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Rigidity and flexibility of entropies of boundary maps associated to Fuchsian groups

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(Recommended by Boris Hasselblatt)

Abstract. Given a closed, orientable surface of constant negative curvature and genus $g \geq 2$, we study the topological entropy and measure-theoretic entropy (with respect to a smooth invariant measure) of generalized Bowen–Series boundary maps. Each such map is defined for a particular fundamental polygon for the surface and a particular multi-parameter.

We survey two strikingly different recent results by the authors: topological entropy is constant in this entire family ("rigidity"), while measure-theoretic entropy varies within Teichmüller space, taking all values ("flexibility") between zero and a maximum. This maximum is achieved on the surface that admits a regular fundamental $(8g-4)$-gon. The rigidity proof uses conjugation to maps of constant slope, while the flexibility proof—valid for a large set of multi-parameters—uses the realization of geodesic flow as a special flow over the natural extension of the boundary map. We obtain explicit formulas for both entropies. We also present some new details pertaining to specific multi-parameters and particular polygons.

1. Introduction

Any closed, orientable, compact surface S of constant negative curvature can be modeled as $S = \Gamma\backslash\mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk endowed with hyperbolic metric $2|\mathrm{d}z|/(1-|z|^2)$ and Γ is a finitely generated Fuchsian group of the first kind acting freely on $\mathbb{D}$ and isomorphic to $\pi_1(S)$. Recall that geodesics in this model are half-circles or diameters orthogonal to $\mathbb{S} = \partial\mathbb{D}$, the circle at infinity. The geodesic flow $\widetilde{\varphi}^t$ on $\mathbb{D}$ is defined as an $\mathbb{R}$-action on the unit tangent bundle $T^1\mathbb{D}$ that moves a tangent vector along the geodesic defined by this vector with unit speed. The geodesic flow $\widetilde{\varphi}^t$ on $\mathbb{D}$ descends to the geodesic flow φ^t on the factor $S = \Gamma\backslash\mathbb{D}$ via the canonical projection of the unit tangent bundles. The orbits of the geodesic flow φ^t are oriented geodesics on S.

A surface S of genus $g \geq 2$ admits an $(8g-4)$-sided fundamental polygon with a particular pairing of sides (1.1) obtained by cutting it with $2g$ closed geodesics that intersect in pairs (g of them go around the "holes" of S and another g go around the "waists"; see Figure 1). The existence of such a fundamental polygon $\mathcal{F}$ is an old result attributed to Dehn, Fenchel, Nielsen, and Koebe [34, 21, 9]. Adler and Flatto [6, Appendix A] give a careful proof of existence and special properties of $\mathcal{F}$.

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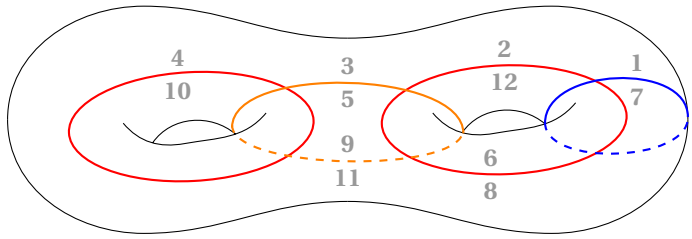

Figure 1. Necklace of $2g$ geodesics on S forming the sides of $\mathcal{F}$ for $g = 2$.

We label the sides of $\mathcal{F}$, which are geodesic segments, in a counterclockwise order by numbers $1 \le k \le 8g-4$ and label the vertices of $\mathcal{F}$ by V_k so that side k connects V_k to $V_{k+1} \pmod{8g-4}$ (this gives us a *marking* of the polygon).

We denote by P_k and Q_{k+1} the endpoints of the oriented infinite geodesic that extends side k to the circle at infinity $\mathbb{S}$. The counter-clockwise order of endpoints on $\mathbb{S}$ is

$$P_1, Q_1, P_2, Q_2, \dots, P_{8g-4}, Q_{8g-4}.$$

The identification of the sides of $\mathcal{F}$ is given by the side pairing rule

$$\sigma(k) := \begin{cases} 4g-k \bmod (8g-4) & \text{if } k \text{ is odd} \\ 2-k \bmod (8g-4) & \text{if } k \text{ is even.} \end{cases} \tag{1.1}$$

We denote by $T_k : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ the Möbius transformation pairing side k with side $\sigma(k)$. The group Γ is generated by $T_1, \dots, T_{8g-4}$.

Note that $\mathcal{F}$ is not regular in general but satisfies the following properties:

(1) the sides k and $\sigma(k)$ have equal length;
(2) the angles at vertices k and $\sigma(k)+1$ add up to π.

We call fundamental $(8g-4)$-gons $\mathcal{F}$ satisfying properties (1) and (2) *canonical*. Property (2) implies the "extension condition", which is crucial for our analysis: the extensions of the sides of $\mathcal{F}$ do not intersect the interior of the tessellation $\gamma\mathcal{F}$, $\gamma \in \Gamma$ (see Figure 2).

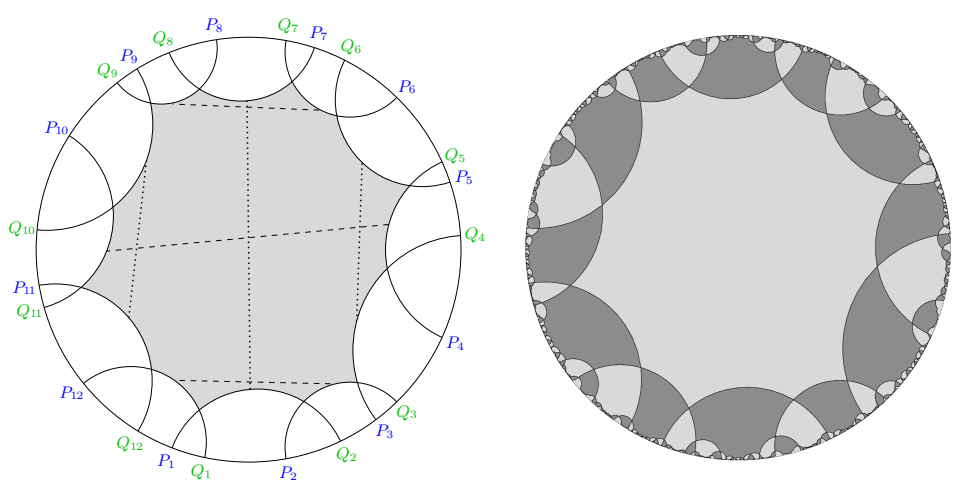

Figure 2. An irregular polygon with dotted lines showing side identifications (left) and tessellation (right), genus 2.

The polygon $\mathcal{F}$ is related to a regular $(8g-4)$-gon $\mathcal{F}_{\text{reg}}$ centered at the origin with all interior angles equal to $\frac{\pi}{2}$ (see [6, Figure 1]) by the following construction. Let $S = \Gamma\backslash\mathbb{D}$

be any compact surface of genus $g \geq 2$ and $S_{\text{reg}} = \Gamma_{\text{reg}} \backslash \mathbb{D}$ be the surface of the same genus admitting the regular fundamental polygon $\mathcal{F}_{\text{reg}}$. By the Fenchel–Nielsen Theorem [32], there exists an orientation-preserving homeomorphism h from $\bar{\mathbb{D}}$ onto $\bar{\mathbb{D}}$ such that $\Gamma = h \circ \Gamma_{\text{reg}} \circ h^{-1}$. The map $h\big|_{\mathbb{S}}$ is a homeomorphism of $\mathbb{S}$ preserving the order of the points $\bar{P} \cup \bar{Q}$, where

$$\bar{P} = \{P_1, ..., P_{8g-4}\} \quad \text{and} \quad \bar{Q} = \{Q_1, ..., Q_{8g-4}\}.^1$$

The sides of $\mathcal{F}$ are constructed by connecting $h(P_k)$ and $h(Q_{k+1})$ by geodesics, where P_k and Q_{k+1} correspond to $\mathcal{F}_{\text{reg}}$. On the other hand, for every marked canonical $(8g-4)$-gon and the associated Fuchsian group Γ, by Euler's formula the genus of $\Gamma \backslash \mathbb{D}$ is g.

This construction gives a less common representation of the Teichmüller space $\mathcal{T}(g)$ for $g \geq 2$: it is the space of marked canonical $(8g-4)$-gons in the unit disk $\mathbb{D}$ up to an isometry of $\mathbb{D}$.[2] The topology on the space of polygons is that $\mathcal{P}_k \to \mathcal{P}$ if and only if the lengths of all sides converge and the measures of all angles converge. It is well-known [16, 17] that $\dim(\mathcal{T}(g)) = 6g-6$; the following is a heuristic argument for this fact using $(8g-4)$-gons. The lengths of the identified pairs of sides are given by $4g-2$ real parameters, and $2g-1$ parameters represent the angles since four angles at each vertex are determined by one parameter. The dimension of the space $SU(1,1)$ of isometries of $\mathbb{D}$ is 3, so we have altogether $(4g-2)+(2g-1)-3 = 6g-6$ parameters.

