} \right\} \subset K,$$

which, acting on B^n on the left by isometries, fixes the geodesic $\mathfrak{I}$ pointwise. The centralizer of A has the form $Z(A) = AW = WA$. Following [**1**] we call an element $g \in G$ *hyperbolic* if it is conjugate to an element in A and *loxodromic* if it is conjugate to an element in $Z(A)$. Note that the loxodromy coefficient of any hyperbolic element is equal to 0 or π. Following [**1**], we call loxodromic elements for which the loxodromy coefficient is equal to 0 or π 'almost hyperbolic'.

A loxodromic element γ_0 fixes a unique geodesic in B^n called its *axis* and denoted by $C(\gamma_0)$ and has two fixed points in the boundary ∂B^n, one *attracting* and *repelling* which correspond to the eigenvectors with eigenvalues (3). Geodesics naturally lift to G/W which can be identified with the unit tangent bundle $S(G/K)$. For any geodesic C

in B^n passing through a point x_0, there exists a $g \in G$ mapping $\mathfrak{I}$ into C in such a way that $g(0) = x_0$.

Let Γ be a lattice in G and $\Gamma\backslash G/K = M$. The action of A descends to the factor $\Gamma\backslash G$ and defines the flow $\tilde{\varphi}_t$ on $\Gamma\backslash G$ which can be interpreted as a flow of unitary frames of tangent vectors or the *unitary frame flow* which, in turn, projects to the geodesic flow on *SM*. The operator of differentiation in the direction of the frame flow is given by the formula

$$\mathfrak{D} f(g) = \frac{d}{dt} f(\tilde{\varphi}_t(g))|_{t=0},$$

defined on the set of functions on $\Gamma\backslash G$ differentiable along the orbits of $\tilde{\varphi}_t$.

2. *Lift of closed geodesics to* $\Gamma\backslash G$

Let γ_0 be a loxodromic element $\gamma_0 \in \Gamma$ with eigenvalues

$$e^{i\psi_1}, \ldots, e^{i\psi_{n-1}}, \lambda e^{-i\psi_0}, \lambda^{-1} e^{-i\psi_0}.$$

By [**1**, Proposition 10], there exists $h \in G$ such that

$$h^{-1}\gamma_0 h = \begin{pmatrix} e^{i\psi_1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{i\psi_2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{i\psi_{n-1}} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cosh T e^{-i\psi_0} & \sinh T e^{-i\psi_0} \\ 0 & 0 & \cdots & 0 & \sinh T e^{-i\psi_0} & \cosh T e^{-i\psi_0} \end{pmatrix}, \tag{13}$$

where T is half the length of the closed geodesic $[\gamma_0]$ and

$$\psi_1 + \psi_2 + \cdots + \psi_{n-1} = 2\psi_0.$$

Then $h(\mathfrak{I}) = ha_t(0) = C(\gamma_0)$, the axis of γ_0, and all lifts of $C(\gamma_0)$ are parametrized by $w \in W$ and are given by

$$C(\gamma_0)_w = \{hwa_t \mid t \in \mathbb{R}\}.$$

We shall denote the family of lifts of $[\gamma_0]$ to $\Gamma\backslash G$ by $\{[\gamma_0]_w \mid w \in W\}$; $w = 1_{n+1}$ gives the *standard* lift to which we have referred in the Introduction.

Let

$$b_t = \begin{pmatrix} e^{i\psi_1 t/T} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{i\psi_2 t/T} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{i\psi_{n-1} t/T} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cosh t e^{-i\psi_0 t/T} & \sinh t e^{-i\psi_0 t/T} \\ 0 & 0 & \cdots & 0 & \sinh t e^{-i\psi_0 t/T} & \cosh t e^{-i\psi_0 t/T} \end{pmatrix}.$$

We see that $\{b_t \mid t \in \mathbb{R}\}$ is a one-parameter subgroup of G, $b_{t+s} = b_t b_s = b_s b_t$, $b_0 = 1_{n+1}$, and $b_T = h^{-1}\gamma_0 h$.

Let

$$w_{\psi_0} = \begin{pmatrix} e^{i\psi_1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{i\psi_2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{i\psi_{n-1}} & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{-i\psi_0} & 0 \\ 0 & 0 & \cdots & 0 & 0 & e^{-i\psi_0} \end{pmatrix}.$$

For a fixed $w \in W$, let $g_t = hb_t w_{\gamma_0}^{-1} w$. We have

$$g_{t+T} = hb_{t+T} w_{\gamma_0}^{-1} w = hb_T b_t w_{\gamma_0}^{-1} w = \gamma_0 hb_t w_{\gamma_0}^{-1} w = \gamma_0 g_t,$$

hence, $\{g_t \mid 0 \le t \le T\}$ is a closed curve on $\Gamma\backslash G$: we denote it by $\{\gamma_0\}_w$. It projects to the closed geodesic $[\gamma_0]$ on SM. Note that the distance with respect to the Bergman metric differs from the parameter t of the frame flow by a factor of two, so T is half the length of $[\gamma_0]$.

Let $\mathbb{T}^{n-1}$ be the Abelian subgroup of W of matrices in the form

$$\left\{ c_{\bar{x}} = \begin{pmatrix} e^{ix_1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{ix_2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{ix_{n-1}} & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{-ix} & 0 \\ 0 & 0 & \cdots & 0 & 0 & e^{-ix} \end{pmatrix} \;\middle|\; 0 \le x_i \le 2\pi \right\},$$

where

$$\bar{x} = (x_1, \ldots, x_{n-1}), \quad x = \frac{x_1 + x_2 + \cdots + x_{n-1}}{2}.$$

Let $\mathbb{T}^1_{\gamma_0} = \{b_t \mid 0 \le t \le T\}$. Then $h(\mathbb{T}^1_{\gamma_0} \times \mathbb{T}^{n-1}) w_{\gamma_0}^{-1} w = \mathbb{T}^n_{\gamma_0, w}$ is a torus on $\Gamma\backslash G$ which projects to the closed geodesic $[\gamma_0]$.

A generic element of this torus has the form

$$hc_{\bar{x}} b_t w_{\gamma_0}^{-1} w = h \times \begin{pmatrix} e^{(i\psi_1 t/T)+ix_1} & 0 & 0 & 0 & 0 \\ 0 & e^{(i\psi_2 t/T)+ix_2} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & e^{(i\psi_{n-1} t/T)+ix_{n-1}} & 0 & 0 \\ 0 & 0 & 0 & \cosh t\, e^{-(i\psi_0 t/T)-ix} & \sinh t\, e^{-(i\psi_0 t/T)-ix} \\ 0 & 0 & 0 & \sinh t\, e^{-(i\psi_0 t/T)-ix} & \cosh t\, e^{-(i\psi_0 t/T)-ix} \end{pmatrix} \times w_{\gamma_0}^{-1} w,$$

where

$$0 \le x_i \le 2\pi, \quad 0 \le t \le T, \quad x = (x_1 + x_2 + \cdots + x_{n-1})/2.$$

Note that if, for $1 \le j \le n-1$,

$$\frac{\psi_j t}{T} + x_j = \psi_j,$$

then

$$\frac{\psi_0 t}{T} + x = \psi_0.$$

It follows that $c_{\bar{x}} b_t = a_t w_{\psi_0}$ and, hence, the lift of the closed geodesic

$$[\gamma_0]_w = \{h a_t w \mid 0 \le t \le T\}$$

belongs to the torus $\mathbb{T}^n_{\gamma_0, w}$ and is given in the coordinates $(x_1, \ldots, x_{n-1}, t)$ by the linear equations

$$\frac{\psi_j t}{T} + x_j = \psi_j, \quad 1 \le j \le n-1.$$

The operator of differentiation along this orbit of the frame flow is given by

$$\mathfrak{D} = -\sum_{j=1}^{n-1} \frac{\psi_j}{T} \left(\frac{\partial}{\partial x_j} \right) + \frac{\partial}{\partial t}.$$