The boundary map. The main object of this paper is the multi-parameter family of generalized Bowen–Series boundary maps studied in [19, 2, 1, 4, 3]. For each $\mathcal{F} \in \mathcal{T}(g)$, recall that side k of $\mathcal{F}$ is contained in the geodesic from P_k to Q_{k+1} and that T_k maps side k to side $\sigma(k)$; we define the *boundary map* $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ by

$$f_{\bar{A}}(x) = T_k(x) \quad \text{if } x \in [A_k, A_{k+1}), \tag{1.2}$$

where

$$\bar{A} := \{A_1, A_2, \ldots, A_{8g-4}\} \quad \text{with} \quad A_k \in [P_k, Q_k].$$

When all $A_k = P_k$ we denote the map by $f_{\bar{P}}$ (see left of Figure 4 for its graph). Similarly, we denote by $f_{\bar{Q}}$ the case where all $A_k = Q_k$. The maps $f_{\bar{P}}$ and $f_{\bar{Q}}$ were extensively studied by Adler and Flatto in [6]; they call them the "Bowen–Series boundary maps", although Bowen and Series' construction [13] used $4g$-gons.

We study two dynamical invariants of $f_{\bar{A}}$: the topological entropy and the measure-theoretic entropy with respect to a smooth (that is, Lebesgue-equivalent) measure. Our two main results are rigidity of the topological entropy (Theorem 1.1) and flexibility of the measure-theoretic entropy (Theorem 1.2).

Theorem 1.1 (Rigidity of topological entropy [3])**.** *Let $S = \Gamma \backslash \mathbb{D}$ be a surface of genus $g \geq 2$ with fundamental polygon $\mathcal{F} \in \mathcal{T}(g)$. For any $\bar{A} = \{A_1, \ldots, A_{8g-4}\}$ with $A_k \in [P_k, Q_k]$, the map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ has topological entropy*

$$h_{\text{top}}(f_{\bar{A}}) = \log\big(4g-3+\sqrt{(4g-3)^2-1}\big). \tag{1.3}$$

Remark. The maps $f_{\bar{A}}$ are not necessarily topologically conjugate since, according to [19], the combinatorial structure of the orbits associated to the discontinuity points A_k can differ.

All $f_{\bar{A}}$ are piecewise continuous and piecewise monotone. Some maps in this family (such as those considered by Bowen and Series [13] and further studied by Adler and

[1] In some works $h|_{\mathbb{S}}$ is also called a "boundary map"; this is different from our usage.

[2] For a related representation of $\mathcal{T}(g)$ as the space of marked canonical $4g$-gons, see [29], following the earlier work [35, 31].

Flatto [6]) are Markov, and so the topological entropy can be calculated as the logarithm of the maximal eigenvalue of a transition matrix [33, Theorem 7.13] in these cases. Not all maps $f_{\overline{A}}$ admit a Markov partition, but (1.3) holds for all $\overline{A}$ regardless. The proof of Theorem 1.1 uses conjugation to maps of constant slope.

The flexibility result requires a restriction on the set of parameters, as defined below. Such restrictions are common in previous work on boundary maps: [6] uses only $\overline{A} = \overline{P}$ and $\overline{A} = \overline{Q}$, [1] focuses on extremal parameters and their so-called duals, and [2] requires short cycles.

Definition. A multi-parameter $\overline{A} = \{A_1, \dots, A_{8g-4}\}$ is *extremal* if for each k either $A_k = P_k$ or $A_k = Q_k$. A multi-parameter $\overline{A}$ satisfies the *cycle property* if each $A_k \in (P_k, Q_k)$ and there exist positive integers m_k, n_k such that $f_{\overline{A}}^{m_k}(T_k A_k) = f_{\overline{A}}^{n_k}(T_{k-1} A_k)$. If $m_k = n_k = 1$ then we say that $\overline{A}$ satisfies the *short cycle property.*

Notice that the above properties are preserved by the Fenchel-Nielsen homeomorphism $h\big|_{\mathbb{S}}$ (see [20]), so it makes sense to talk about a multi-parameter $\overline{A}$ satisfying these properties in Teichmüller space $\mathcal{T}(g)$.

If $\overline{A}$ is extremal or has short cycles, then $f_{\overline{A}}$ admits a unique smooth invariant ergodic measure $\mu_{\overline{A}}$ obtained as a projection of the Liouville measure for geodesic flow.

Theorem 1.2 (Flexibility of measure-theoretic entropy [4])**.** *Let $S = \Gamma\backslash\mathbb{D}$ be a surface of genus $g \geq 2$, and let $\overline{A}$ be extremal or satisfy the short cycle property.*[3]

(i) For each $\mathcal{F} \in \mathcal{T}(g)$, the measure-theoretic entropy of $f_{\overline{A}}$ with respect to its smooth invariant probability measure $\mu_{\overline{A}}$ is

$$h_{\mu_{\overline{A}}}(f_{\overline{A}}) = \frac{\pi^2(4g-4)}{\text{Perimeter}(\mathcal{F})} = \pi \cdot \frac{\text{Area}(\mathcal{F})}{\text{Perimeter}(\mathcal{F})}. \tag{1.4}$$

(ii) Among all $\mathcal{F} \in \mathcal{T}(g)$, the maximum value of $h_{\mu_{\overline{A}}}(f_{\overline{A}})$ is achieved on the surface for which $\mathcal{F}$ is regular; this value is

$$H(g) := h_{\mu_{\overline{A}}^{\text{reg}}}(f_{\overline{A}}^{\text{reg}}) = \frac{\pi^2(4g-4)}{(8g-4)\,\text{arccosh}(1+2\cos\frac{\pi}{4g-2})}.$$

(iii) For any value $h \in (0, H(g)]$ there exists $\mathcal{F} \in \mathcal{T}(g)$ such that $h_{\mu_{\overline{A}}}(f_{\overline{A}}) = h$.

Remark. The formula (1.4) shows that the measure-theoretic entropy remains constant under the change of the multi-parameter $\overline{A}$ within the considered class, so there is an aspect of rigidity to Theorem 1.2 as well. We emphasize that $h_{\mu_{\overline{A}}}(f_{\overline{A}})$ is flexible in the Teichmüller space.

The paper is organized as follows. Sections 2 and 3 summarize the proofs of Theorems 1.1 and 1.2, respectively. Section 4.1 compares the two entropies (as was done in [4]), and Sections 4.2 and 4.3 contain results about genus 2 examples that are presented here for the first time. Section 5 lists some conjectures and open questions.

2. Rigidity of topological entropy

The notion of topological entropy was introduced by Adler, Konheim, and McAndrew in [5]. Their definition used covers and applied to compact Hausdorff spaces; Dinaburg [14] and Bowen [11] gave definitions involving distance functions and separated sets, which are often more suitable for calculation. While these formulations of topological entropy were originally intended for continuous maps acting on compact

[3] In [4] the main theorem is stated only for the classical Bowen–Series case $\overline{A} = \overline{P}$. As explained in [4, Remark 5], the results also hold for $\overline{A}$ extremal or with short cycles, so we state the broader result here.

spaces, Bowen's definition can actually be applied to piecewise continuous, piecewise monotone maps on an interval, as explained in [25]. The theory naturally extends to maps of the circle, where piecewise monotonicity is understood to mean local monotonicity or, equivalently, having a piecewise monotone lift to $\mathbb{R}$.

In [26], Parry introduced two probability measures for topological Markov chains: the first is the measure of maximal entropy and is commonly known as the "Parry measure"; the second measure is not shift-invariant but is uniformly expanding on cylinders. In [27], Parry used this second measure to conjugate a piecewise monotone, piecewise continuous, strongly transitive interval map (not necessarily Markov) with positive topological entropy to a map with constant slope. Alsedà, Llibre, and Misiurewicz [7, 8] generalized this construction, obtaining semi-conjugacy for non-transitive maps. We apply the results of [27, 8] to the family of maps $f_{\overline{A}}$, defining the second of Parry's measures, which we denote by $\rho_{\overline{A}}$, in (2.2).

Our original proof of Theorem 1.1 in [3] used several symmetry properties of the regular fundamental polygon $\mathcal{F}_{\text{reg}}$ and the conjugacy $\psi_{\overline{P}}$ constructed for the boundary map $f_{\overline{P}}$ that is associated to the regular polygon. The main fact required for Theorem 1.1 is [3, Lemma 15(a)], and in hindsight this can be proved without appealing to any symmetry. The proof presented in this paper applies directly to any polygon $\mathcal{F}$, without using the Fenchel–Nielsen homeomorphism to relate the setup to the regular polygon. (Figures 4 and 5 were made using the regular polygon with genus $g = 2$.)