3. *Cusp forms and their lifts to G*

As in [**1**] we use the following local (partial) coordinates (z, η, ζ) on G:

$$z(g) = g(0) \in B^n, \quad \eta(g) = J(g, 0) \cdot \iota \in S_z(B^n), \tag{14}$$

where ι is the unit tangent vector at 0 to the 'standard geodesic' $\mathfrak{I}$, introduced in §1 and

$$\zeta(g) = j(g, 0) = d^{-1} \neq 0.$$

Left multiplication by $g_0 \in G$ corresponds to the action on $z \in B^n$ by a biholomorphic transformation $g_0(z)$ (10), on $\eta \in S_z(B^n)$ by the Jacobian matrix $J(g_0, z)$, and on ζ by

$$j(g_0, z) = (\det J(g_0, z))^{1/(n+1)}.$$

Let $\kappa \ge 1$ be a half-integer such that $(n+1)\kappa$ is an integer, and $p_0 = 2(n+1)\kappa$. The lift of a cusp form $f \in S_{p_0}(\Gamma)$ to G (see [**1**]) has a nice expression in these coordinates:

$$\tilde{f}(g) = f(z)\zeta^{p_0}.$$

Let us fix a loxodromic $\gamma_0 \in \Gamma$ and

$$w = \begin{pmatrix} u & 0 & 0 \\ 0 & e^{-i\psi} & 0 \\ 0 & 0 & e^{-i\psi} \end{pmatrix} \in W. \tag{15}$$

The restriction of $\tilde{f}$ to the torus $\mathbb{T}^n_{\gamma_0, w}$ is given by

$$\begin{aligned} \tilde{f}(h c_{\bar{x}} b_t w_{\psi_0}^{-1} w) &= f(h c_{\bar{x}} b_t w_{\psi_0}^{-1} w(0)) j(h c_{\bar{x}} b_t w_{\psi_0}^{-1} w, 0)^{p_0} \\ &= e^{i p_0 (\psi_0 - \psi)} f(h(x_t)) j(h, x_t)^{p_0} \zeta_{x_t}^{p_0} \end{aligned}$$

where

$$x_t = c_{\bar{x}} b_t(0) = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \tanh t \\ 1 \end{pmatrix}, \quad \zeta_{x_t} = j(c_{\bar{x}} b_t, 0) = \frac{e^{(i\psi_0 t/T) + ix}}{\cosh t}. \tag{16}$$

Thus it can be rewritten in coordinates $(\bar{x}, t)$ as

$$\begin{aligned}\tilde{f}(\bar{x}, t) = \tilde{f}(hc_{\bar{x}}b_t w) &= e^{ip_0(\psi_0-\psi)}\frac{f(h(x_t))}{\cosh^{p_0} t} j(h, x_t)^{p_0} e^{ip_0\psi_0 t/T} e^{ip_0 x} \\ &= \tilde{f}(0, t)e^{ip_0 x}. \end{aligned} \tag{17}$$

Note that the restrictions to the tori corresponding to different lifts of the closed geodesic $[\gamma_0]_w$ to $\Gamma\backslash G$ differ by a constant multiplicative factor $e^{-ip_0\psi}$ where ψ comes from (15).

We write the Fourier expansion of the restriction of $\tilde{f}$ to the torus $\mathbb{T}^n = \mathbb{T}^n_{\gamma_0,w}$,

$$\tilde{f}(\bar{x}, t) = \sum_{\bar{p},q} \tilde{f}_{\bar{p},q}\chi_{\bar{p},q}, \quad \bar{p} \in \mathbb{Z}^{n-1}, q \in \mathbb{Z},$$

where

$$\chi_{\bar{p},q}(\bar{x}, t) = e^{i(\bar{p}\cdot\bar{x}+(2\pi q/T)t)}$$

and

$$\begin{aligned}\tilde{f}_{\bar{p},q} &= \frac{1}{(2\pi)^{n-1}T}\int_{\mathbb{T}^n} \tilde{f}(\bar{x}, t)\overline{\chi_{\bar{p},q}(\bar{x}, t)}\, d\bar{x}\, dt \\ &= \frac{1}{(2\pi)^{n-1}T}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^T \tilde{f}(0, t)e^{ip_0 x}e^{-i(\bar{p}\cdot\bar{x}+(2\pi q/T)t)}\, d\bar{x}\, dt.\end{aligned}$$

It follows from (17) that $\tilde{f}_{\bar{p},q} = 0$ for $\bar{p} \neq p*$, where

$$p* = \tfrac{1}{2}(p_0, \ldots, p_0), \tag{18}$$

and

$$\tilde{f}_{\bar{p}*,q} = \frac{1}{T}\int_0^T \tilde{f}(0, t)e^{-2\pi iqt/T}\, dt = \frac{1}{T}\int_{\{\gamma_0\}_w} \tilde{f}(g_t)e^{-2\pi iqt/T}\, dt.$$

It follows from the remark after (17) that

$$\int_{\{\gamma_0\}_w} \tilde{f}(g_t)e^{-2\pi iqt/T}\, dt = e^{-ip_0\psi}\int_{\{\gamma_0\}} \tilde{f}(g_t)e^{-2\pi iqt/T}\, dt, \tag{19}$$

where $\{\gamma_0\} = \{\gamma_0\}_{1_{n+1}}$.

4. *Proof of Theorem 1*

First, for each loxodromic $\gamma_0 \in \Gamma$, we solve the cohomological equation (6) on the torus $\mathbb{T}^n_{\gamma_0,w}$.

PROPOSITION 2. *Given $c > 0$, let $\tilde{f} = f(z)\zeta^{p_0}$ be the lift of the cusp form $f(z) \in S_{p_0}(\Gamma)$ to $\Gamma\backslash G$, such that Fourier coefficients of its restriction to $\mathbb{T}^n_{\gamma_0,w}$, $\tilde{f}_{\bar{p},k}$, vanish for all $k \in \Phi_{\gamma_0,c}$, where the set $\Phi_{\gamma_0,c}$ was defined in (4). Then the cohomological equation*

$$\tilde{f} = \mathfrak{D}F \tag{20}$$

has a Lipschitz solution F on $\mathbb{T}^n_{\gamma_0,w}$ whose Lipschitz constant depends only on Γ and the function $\tilde{f}$, but not on γ_0, in particular, not on T.

Proof. First let us find a solution formally as a Fourier series

$$F(x,t)=\sum_{\bar{p},q}F_{\bar{p},q}\chi_{\bar{p},q}.$$

We have

$$\mathfrak{D}\chi_{\bar{p},q}=\frac{i(2\pi q-\bar{p}\cdot\bar{\psi})}{T}\chi_{\bar{p},q},$$

where $\bar{\psi}=(\psi_1,\ldots,\psi_{n-1})$. Equation (20) implies the following relations between the Fourier coefficients of F and $\tilde{f}$:

$$F_{\bar{p},q}=\frac{\tilde{f}_{\bar{p},q}T}{i(2\pi q-\bar{p}\cdot\bar{\psi})},$$

and since $\tilde{f}_{\bar{p},q}=0$ if $\bar{p}\neq\bar{p}*$ (18), we set $F_{\bar{p},q}=0$ for $\bar{p}\neq\bar{p}*$ and

$$F_{\bar{p}*,q}=\frac{\tilde{f}_{\bar{p}*,q}T}{i(2\pi q-\bar{p}*\cdot\bar{\psi})}=\frac{\tilde{f}_{\bar{p}*,q}T}{i(2\pi q-p_0\psi_0)}.$$

The function F on $\mathbb{T}^n_{\gamma_0,w}$, defined by its Fourier series

$$F\sim\sum_{q\in\mathbb{Z}}F_{\bar{p}*,q}\chi_{\bar{p}*,q}$$

with $F_{\bar{p}*,q}=0$ for $q\in\Phi_{\gamma_0,c}$, and

$$F_{\bar{p}*,q}=\frac{\tilde{f}_{\bar{p}*,q}T}{i(2\pi q-\psi_0 p_0)}$$

for other q, satisfies (20).

If $|q|\le cT$ or $q=\psi_0 p_0/2\pi$ we have $\tilde{f}_{\bar{p}*,q}=0$. For $q\neq\psi_0 p_0/2\pi$, the denominators do not vanish and, for $|q|>cT$, $|2\pi q-\psi_0 p_0|>c_1 T$ for some constant c_1. Therefore, for some constant c_2, we obtain

$$|F_{\bar{p}*,q}|\le c_2|\tilde{f}_{\bar{p}*,q}|.$$

Since the function $\tilde{f}$ is smooth and belongs to $L^2(\mathbb{T}^n_{\gamma_0,w})$, we conclude that the function $F\in L^2(\mathbb{T}^n_{\gamma_0,w})$ and its Fourier series converges to it absolutely. So

$$F(\bar{x},t)=\sum_{|q|>cT}F_{\bar{p}*,q}e^{i(p_0x+(2\pi q/T)t)}=e^{ip_0x}\sum_{|q|>cT}F_{\bar{p}*,q}e^{(2\pi iq/T)t}=e^{ip_0x}F(0,t).$$