Step 1: topological entropy for extremal parameters. All $f_{\overline{A}}(P_k)$ and $f_{\overline{A}}(Q_k)$ belong to the set $\overline{P} \cup \overline{Q}$ (see [19, Proposition 2.2], originally [6, Theorem 3.4]), so the partition of $\mathbb{S}$ into intervals $I_1, \dots, I_{16g-8}$ given by

$$I_{2k-1} := [P_k, Q_k], \qquad I_{2k} := [Q_k, P_{k+1}], \qquad k = 1, \dots, 8g-4,$$

is a Markov partition for $f_{\overline{A}}$ for every extremal $\overline{A}$. Although they share a Markov partition, each extremal $\overline{A}$ has its own transition matrix $M_{\overline{A}} = (m_{i,j})$ given by

$$m_{i,j} := \begin{cases} 1 & \text{if } f_{\overline{A}}(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The transition matrices $M_{\overline{P}}$ and $M_{\overline{Q}}$ for genus 2 are shown in Figure 3. A word $\omega = (\omega_0, \dots, \omega_n)$ in the alphabet $\{1, \dots, 16g-8\}$ is called $\overline{A}$-*admissible* if $m_{\omega_i, \omega_{i+1}} = 1$ for $i = 0, \dots, n-1$.

$$\begin{pmatrix}
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\end{pmatrix}$$

FIGURE 3. Transition matrices $M_{\overline{P}}$ (left) and $M_{\overline{Q}}$ (right) for $g = 2$.

We compute the maximal eigenvalue of the transition matrices for all extremal parameters and hence the topological entropy for these Markov cases [3, Proposition 7 and Corollary 8]:

- For any extremal multi-parameter $\overline{A}$, the number of $\overline{A}$-admissible words of length n grows as approximately λ^n, where
$$\lambda = 4g - 3 + \sqrt{(4g-3)^2 - 1},$$
and therefore the maximal eigenvalue of $M_{\overline{A}}$ is exactly λ.
- Thus $h_{\text{top}}(f_{\overline{A}}) = \log \lambda$ if $\overline{A}$ is extremal.

Step 2: conjugacy to a constant slope map. The following theorem combines several results of [27, 8], stated here for circle maps (as in [24]) instead of interval maps:

Theorem 2.1. *Given a piecewise monotone, piecewise continuous, topologically transitive map $f : \mathbb{S} \to \mathbb{S}$ of positive topological entropy $h > 0$, there exists a unique (up to rotation of $\mathbb{S}$) increasing homeomorphism $\psi : \mathbb{S} \to \mathbb{S}$ conjugating f to a piecewise continuous map with constant slope e^h.*

Although for each parameter $\overline{A}$ (not necessarily extremal) the map $f_{\overline{A}}$ has, a priori, its own conjugacy and its own corresponding map of constant slope, we are especially interested in the cases $\overline{A} = \overline{P}$ and $\overline{A} = \overline{Q}$.

- The map $f_{\overline{P}} : \mathbb{S} \to \mathbb{S}$ is piecewise monotone, piecewise continuous, topologically transitive (see [13, Lemma 2.5]), and has positive topological entropy (by Step 1), so by Theorem 2.1 there exists an increasing homeomorphism $\psi_{\overline{P}} : \mathbb{S} \to \mathbb{S}$ conjugating it to a map
$$\ell_{\overline{P}} := \psi_{\overline{P}} \circ f_{\overline{P}} \circ \psi_{\overline{P}}^{-1}$$
with constant slope, see Figure 4. The map $\psi_{\overline{P}}$ is unique up to rotation of $\mathbb{S}$, and the slope of $\ell_{\overline{P}}$ is exactly $\lambda = e^{h_{\text{top}}(f_{\overline{P}})}$.
- The map $f_{\overline{Q}} : \mathbb{S} \to \mathbb{S}$ also satisfies the conditions of Theorem 2.1, so there exists an increasing homeomorphism $\psi_{\overline{Q}} : \mathbb{S} \to \mathbb{S}$, unique up to rotation of $\mathbb{S}$, conjugating it to a map $\ell_{\overline{Q}}$ of constant slope. By Step 1, $\ell_{\overline{P}}$ and $\ell_{\overline{Q}}$ have the same slope.

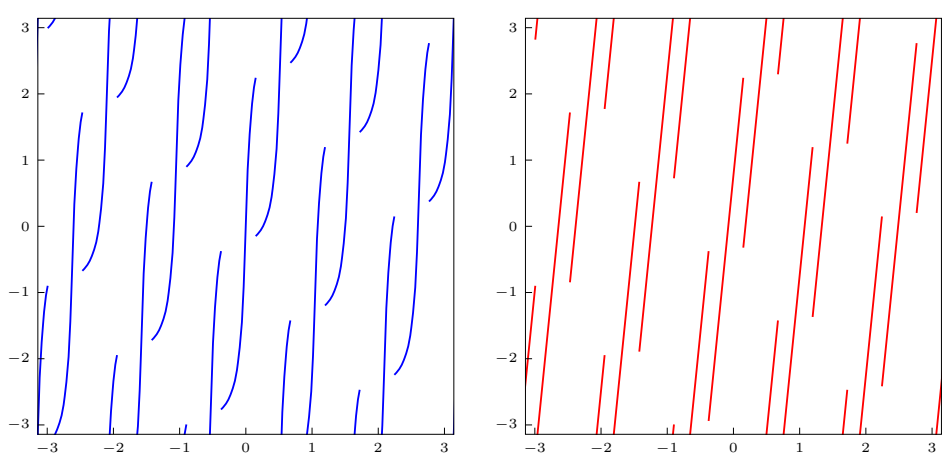


Figure 4. Plots of $f_{\overline{P}}(x)$ (left) and $\ell_{\overline{P}}(x)$ (right) for $g = 2$.

When $f_{\overline{A}}$ is Markov—in particular when $\overline{A}$ is extremal—the construction of the conjugacy $\psi_{\overline{A}}$ follows the classical work of Parry [27], also used in the proof of [8, Lemma 5.1]: let λ, v be the maximal eigenpair for the transition matrix $M_{\overline{A}}$ and define the measure $\rho_{\overline{A}}$ on the symbolic shift space $X_{\overline{A}}$ for any non-empty cylinder $C^{\omega}_{\overline{A}}$ (see [3, Section 4] for precise definitions of $X_{\overline{A}}$ and $C^{\omega}_{\overline{A}}$) as

$$\rho_{\overline{A}}\big(C^{(\omega_0,\dots,\omega_n)}_{\overline{A}}\big) = \frac{v_{\omega_n}}{\lambda^n}. \tag{2.2}$$

One defines the push-forward measure $\rho'_{\overline{A}}$ on $\mathbb{S}$ as

$$\rho'_{\overline{A}}(E) = \rho_{\overline{A}}\big(\phi^{-1}_{\overline{A}}(E)\big) \qquad \text{for Borel } E,$$

where $\phi_{\overline{A}} : X_{\overline{A}} \to \mathbb{S}$ is the symbolic coding map, that is, $\phi_{\overline{A}}(\omega) = \bigcap_{i=0}^{\infty} f^{-i}_{\overline{A}}(I_{\omega_i})$. In Markov cases, the conjugacy $\psi_{\overline{A}} : \mathbb{S} \to \mathbb{S}$ with $\mathbb{S}$ modeled as $[-\pi,\pi]$ is then given by

$$\psi_{\overline{A}}(x) := -\pi + 2\pi \cdot \rho'_{\overline{A}}\big([-\pi, x]\big). \tag{2.3}$$

It turns out that the maps $\psi_{\overline{P}}$ and $\psi_{\overline{Q}}$ thus constructed coincide:

Theorem 2.2 ([3, Theorem 12])**.** *For all $x \in \mathbb{S}$, $\psi_{\overline{P}}(x) = \psi_{\overline{Q}}(x)$.*

This is the most technically difficult part of the argument. The proof is presented in [3, Appendix A]. We also prove many symmetric properties of $\psi_{\overline{P}}$ ([3, Propositions 14 and 15]) that are visually evident in [3, Figure 4].

Step 3: completion of the proof. Denote $P'_k = \psi_{\overline{P}}(P_k)$ and $Q'_k = \psi_{\overline{P}}(Q_k)$. By construction, the map $\psi_{\overline{P}} \circ T_k \circ \psi^{-1}_{\overline{P}}$ is linear (with slope λ) on $[P'_k, P'_{k+1})$ since $[P_k, P_{k+1})$ is the interval where $f_{\overline{P}}$ acts by T_k. The crucial fact that is necessary for our proof of rigidity is that $\psi_{\overline{P}} \circ T_k \circ \psi^{-1}_{\overline{P}}$ is linear on the longer interval $[P'_k, Q'_{k+1}) \supset [P'_k, P'_{k+1})$.