We shall show that

$$\|F\|_{C^1}=\sup|F|+\sup\left|\frac{\partial F}{\partial x}\right|+\sup|\mathfrak{D}F|$$

is bounded from above by a constant independent of T. Since

$$\frac{\partial F}{\partial x_j}=i\frac{p_0}{2}F\quad\text{and}\quad\mathfrak{D}F=\tilde{f},$$

it is sufficient to prove that C-norms are bounded, i.e. that $|F(0,t)|$ and $|\tilde{f}(0,t)|$ are bounded from above on $[0,T]$ by a constant independent of T. Since $\tilde{f}(0,t)$ is a restriction of a smooth function on $\Gamma\backslash G$, which is globally bounded, it remains to estimate $|F(0,t)|$.

We have

$$F(0,t) = \sum_{|q|>cT} F_{\bar{p}*,q} e^{(2\pi i q/T)t}.$$

We make change of variables $s = 2\pi t/T$ and let $F_0(s) = F(0, sT/2\pi)$ and $\tilde{f}_0(s) = \tilde{f}(0, sT/2\pi)$. The functions F_0 and $\tilde{f}_0$ are periodic with period 2π and have Fourier expansions

$$F_0(s) = \sum_{|q|>cT} F_{\bar{p}*,q} e^{iqs} \quad \text{and} \quad \tilde{f}_0(s) = \sum_{|q|>cT} \tilde{f}_{\bar{p}*,q} e^{iqs},$$

respectively. We have

$$\mathfrak{D}F(0,t) = F'(0,t) = F_0'(s)\frac{2\pi}{T} = \tilde{f}(0,t) = \tilde{f}_0(s),$$

which implies $F_0'(s) = (T/2\pi)\tilde{f}_0(s)$. It follows from the 'reverse Bernstein inequality' [**5**, §8.4], with $B = C[0, 2\pi]$ and $m = 1$ that

$$\max_t |F(0,t)| = \|F_0\|_B \le 12(cT)^{-1}\|F_0'\|_B = 12(cT)^{-1}\frac{T}{2\pi}\|\tilde{f}_0\|_B.$$

Since $\|\tilde{f}_0\|_B = \max_s |\tilde{f}_0| \le K$ with the constant K independent of T, we have received the desired estimate. □

Now we construct a solution F on a dense orbit by integrating $\tilde{f}$ along it and prove that it satisfies a Lipschitz condition using a modification of Livshitz's original argument similar to the argument in the proof of Theorem 8 in [**1**]. Assume that the orbit is ϵ-closed. Then by Lemma 9 of [**1**], there exists a lift of a closed geodesic on *SM* which exponentially ϵ-approximates this piece of the dense orbit. On the torus where this lift belongs, we have a solution satisfying the Lipschitz condition with a constant K independent of the length of the closed geodesic on *SM*. We have, on the dense orbit,

$$F(x_1) = \int_0^{t_1} \tilde{f}(\tilde{\varphi}_t(x_0))\,dt, \quad F(x_2) = \int_0^{t_2} \tilde{f}(\tilde{\varphi}_t(x_0))\,dt,$$

and $d(x_1, x_2) < \epsilon$ and, on the lift of the closed orbit, we have two points x_1' and x_2' such that $d(x_1, x_1') < \epsilon$, $d(x_2, x_2') < \epsilon$, and

$$F(x_2') - F(x_1') = \int_0^T \tilde{f}(\tilde{\varphi}_t(x_1'))\,dt.$$

We have

$$F(x_2) - F(x_1) = \int_{t_1}^{t_2} \tilde{f}(\tilde{\varphi}_t(x_0))\,dt.$$

By Lemma 9 of [**1**], the last two integrals are ϵ-close:

$$\left| \int_0^T \tilde{f}(\tilde{\varphi}_t(x_1'))\,dt - \int_{t_1}^{t_2} \tilde{f}(\tilde{\varphi}_t(x_0))\,dt \right| = O(\epsilon).$$

Since $|F(x_2') - F(x_1')| \le K d(x_1', x_2') \le K\epsilon$, we conclude that

$$|F(x_2) - F(x_1)| = O(\epsilon).$$

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A. *Appendix: Spanning sets for cusp forms on complex hyperbolic spaces*

The purpose of this note is to correct an error found in [**1**]. We construct a family of new relative Poincaré series and prove that they span the space of cusp forms. The dynamical ingredient in the proof is Theorem 1.