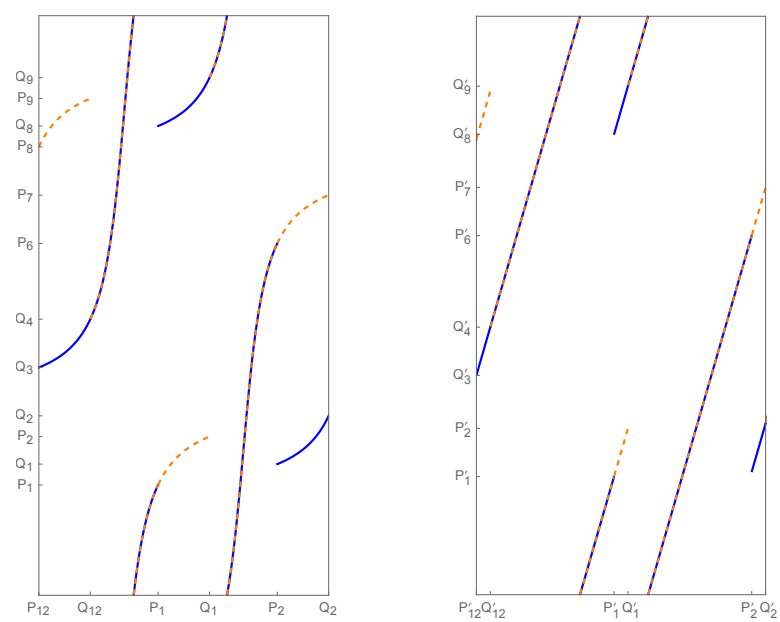


FIGURE 5. Left: part of the graphs of $f_{\overline{P}}$ (blue) and $f_{\overline{Q}}$ (orange and dashed). Right: $\ell_{\overline{P}}$ (blue) and $\ell_{\overline{Q}}$ (orange and dashed).

To see this, note that $\psi_{\overline{Q}} \circ T_k \circ \psi_{\overline{Q}}^{-1}$ is linear on $[Q'_k, Q'_{k+1})$ by construction and that, by Theorem 2.2, $\psi_{\overline{P}} = \psi_{\overline{Q}}$. Thus $\psi_{\overline{P}} \circ T_k \circ \psi_{\overline{P}}^{-1} = \psi_{\overline{Q}} \circ T_k \circ \psi_{\overline{Q}}^{-1}$ is linear on $[P'_k, P'_{k+1}] \cup [Q'_k, Q'_{k+1}] = [P'_k, Q'_{k+1}]$. See Figure 5.

Proof of Theorem 1.1. Let $\overline{A} = \{A_1, ..., A_{8g-4}\}$ with each $A_k \in [P_k, Q_k]$ be arbitrary. For each k, the map

$$\psi_{\overline{P}} \circ f_{\overline{A}} \circ \psi_P^{-1}$$

(note the use of $\psi_{\overline{P}}$ with $f_{\overline{A}}$) is linear with slope λ on $[A'_k, A'_{k+1})$ because $\psi_{\overline{P}} \circ T_k \circ \psi_{\overline{P}}^{-1}$ is linear on $[A'_k, A'_{k+1}) \subset [P'_k, Q'_{k+1})$. Thus on whole set $\bigcup_{k=1}^{8g-6} [A'_k, A'_{k+1}) = \mathbb{S}$ the map $\psi_{\overline{P}} \circ f_{\overline{A}} \circ \psi_{\overline{P}}^{-1}$ is piecewise linear with slope λ, and so, by [24, Theorem 3′], the topological entropy of $f_{\overline{A}}$ is $\log \lambda$. □

3. Flexibility of measure-theoretic entropy

A few years ago, Anatole Katok suggested a new area of research—or, at the very least, a new viewpoint—called the "flexibility program", which can be broadly formulated as follows: under properly understood general restrictions, within a fixed class of smooth dynamical systems, some dynamical invariants take arbitrary values. Taking this point of view, it is natural to ask how the measure-theoretic entropy $h_{\mu_{\overline{P}}}(f_{\overline{P}})$ changes in $\mathcal{T}(g)$. Theorem 1.2 addresses this question, and we sketch its proof in this section.

Step 1: the smooth invariant measure. If $\overline{A}$ is extremal (e.g., $\overline{A} = \overline{P}$) or has short cycles, then the smooth invariant measure $\mu_{\overline{A}}$ for $f_{\overline{A}}$ can be described as two-step projection of the Liouville measure for the geodesic flow [2, 1].

- On the unit tangent bundle $T^1\mathbb{D}$, parameterized by (x, y, θ) with $x + iy \in \mathbb{D}$ the base-point of tangent vector and θ its angle with the real axis, the Liouville volume
$$\mathrm{d}\omega = \frac{4\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta}{(1 - x^2 - y^2)^2}$$
comes from the hyperbolic measure on $\mathbb{D}$ and is invariant under geodesic flow.
- The measure
$$\mathrm{d}\nu = \frac{|\mathrm{d}u|\,|\mathrm{d}w|}{|u - w|^2}$$
is a smooth measure on the space of oriented geodesics on $\mathbb{D}$ (modeled as $\{(u, w) \in \mathbb{S} \times \mathbb{S} : u \neq w\}$) and is preserved by Möbius transformations.[4] Sometimes called "geodesic current", this measure was most probably first considered by E. Hopf [18] and was later used by Sullivan [30], Bonahon [10], Adler–Flatto [6], and the current authors [19, 2].
- Using coordinates (u, w, s) on $T^1\mathbb{D}$, where s is arclength along a geodesic, instead of coordinates (x, y, θ), the measure
$$\mathrm{d}m = \mathrm{d}\nu\,\mathrm{d}s$$
is also preserved by geodesic flow and is a multiple of the Liouville volume $\mathrm{d}\omega$. Specifically, $\mathrm{d}m = \frac{1}{2}\mathrm{d}\omega$ (see [10, Appendix A2]).[5]

[4] The formula $\mathrm{d}\nu = |\mathrm{d}u||\mathrm{d}w|/|u - w|^2$ is used when $u, w \in \{z \in \mathbb{C} : |z| = 1\}$. If we consider $u, w \in (-\pi, \pi]$ as arguments, then $\mathrm{d}\nu = \mathrm{d}u\mathrm{d}w/(2 - 2\cos(u - w))$.

[5] The constant relating $d\omega$ and $\mathrm{d}m$ was given incorrectly as 1/4 in [6, page 250]. Following that, 1/4 was used in [2, Proposition 10.1].

Generalizing Adler and Flatto's "rectilinear map" from [6], we define the map $F_{\overline{A}} : \mathbb{S} \times \mathbb{S} \setminus \Delta \to \mathbb{S} \times \mathbb{S} \setminus \Delta$, where Δ is the diagonal $\{(w, w) : w \in \mathbb{S}\}$, by

$$F_{\overline{A}}(u, w) = (T_k u, T_k w) \qquad \text{if } w \in [A_k, A_{k+1}). \tag{3.1}$$

In [19], the authors showed that $F_{\overline{A}}$ admits a global attractor $\Omega_{\overline{A}}$ with finite rectangular structure if $\overline{A} = \overline{P}$ or if $\overline{A}$ satisfies the short cycle property. Adler and Flatto had previously shown that $F_{\overline{P}}$ has as an invariant domain with finite rectangular structure (which we call $\Omega_{\overline{P}}$, see Figure 6), and this was extended to $F_{\overline{A}}$ for all extremal parameters in [1]. The restriction of $F_{\overline{A}}$ to $\Omega_{\overline{A}}$ is the natural extension map of $f_{\overline{A}}$; we will often denote $F_{\overline{A}}|_{\Omega_{\overline{A}}}$ by simply $F_{\overline{A}}$.

- The normalized measure

$$\mathrm{d}\nu_{\overline{A}} := \frac{\mathrm{d}\nu}{\int_{\Omega_{\overline{A}}} \mathrm{d}\nu}$$

  is by construction a smooth invariant probability measure for $F_{\overline{A}}$.

Because $f_{\overline{A}}$ is a factor of $F_{\overline{A}}$ (projecting on the second coordinate), its smooth invariant probability measure $\mu_{\overline{A}}$ is obtained as a projection of $\nu_{\overline{A}}$.

Step 2: formula for the entropy. The geodesic flow on S can be realized as a special flow over a cross-section that is parametrized by $\Omega_{\overline{A}}$, and the first return map to this cross-section acts exactly as $F_{\overline{A}} : \Omega_{\overline{A}} \to \Omega_{\overline{A}}$ (see [2, Section 4] for short cycles and [1, Section 2.2] for extremal). Using this realization along with Abramov's formula and the Ambrose–Kakutani theorem, we have from [2, Proposition 10.1] (with a corrected constant) that

$$h_{\nu_{\overline{A}}}(F_{\overline{A}}) = \frac{\pi^2(4g-4)}{\int_{\Omega_{\overline{A}}} \mathrm{d}\nu}.$$

Since $F_{\overline{A}}$ is the natural extension of $f_{\overline{A}}$, the entropies $h_{\nu_{\overline{A}}}(F_{\overline{A}})$ and $h_{\mu_{\overline{A}}}(f_{\overline{A}})$ are equal, and since $\mathrm{Area}(\mathcal{F}) = 2\pi(2g-2)$ by the Gauss–Bonnet formula, we have

$$h_{\mu_{\overline{A}}}(f_{\overline{A}}) = \pi \cdot \frac{\mathrm{Area}(\mathcal{F})}{\int_{\Omega_{\overline{A}}} \mathrm{d}\nu}. \tag{3.2}$$

The integral $\int_{\Omega_{\overline{A}}} \mathrm{d}\nu$ can be explicitly computed using the finite rectangular structure of $\Omega_{\overline{A}}$, but this does not easily relate to the (hyperbolic) perimeter of $\mathcal{F}$.