Let

$$\gamma_0 = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & b_n \\ c_1 & \dots & c_n & d \end{pmatrix}$$

be a loxodromic element in Γ. As an automorphism of B^n, it has two fixed points on the boundary ∂B^n, X and Y.

In [**1**] we introduced the function $Q_{\gamma_0}(z) = \langle z, X\rangle\langle z, Y\rangle \neq 0$, where $\langle\cdot,\cdot\rangle$ has the same meaning as in (9). However, it transforms according to the formula

$$Q_{\gamma_0}(\gamma_0 z) = j(\gamma_0, z)^2 Q_{\gamma_0}(z) = (c_1 z_1 + \dots + c_n z_n + d)^{-2} Q_{\gamma_0}(z) \tag{21}$$

only if γ_0 is 'almost hyperbolic', i.e. if its loxodromy coefficient is equal to 0 or π. If γ_0 is a general loxodromic element with the eigenvalues (3), corresponding to X and Y, respectively, the transformation law is

$$Q_{\gamma_0}(\gamma_0 z) = j(\gamma_0, z)^2 e^{-2i\psi_0} Q_{\gamma_0}(z),$$

and $Q_{\gamma_0}(z)$ cannot be used for the construction of a relative Poincaré series.

In order to correct this, we construct a function that transforms according to (21) for each loxodromic element in Γ. As a matter of fact, we construct infinitely many such functions for each loxodromic element.

For each integer k, we set

$$\tau_k = \frac{\psi_0 - (\pi k/(n+1)\kappa)}{T}$$

and

$$Q^{(k)}_{\gamma_0}(z) = \langle z, X\rangle^{1-i\tau_k}\langle z, Y\rangle^{1+i\tau_k}. \tag{22}$$

We have

$$\gamma_0 \cdot X = \lambda e^{-i\psi_0} X \quad \text{and} \quad \gamma_0 \cdot Y = \lambda^{-1} e^{-i\psi_0} Y.$$

Then

$$\begin{aligned} Q^{(k)}_{\gamma_0}(\gamma_0(z)) &= \langle \gamma_0(z), X\rangle^{1-i\tau_k}\langle \gamma_0(z), Y\rangle^{1+i\tau_k} \\ &= j(\gamma_0, z)^2 (\lambda^{-1} e^{-i\psi_0})^{1-i\tau_k} (\lambda e^{-i\psi_0})^{1+i\tau_k} \cdot \langle \gamma_0 \cdot z, \gamma_0 \cdot X\rangle^{1-i\tau_k}\langle \gamma_0 \cdot z, \gamma_0 \cdot Y\rangle^{1+i\tau_k} \\ &= j(\gamma_0, z)^2 \lambda^{2i\tau_k} e^{-2i\psi_0} \langle z, X\rangle^{1-i\tau_k}\langle z, Y\rangle^{1+i\tau_k} \\ &= j(\gamma_0, z)^2 e^{2i(T\tau_k - \psi_0)} Q^{(k)}_{\gamma_0}(z) \\ &= j(\gamma_0, z)^2 e^{-2i(\pi k/(n+1)\kappa)} Q^{(k)}_{\gamma_0}(z) \end{aligned}$$

and, hence,

$$Q^{(k)}{}_{\gamma_0}^{(n+1)\kappa}(\gamma_0(z)) = j(\gamma_0, z)^{2(n+1)\kappa} Q^{(k)}{}_{\gamma_0}^{(n+1)\kappa}(z).$$