Adler–Flatto [6] introduced another map, called the "curvilinear map" (or "geometric map" in [2]): denoting by uw the geodesic from u to w, the map is defined on the set

$$\Omega_{\mathrm{geo}} := \{(u, w) : uw \text{ intersects } \mathcal{F}\} \subset \mathbb{S} \times \mathbb{S} \setminus \Delta$$

(shown in Figure 6) and is given by

$$F_{\mathrm{geo}}(u, w) = (T_k u, T_k w) \quad \text{if } uw \text{ exits } \mathcal{F} \text{ through side } k.$$

There is a bijection $\Phi_{\overline{A}} : \Omega_{\mathrm{geo}} \to \Omega_{\overline{A}}$ that acts piecewise by Möbius transformations (see [2, Proposition 3.4] for short cycles and [1, Proposition 12] for extremal) and since Möbius transformations preserve the measure ν, it follows that

$$\int_{\Omega_{\overline{A}}} \mathrm{d}\nu = \int_{\Omega_{\mathrm{geo}}} \mathrm{d}\nu. \tag{3.3}$$

Given (3.3), we now want to show that $\int_{\Omega_{\mathrm{geo}}} \mathrm{d}\nu$ is equal to the (hyperbolic) perimeter of $\mathcal{F}$. We use the following fact proved by F. Bonahon [10, Appendix A3]:

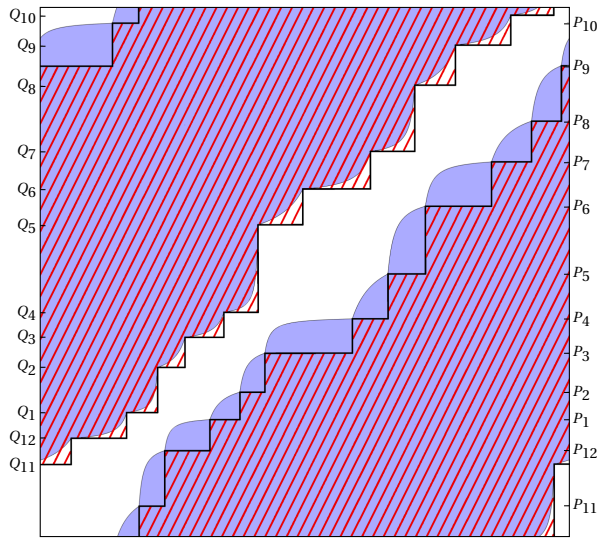


Figure 6. Arithmetic set $\Omega_{\overline{P}}$ in striped red, geometric set Ω_{geo} in solid blue (for an irregular polygon with $g = 2$).

- For any oriented geodesic segment s on $\mathbb{D}$,
$$\int_{\Psi^+(s)} \mathrm{d}\nu = \text{length}(s),$$
where $\Psi^+(s)$ is the set of oriented geodesics intersecting s with the oriented angle at the intersection between 0 and π.

The domain Ω_{geo} of the geometric map F_{geo} can be decomposed as $\Omega_{\text{geo}} = \bigcup_{k=1}^{8g-4} \mathcal{G}_k$ using the "strips"
$$\mathcal{G}_k = \left\{(u, w) : uw \text{ exits } \mathcal{F} \text{ through side } k\right\} = \Psi_+(\text{side } k)$$
(these are shown in [2, Figure 3]). Thus from Bohanon's result we immediately get
$$\int_{\Omega_{\text{geo}}} \mathrm{d}\nu = \sum_{k=1}^{8g-4} \int_{\mathcal{G}_k} \mathrm{d}\nu = \sum_{k=1}^{8g-4} \text{length}(\text{side } k) = \text{Perimeter}(\mathcal{F}).$$
Combining this with (3.3), one can replace $\int_{\Omega_{\overline{A}}} \mathrm{d}\nu$ by the perimeter of $\mathcal{F}$ in the denominator of (3.2), and we obtain the formula (1.4), that is,
$$h_{\mu_{\overline{A}}}(f_{\overline{A}}) = \frac{\pi^2(4g-4)}{\text{Perimeter}(\mathcal{F})} = \pi \cdot \frac{\text{Area}(\mathcal{F})}{\text{Perimeter}(\mathcal{F})}.$$

Step 3: maximum of the entropy. To prove that Theorem 1.2(ii) follows from (1.4) we only need to show that for each genus g the perimeter of $\mathcal{F}$ in $\mathcal{T}(g)$ is minimized on the regular polygon.

- Isoareal Inequality: *among all hyperbolic polygons with a given area and number of sides, the regular polygon has the smallest perimeter.* More precisely, Ku-Ku-Zhang prove [22, Theorem 1.2(a)] that for a hyperbolic n-gon $\mathcal{P}_n$,
$$\text{Perimeter}(\mathcal{P}_n)^2 \geq 4d_n \text{Area}(\mathcal{P}_n), \text{ where } d_n = n \tan\left(\frac{\text{Area}(\mathcal{P}_n)}{2n}\right),$$
with equality achieved on a regular polygon. The Isoareal Inequality follows immediately: $\text{Area}(\mathcal{P}_n)$ and n are constant, so the right-hand side $4d_n\text{Area}(\mathcal{P}_n)$ is constant and thus the perimeter is minimized when $\mathcal{P}_n$ is regular.

- In our setting, $\mathcal{F} = \mathcal{P}_n$ with $n = 8g-4$, and $\mathrm{Area}(\mathcal{F}) = 2\pi(2g-2)$ is constant in $\mathcal{T}(g)$, so the Isoareal Inequality implies that the perimeter of $\mathcal{F}$ is minimized when $\mathcal{F}$ is regular.

The expression for the maximum value $H(g)$ in Theorem 1.2 comes directly from (1.4), with

$$\operatorname{arccosh}\left(1 + 2\cos\tfrac{\pi}{4g-2}\right)$$

being the length of a single side of the regular $(8g-4)$-gon. This completes the proof of Theorem 1.2(ii).

Step 4: flexibility of the entropy. The space $\mathcal{T}(g)$ is homeomorphic to $\mathbb{R}^{6g-6}$, and a standard way to parametrize $\mathcal{T}(g)$ is through Fenchel–Nielsen coordinates (see the classical manuscript recently published in [15]). The surface S can be decomposed into $2g-2$ pairs of pants by $3g-3$ non-intersecting closed geodesics; this decomposition is shown for $g = 2$ in the bottom of Figure 7 (for genus 3, see [4, Figure 4]). The lengths of these geodesics can be manipulated independently (they form $3g-3$ of the $6g-6$ coordinates) and can take arbitrarily large values. We take one of these geodesics to also be a geodesic from the necklace described in Section 1 that corresponds to one entire side of $\mathcal{F}$ (this shared geodesic is on the far right in both parts of Figure 7). Since the length of this side—one of the Fenchel–Nielsen coordinates—can be made arbitrarily large, the perimeter of $\mathcal{F}$ can also be made arbitrarily large, which by (1.4) means that $h_{\mu_{\bar{A}}}(f_{\bar{A}})$ can be made arbitrarily small.

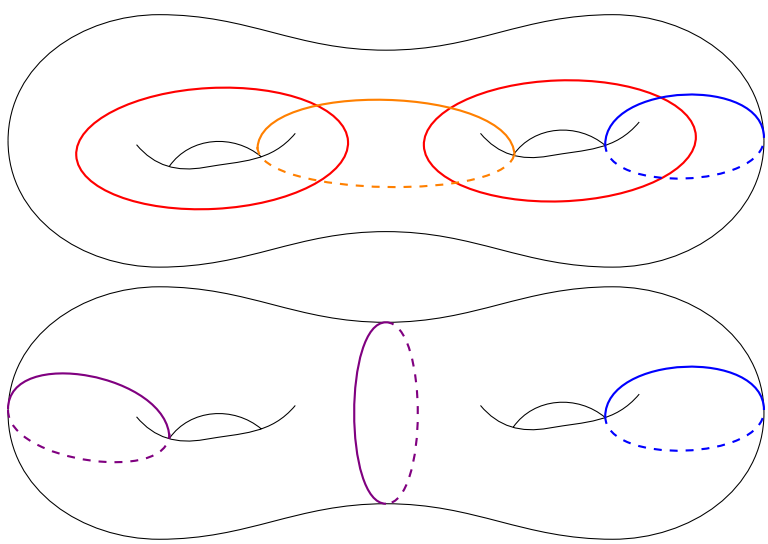


FIGURE 7. Necklace of $2g$ geodesics on S forming the sides of $\mathcal{F}$ (top) and decomposition of S into $2g-2$ pairs of pants by $3g-3$ non-intersecting geodesics (bottom) for $g = 2$.

Using the topology on the Teichmüller space $\mathcal{T}(g)$ as the space of marked canonical $(8g-4)$-gons, we see that the perimeter of $\mathcal{F}$ varies continuously within $\mathcal{T}(g)$. From (1.4) we conclude the continuity of the entropy $h_{\mu_{\bar{A}}}(f_{\bar{A}})$ within $\mathcal{T}(g)$. By the Intermediate Value Theorem, $h_{\mu_{\bar{A}}}(f_{\bar{A}})$ must take on all values between 0 and its maximum; this is precisely the claim of Theorem 1.2(iii).

4. Additional results

4.1. Comparison of entropies. By the Variational Principle, the measure-theoretic entropy of a map can never exceed its topological entropy. For some classical systems, the Lebesgue measure is both a smooth invariant measure and also the measure of maximal entropy, but the boundary maps $f_{\overline{A}} : \mathbb{S} \to \mathbb{S}$ provide examples where the ergodic smooth invariant probability measure is not the measure of maximal entropy.

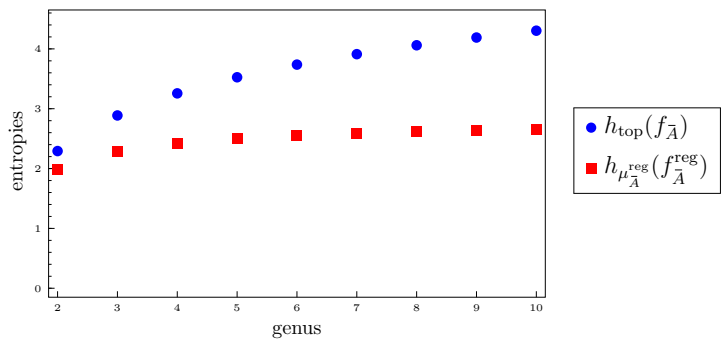


Figure 8. Topological entropy and measure-theoretic entropy for different genera.

Proposition 4.1 ([4, Corollary 7])**.** *If $\overline{A}$ is extremal or has short cycles, the measure-theoretic entropy of $f_{\overline{A}}$ with respect to its smooth invariant measure $\mu_{\overline{A}}$ is strictly less than the topological entropy of $f_{\overline{A}}$.*

From the expression for $H(g)$ in Theorem 1.2(ii), the maximum possible value of the measure-theoretic entropy with respect to the smooth invariant measure is

$$h_{\mu^{\mathrm{reg}}_{\overline{A}}}(f^{\mathrm{reg}}_{\overline{A}}) = \frac{\pi^2(4g-4)}{(8g-4)\operatorname{arccosh}(1+2\cos\frac{\pi}{4g-2})},$$

and we show in [4, Section 4] that this is strictly larger than the topological entropy

$$h_{\mathrm{top}}(f_{\overline{A}}) = \log\left(4g-3+\sqrt{(4g-3)^2-1}\right)$$

for all $g \geq 2$. See Figure 8 for a graph of these values.

4.2. Examples of other Markov partitions. For generic $\overline{A}$, the map $f_{\overline{A}}$ is not necessarily Markov. However, a Markov structure does occur when the right and left orbits of A_k, that is,

$$\{\lim_{\varepsilon\to 0^+} f^n_{\overline{A}}(A_k+\varepsilon)\}_{n=0}^{\infty} \quad \text{and} \quad \{\lim_{\varepsilon\to 0^+} f^n_{\overline{A}}(A_k-\varepsilon)\}_{n=0}^{\infty},$$

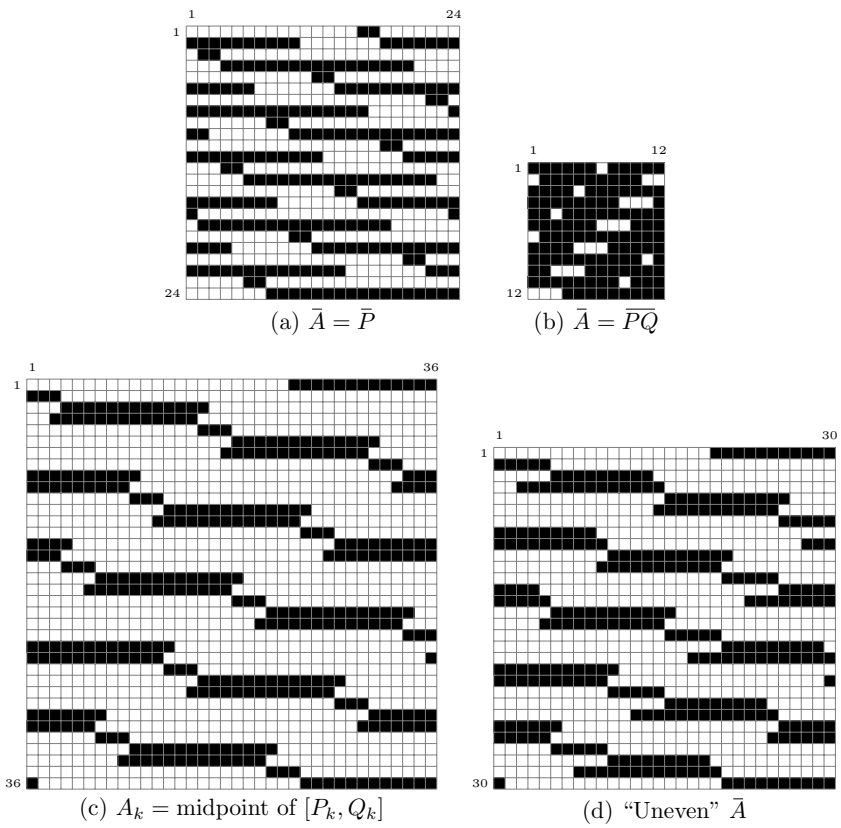
are eventually periodic for each $\overline{A}_k$ (these are called "upper" and "lower" orbits in [19]). For extremal $\overline{A}$, this is immediate. If $\overline{A}$ has the cycle property, it is sufficient that the "cycle ends" are discontinuity points, as stated for short cycles in [2, Proposition 7.2].

When $f_{\overline{A}}$ is Markov, the maximal eigenvalue of the associated transition matrix must be $\lambda = 4g-3+\sqrt{(4g-3)^2-1}$ because we know that $h_{\mathrm{top}}(f_{\overline{A}})$ is the log of this value for all parameters $\overline{A}$. Given a Markov partition $\{I_1,\ldots,I_n\}$ with n elements, the entries of the associated $n\times n$ transition matrix are described in (2.1). In this section, we give four examples of maps and transition matrices with, respectively,

(a) $n = 2(8g-4)$, **(b)** $n = 8g-4$, **(c)** $n = 3(8g-4)$, **(d)** $n = \frac{5}{2}(8g-4)$

partition elements.

Figure 9 shows these matrices for genus 2 (each white cell is a 0 in the matrix, and each black cell is a 1); each matrix is for a different map $f_{\overline{A}}$ and uses a different partition $\{I_1, ..., I_n\}$, but all have the same maximal eigenvalue.



(a) $\overline{A} = \overline{P}$ (b) $\overline{A} = \overline{PQ}$

(c) $A_k =$ midpoint of $[P_k, Q_k]$ (d) "Uneven" $\overline{A}$

FIGURE 9. Four transition matrices with maximal eigenvalue $5 + 2\sqrt{6}$.

(a) Let $A_k = P_k$ for all k. As described in [6, 4] and Step 1 of Section 2, the partition $\{I_1, \dots, I_{16g-8}\}$ with

$$I_{2k-1} := [P_k, Q_k], \quad I_{2k} := [Q_k, P_{k+1}], \qquad k = 1, \dots, 8g-4,$$

is a Markov partition for $f_{\overline{P}}$ (and, in fact, for every extremal $\overline{A}$). For $g = 2$, the matrices $M_{\overline{P}}$ and $M_{\overline{Q}}$ for this partition are shown in Figure 3, and $M_{\overline{P}}$ is shown again in Figure 9(a) as a colored grid.

(b) For $\overline{A} = \overline{PQ} = \{P_1, Q_2, P_3, Q_4, ...,\}$ we *could* use the partition $\{I_1, \dots, I_{16g-8}\}$ from (a) and [3], but the coarser partition $\{I_1, ..., I_{8g-4}\}$ with

$$I_k = [P_k, Q_{k+1}] \text{ for odd } k, \quad I_k = [Q_k, P_{k+1}] \text{ for even } k$$

is also Markov for $f_{\overline{PQ}}$. Using this partition we get a much smaller transition matrix (it is $(8g-4) \times (8g-4)$ rather than $(16g-8) \times (16g-8)$) with exactly the same maximal eigenvalue. For $g = 2$, this 12×12 matrix is shown in Figure 9(b).