The function $q_k(z) = 1/Q^{(k)}{}_{\gamma_0}^{(n+1)\kappa}(z)$ is an automorphic form of type $\mu(g, z) = j(g, z)^{2(n+1)\kappa}$ for the subgroup $\Gamma_0 = \langle\gamma_0\rangle$ and it satisfies the conditions (1) and (2) of Theorem 5 of [**1**]. Thus, for any $\kappa \in \mathcal{K}_n = \{\kappa \geq 1 \mid \kappa \in \frac{1}{2}\mathbb{Z},\ (n+1)\kappa \in \mathbb{Z}\}$ and any $k \in \mathbb{Z}$, we produce a relative Poincaré series

$$\Theta^{(k)}_{\gamma_0,\kappa}(z) = \sum_{\gamma \in \Gamma_0\backslash\Gamma} (q_k|\gamma)(z) \tag{23}$$

of weight $2(n+1)\kappa = p_0$ which belongs to $L^1(\Gamma\backslash G)$ and, by Satake's theorem, is a cusp form.

The period formula (Theorem 11 of [**1**]) now holds for all relative Poincaré series.

THEOREM 3. *For any $f \in S_{p_0}$, any $k \in \mathbb{Z}$ and any loxodromic $\gamma_0 \in \Gamma$,*

$$(f, \Theta^{(k)}_{\gamma_0,\kappa}) = (\alpha^{1+i\tau_k}\bar{\alpha}^{1-i\tau_k})^{-(n+1)\kappa} C \int_{z_0}^{\gamma_0 z_0} f(z) Q^{(k)}_{\gamma_0,\kappa}(z)^{(n+1)\kappa}\, dt. \tag{24}$$

Here $z_0 \in C(\gamma_0)$, and the integration is over $C(\gamma_0)$ in B^n, t is the parameter of the geodesic flow, $\alpha = -\langle X, Y\rangle/2 \neq 0$, and

$$C = \frac{\pi^{n-1}2^{2h-1}(p_0 - n - 1)!}{(\rho_0 - 2)!} \int_0^{\pi} (\sin t)^{\rho_0 - 2} e^{\rho_0\tau_k t}\, dt > 0.$$

This formula can be lifted to $\Gamma\backslash G$ as follows.

THEOREM 4.

$$(f, \Theta^{(k)}_{\gamma_0,\kappa}) = (\alpha^{1+i\tau_k})^{-(n+1)\kappa} C e^{ip_0(\psi-\psi_0)} \int_{\{\gamma_0\}_w} \tilde{f}(g_t) e^{-2\pi ikt/T}\, dt. \tag{25}$$

Proof. Let $z_t \in C(\gamma_0)$. Using (13) and (16), we have

$$z_t = h(x_t), \quad X = h(X_0), \quad Y = h(Y_0),$$

where

$$x_t \in \mathfrak{I}, \quad X_0 = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} 0 \\ \dots \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} Q^{(k)}_{\gamma_0}(z_t) &= \langle z_t, X\rangle^{1-i\tau_k}\langle z_t, Y\rangle^{1+i\tau_k} \\ &= \langle h(x_t), h(X_0)\rangle^{1-i\tau_k}\langle h(x_t), h(Y_0)\rangle^{1+i\tau_k} \\ &= j(h, x_t)^2 L\overline{j(h, X_0)}^{1-i\tau_k}\overline{j(h, Y_0)}^{1+i\tau_k}\langle h\cdot x_t, h\cdot X_0\rangle^{1-i\tau_k}\langle h\cdot x_t, h\cdot Y_0\rangle^{1+i\tau_k} \\ &= j(h, x_t)^2\bar{\alpha}^{1-i\tau_k}\langle x_t, X_0\rangle^{1-i\tau_k}\langle x_t, Y_0\rangle^{1+i\tau_k} \end{aligned}$$

and

$$\begin{aligned} Q_{\gamma_0}^{(k)}(z_t)^{(n+1)\kappa} &= j(h, x_t)^{2(n+1)\kappa}(\bar{\alpha}^{1-i\tau_k})^{(n+1)\kappa}((1-\tanh t)^{1-i\tau_k}(1+\tanh t)^{1+i\tau_k})^{(n+1)\kappa} \\ &= j(h, x_t)^{2(n+1)\kappa}(\bar{\alpha}^{1-i\tau_k})^{(n+1)\kappa}(1-\tanh t)^{(n+1)\kappa}e^{2i\tau_k t(n+1)\kappa} \\ &= j(h, x_t)^{2(n+1)\kappa}(\bar{\alpha}^{1-i\tau_k})^{(n+1)\kappa}(1-\tanh t)^{(n+1)\kappa}e^{ip_0\psi_0 t/T}e^{-2\pi ikt/T}. \end{aligned}$$