(c) Let A_k be the midpoint of $[P_k, Q_k]$ for each k. As described in [2, Sections 7-8], this $f_{\overline{A}}$ has a Markov partition $\{I_1, ..., I_{24g-12}\}$ with

$$I_{3k-2} := [A_k,], \quad I_{3k-1} := [C_k, B_k], \quad I_{3k} := [B_k, A_{k+1}], \qquad k = 1, \dots, 8g-4,$$

where

$$B_k := T_{\sigma(k-1)} A_{\sigma(k-1)} \quad \text{and} \quad C_k := T_{\sigma(k+1)} A_{\sigma(k+1)+1}$$

are the first iterates in the right and left orbits. For $g = 2$, the corresponding 36×36 transition matrix is shown in Figure 9(c).

(d) For odd k let A_k be the midpoint of $[P_k, Q_k]$, and for even k let A_k be the image under $T_{\sigma(k)+1} \circ T_k$ of the midpoint of $[P_{k+1}, Q_{k+1}]$. In [2, Section 8, Example 3] we show that this $f_{\bar{A}}$ has a partition with $3(8g-4)$ intervals following the same construction as (c). However, for this particular $\bar{A}$ we have that $C_{k+1} = B_{k+1}$ if k is odd, which means that the intervals $I_5 = I_{3(2)-1} = [C_2, B_2]$, $I_{11} = I_{3(4)-1} = [C_4, B_4]$, etc., are trivial. These comprise $4g-2$ of the $24g-12$ intervals.

Thus we can make a Markov partition for this $f_{\bar{A}}$ using only $20g-10$ intervals. Explicitly, we use $\{I_1, ..., I_{20g-10}\}$ given by

$$\begin{aligned} I_{5k-4} &:= [A_k, C_k], \quad I_{5k-3} := [C_k, B_k], \quad I_{5k-2} := [B_k, A_{k+1}], \\ I_{5k-1} &:= [A_{k+1}, B_{k+1}], \quad I_{5k} := [B_{k+1}, A_{k+2}], \qquad k = 1, \ldots, 4g-2. \end{aligned}$$

For $g = 2$, the corresponding 30×30 transition matrix is shown in Figure 9(d).

4.3. Teichmüller parameters for genus 2. We now provide some geometric results specific to genus 2. We use the parameterization of $\mathcal{T}(2)$ given by Maskit in [23]:

Theorem 4.2 ([23, Theorem 8.1])**.** *For any $(\alpha, \beta, \gamma, \sigma, \tau, \rho) \in \mathbb{R}^6$, define*

$$\begin{aligned} \mu &:= \operatorname{arccosh}(\coth\beta \cosh\sigma \cosh\tau + \sinh\sigma \sinh\tau), \\ \delta &:= \operatorname{arccoth}\Big(\frac{\cosh\gamma \cosh\mu - \coth\alpha \sinh\gamma \sinh\mu - \sinh\rho \sinh\sigma}{\cosh\rho \cosh\sigma}\Big), \\ \epsilon &:= \operatorname{arccosh}(\coth\mu \sinh\alpha \sinh\gamma - \cosh\alpha \cosh\gamma), \\ \phi &:= \operatorname{arccosh}\Big(\frac{\sinh\beta \sinh\delta(\sinh\rho \sinh\tau + \cosh\gamma)}{\cosh\rho \cosh\tau} - \cosh\beta \cosh\delta\Big). \end{aligned} \tag{4.1}$$

There exists a real-analytic diffeomorphism between

$$\{(\alpha, \beta, \gamma, \sigma, \tau, \rho) \in \mathbb{R}^6 : \alpha > 0, \beta > 0, \gamma > 0, \delta > 0, \epsilon > 0\}$$

and the Teichmüller space $\mathcal{T}(2)$. (Note that $\delta > 0$ and $\epsilon > 0$ implicitly are requirements on $\alpha, \beta, \gamma, \sigma, \tau, \rho$ and that the stated inequalities imply $\phi > 0$.)

In [23], μ and δ are defined geometrically and are then proven to satisfy [23, equation (12)], that is,

$$\coth\delta = \frac{\cosh\gamma \cosh\mu - \coth\alpha \sinh\gamma \sinh\mu - \sinh\sigma \sinh\rho}{\cosh\sigma \cosh\rho},$$

when all parameters satisfy certain necessary inequalities. Here, we are defining μ and δ by (4.1) and making $\delta > 0$ a requirement of valid parameters. This condition $\delta > 0$ is equivalent to the inequality [23, equation (13)]. Maskit does not explicitly use ϵ or ϕ at all, but algebraic manipulation shows that [23, equation (3)] is equivalent to $\epsilon > 0$. Additionally, the inclusion of ϵ and ϕ allows for the concise statement of Proposition 4.3.

The correspondence between parameters $(\alpha,\beta,\gamma,\sigma,\tau,\rho)$ and polygons $\mathcal{F}$ can be made explicit as follows. Let

$$A = X_A\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha}\end{pmatrix}X_A^{-1} \qquad \text{with } X_A = \begin{pmatrix} i & 1 \\ 1 & i\end{pmatrix}\begin{pmatrix} 1 & 1 \\ e^{\mu} & e^{-\mu}\end{pmatrix},$$

$$B = X_B\begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta}\end{pmatrix}X_B^{-1} \qquad \text{with } X_B = \begin{pmatrix} i & 1 \\ 1 & i\end{pmatrix}\begin{pmatrix} e^{\sigma} & e^{\sigma} \\ e^{-\tau} & -e^{\tau}\end{pmatrix},$$

$$C = X_C\begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma}\end{pmatrix}X_C^{-1} \qquad \text{with } X_C = \begin{pmatrix} i & 1 \\ 1 & i\end{pmatrix},$$

$$D = X_D\begin{pmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta}\end{pmatrix}X_D^{-1} \qquad \text{with } X_D = \begin{pmatrix} i & 1 \\ 1 & i\end{pmatrix}\begin{pmatrix} e^{\sigma+\gamma} & e^{\sigma+\gamma} \\ -e^{\rho} & e^{-\rho}\end{pmatrix},$$

$E = A^{-1}C^{-1}$, and $F = D^{-1}B^{-1}$ (the eigenvalues of E and F are, respectively, $e^{\pm\epsilon}$ and $e^{\pm\phi}$, but the matrices that diagonalize E and F do not seem to have simple expressions). Then the sides of the polygon $\mathcal{F}$ lie along axes of Möbius transformations that are products of these matrices; explicitly, if s_k is the transformation for which P_k is the repelling fixed point and Q_{k+1} is the attracting fixed point, then

$$\begin{aligned}
&s_1 = C^{-1}D^{-1}C, && s_2 = AC, && s_3 = AF^{-1}A^{-1}, && s_4 = A^{-1},\\
&s_5 = F, && s_6 = E^{-1}, && s_7 = D, && s_8 = DED^{-1},\\
&s_9 = B^{-1}D^{-1}, && s_{10} = B^{-1}AB, && s_{11} = s_1^{-1}B, && s_{12} = C^{-1}s_{10}^{-1}.
\end{aligned}$$

These expressions for s_k are from [4, Appendix] (called "S_k" there), and the expressions for A,B,C,D are newly-discovered formulas that are easily shown to be equivalent to the original descriptions of the matrices in [23, Theorem 5.1].

Proposition 4.3. *Let $(\alpha,\beta,\gamma,\sigma,\tau,\rho)$ be Maskit's parameters for a point in $\mathcal{T}(2)$, and let $\mathcal{F}$ be the associated fundamental polygon. Then*

$$\text{Perimeter}(\mathcal{F}) = 4(\alpha+\delta+\epsilon+\phi)$$

with δ,ϵ,ϕ defined by (4.1)*. This immediately implies $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = \pi^2/(\alpha+\delta+\epsilon+\phi)$.*

Proof. For a Möbius transformation $M \in \Gamma$, the axis of M on $\mathbb{D}$ projects to a closed geodesic on $\Gamma\backslash\mathbb{D}$, and the length of this closed geodesic is $2\operatorname{arccosh}|\operatorname{tr}(M)/2|$, where $\operatorname{tr}(M)$ is the trace of M.

Sides $1,4,7,10$ of $\mathcal{F}$ are closed geodesics on the surface S, while the other sides come in pairs: the closed geodesic along the axis of s_k for $k \notin \{1,4,7,10\}$ consists of side k and side $k+6$.