Then

$$\begin{aligned} (f, \Theta_{\gamma_0,\kappa}^{(k)}) &= (\alpha^{1+i\tau_k}\bar{\alpha}^{1-i\tau_k})^{-(n+1)\kappa}C\int_{z_0}^{\gamma_0 z_0} f(z_t)Q_{\gamma_0,\kappa}^{(k)}(z_t)^{(n+1)\kappa}\,dt \\ &= (\alpha^{1+i\tau_k})^{-(n+1)\kappa}C\int_0^T \frac{f(h(x_t))}{\cosh^{p_0} t} j(h, x_t)^{p_0}e^{ip_0\psi_0 t/T}e^{-2\pi ikt/T}\,dt. \end{aligned}$$

However, using (17) with $\bar{x} = 0$, we obtain that the restriction of the cusp form $\tilde{f}$ to $\{\gamma_0\}_w$ is given by the formula

$$\tilde{f}(0, t) = \tilde{f}(g_t) = e^{ip_0(\psi_0-\psi)}\left(\frac{f(h(x_t))}{\cosh^{p_0} t} j(h, x_t)^{p_0}e^{ip_0\psi_0 t/T}\right)$$

and we obtain

$$(f, \Theta_{\gamma_0,\kappa}^{(k)}) = (\alpha^{1+i\tau_k})^{-(n+1)\kappa}Ce^{ip_0(\psi-\psi_0)}\int_{\{\gamma_0\}_w} \tilde{f}(g_t)e^{-2\pi ikt/T}\,dt. \qquad \square$$

Theorem 1 of [**1**] holds in a modified form.

THEOREM 5. *Let $\Phi_{\gamma_0,c}$ be the set defined in (4). For any constant $c > 0$, the relative Poincaré series*

$$\{\Theta_{\gamma_0,\kappa}^{(k)} \mid \gamma_0 \in \Gamma \textit{ loxodromic}, k \in \Phi_{\gamma_0,c}\}$$

span $S_{p_0}(\Gamma)$.

The proof follows the same scheme as in [**1**]. We suppose that there is a cusp form $f \in S_{p_0}(\Gamma)$ which is orthogonal to all $\Theta_{\gamma_0,\kappa}^{(k)}$ above. By Theorem 4, which replaces Theorem 14 of [**1**], and (19), the function $\tilde{f}$ satisfies Theorem 1, which replaces Theorem 8 of [**1**]. The rest of the proof remains the same.

In the Fuchsian groups case ($n = 1$), all loxodromic elements are hyperbolic. However, the new relative Poincaré series (23) can still be constructed for any hyperbolic $\gamma_0 \in \Gamma$ (and $\psi_0 = 0$). Moreover, they already give us a spanning set for cusp forms.

THEOREM 6. *Let Γ be a lattice in $G = SU(1, 1)$, and $\gamma_0 \in \Gamma$ a hyperbolic element. Then the relative Poincaré series $\{\Theta_{\gamma_0,\kappa}^{(k)},\ k \in \mathbb{Z}\}$ span $S_{p_0}(\Gamma)$.*

Proof. Let us assume that there is a cusp form $f \in S_{p_0}(\Gamma)$ such that $(f, \Theta_{\gamma_0,\kappa}^{(k)}) = 0$. Then by Theorem 4, all Fourier coefficients of $\tilde{f}$ restricted to $\{\gamma_0\}$ (which, in this case, is simply the closed geodesic $[\gamma_0]$ lifted to the unit tangent bundle $SM = \Gamma\backslash G$) are equal to 0. Therefore, $\tilde{f} = 0$ on $[\gamma_0]$. But $\tilde{f} = f(z)\zeta^{p_0}$ and since $\zeta \neq 0$, we conclude that $f(z) = 0$ on $[\gamma_0]$. Since $f(z)$ is holomorphic, this implies that $f(z) = 0$ for all z. $\square$

Remark. These new relative Poincaré series are exact 'hyperbolic' analogues of the classical Poincaré–Eisenstein series associated with a cusp of $\Gamma\backslash G$, i.e. a conjugacy class of parabolic elements in Γ.

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