The eigenvalues of A,B,C,D,E,F are $e^{\pm\alpha}, e^{\pm\beta}, e^{\pm\gamma}, e^{\pm\delta}, e^{\pm\epsilon}, e^{\pm\phi}$, respectively. Thus the length of side 1 is

$$\begin{aligned}
2\operatorname{arccosh}|\operatorname{tr}(s_1)/2| &= 2\operatorname{arccosh}|\operatorname{tr}(C^{-1}D^{-1}C)/2| = 2\operatorname{arccosh}|\operatorname{tr}(D^{-1})/2|\\
&= 2\operatorname{arccosh}|\tfrac{1}{2}(e^{\delta}+e^{-\delta})| = 2\operatorname{arccosh}|\cosh\delta| = 2\delta
\end{aligned}$$

and the combined length of sides 2 and 8 is

$$\begin{aligned}
2\operatorname{arccosh}|\operatorname{tr}(s_2)/2| &= 2\operatorname{arccosh}|\operatorname{tr}(AC)/2| = 2\operatorname{arccosh}|\operatorname{tr}(CA)/2|\\
&= 2\operatorname{arccosh}|\operatorname{tr}(E^{-1})/2| = 2\operatorname{arccosh}|\tfrac{1}{2}(e^{\epsilon}+e^{-\epsilon})| = 2\epsilon.
\end{aligned}$$

Similarly, the combined length of sides 3 and 9 is $2\operatorname{arccosh}|\operatorname{tr}(F)/2| = 2\phi$, and the length of side 4 alone is $2\operatorname{arccosh}|\operatorname{tr}(A)/2| = 2\alpha$. Sides 6, 7, 10, 11, and 12 have the same length as, respectively, sides 8, 1, 4, 9, and 2 because those respective side pairs are identified. Thus the total perimeter of $\mathcal{F}$ is

$$2(2\delta+2\epsilon+2\phi+2\alpha) = 4(\alpha+\delta+\epsilon+\phi). \qquad \square$$

Remark 4.4. For many surfaces (e.g., all those with $\sigma = \tau = \rho = 0$) sides 2 and 8 will each have length ϵ, sides 3 and 9 will each have length ϕ, etc., but in general we require only that sides 2 and 8 together have total length 2ϵ, and so on, with only sides $1, 4, 7, 10$ having simple expressions for their individual lengths.

The subspace of $\mathcal{T}(2)$ in which $\sigma = \tau = 0$ contains several notable surfaces (in these cases, the origin of $\mathbb{D}$ is a "Weierstrass point" of the surface):

- The regular polygon corresponds to $\alpha = \frac{1}{2}\operatorname{arccosh}(1+\sqrt{3})$, $\beta = \gamma = 2\alpha$, and $\rho = 0$. The perimeter of the regular polygon for $g = 2$ is $12\operatorname{arccosh}(1+\sqrt{3}) \approx 19.955$, and the associated entropy is $H(2) = 4\pi^2/(12\operatorname{arccosh}(1+\sqrt{3})) \approx 1.978$.
- Maskit's "base surface" [23, Sec. 3] corresponds to $\alpha = \beta = \gamma = \operatorname{arccosh}(2)$ and $\rho = 0$. The perimeter of $\mathcal{F}$ for this surface is $16\operatorname{arccosh}(2) \approx 21.071$, and so the entropy of the boundary map is ≈ 1.874.
- The Bolza surface is the hyperbolic genus 2 surface that maximizes the systole (the length of the shortest closed geodesic on the surface) [28]. Denoting

  $$\ell_1 = 2\operatorname{arccosh}(1+\sqrt{2}) \approx 3.057, \qquad \ell_2 = 2\operatorname{arccosh}(3+2\sqrt{2}) \approx 4.897,$$

  this systole is exactly ℓ_1. This surface corresponds to $\alpha = \beta = \gamma = \frac{1}{2}\ell_1$ and $\rho = \operatorname{arcsinh}(1)$.[6] Usually the Bolza surface is described as a gluing of a regular *octagon*, but it can also be described as a gluing of a 12-gon, as shown in Figure 10. Sides 1, 4, 7, 10 have length ℓ_1; sides 2, 6, 8, 12 have length $\frac{1}{2}\ell_2$; and sides 3, 5, 9, 11 have length $\frac{1}{2}\ell_1$. Thus the perimeter of $\mathcal{F}$ is $6\ell_1 + 2\ell_2 \approx 28.137$, and the entropy of the boundary map is ≈ 1.403.

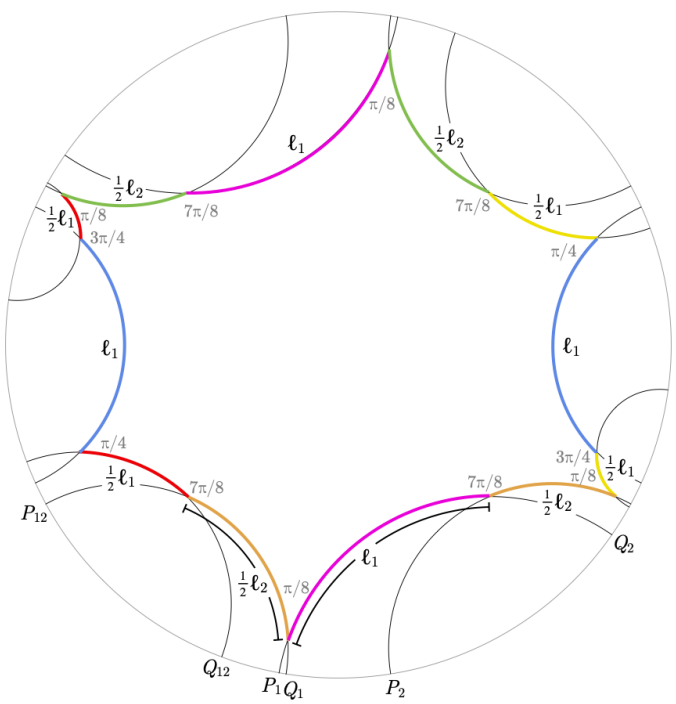


Figure 10. Fundamental 12-gon for the Bolza surface (sides of the same color are identified).

[6]The authors thank Polina Vytnova for suggesting and assisting with the question of parameters for the Bolza surface.

5. Open questions

The flexibility results of Section 3 are proved under the assumption that the multi-parameter $\overline{A}$ is extremal or has short cycles. In order to say anything about the measure-theoretic entropy $h_{\mu_{\overline{A}}}(f_{\overline{A}})$, we must have a well-defined measure $\mu_{\overline{A}}$, and this is not guaranteed in all cases. This could be established directly—Conjecture 5.1—or through the connection to geodesic flow and natural extension maps—Conjectures 5.2 and 5.3. Additionally, we use the conjugacy

$$\Phi_{\overline{A}} : \Omega_{\text{geo}} \to \Omega_{\overline{A}}$$

between F_{geo} and $F_{\overline{A}}\big|_{\Omega_{\overline{A}}}$ to prove that $h_{\mu_{\overline{A}}}(f_{\overline{A}}) = \pi \cdot \text{Area}(\mathcal{F})/\nu(\Omega_{\overline{A}})$ from Step 2 of Section 3, and this is currently known only for extremal and short cycle multi-parameters.

Conjecture 5.1. *For any $\mathcal{F} \in \mathcal{T}(g)$ and any $\overline{A}$ with $A_k \in [P_k, Q_k]$, there exists a smooth $f_{\overline{A}}$-invariant ergodic probability measure $\mu_{\overline{A}}$.*

The existence of this measure for maps $f_{\overline{A}}$ admitting a Markov partition can be established directly using Adler's "Folklore Theorem" similarly to the case $\overline{A} = \overline{P}$ discussed in [6, 13, 12]. For regular fundamental polygons, existence can be established using results about expanding, non-Markov maps in [36].

Conjecture 5.2. *For any $\overline{A}$ with $A_k \in [P_k, Q_k]$, there exists a set $\Omega_{\overline{A}} \subset \mathbb{S} \times \mathbb{S} \setminus \Delta$ with finite rectangular structure that is a domain of bijectivity for $F_{\overline{A}}$ and moreover the global attractor of $F_{\overline{A}} : \mathbb{S} \times \mathbb{S} \setminus \Delta \to \mathbb{S} \times \mathbb{S} \setminus \Delta$.*

Conjecture 5.2 is part of the "Reduction Theory" proposed by Don Zagier and described for Fuchsian groups in [19, Introduction]. Additionally, understanding the structure of $\Omega_{\overline{A}}$ may help in proving the following:

Conjecture 5.3. *For any $\overline{A}$ with $A_k \in [P_k, Q_k]$, the map $F_{\overline{A}}\big|_{\Omega_{\overline{A}}}$ is conjugate to F_{geo} by a map $\Phi_{\overline{A}} : \Omega_{\text{geo}} \to \Omega_{\overline{A}}$ that acts piecewise by Möbius transformations.*

If Conjecture 5.3 is true then Theorem 1.2 in fact holds for all $\overline{A}$ with $A_k \in [P_k, Q_k]$.

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