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Reduction Theory for Fuchsian Groups

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0. Introduction

Let Γ be any Fuchsian group, i.e., a discrete subgroup of the group of isometries of the hyperbolic plane H^2. We consider the unit disc $D=\{z\in \mathbf{C}, |z|<1\}$ endowed with the Poincaré metric $ds=\frac{2|dz|}{1-|z|^2}$ as a model of the hyperbolic plane. The geodesics for this metric are circular arcs orthogonal to $S^1=\partial D=\{z\in \mathbf{C}, |z|=1\}$. Two geodesics can intersect at most once. The group Γ acts on D by linear fractional transformations and can be represented by matrices $\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$ with $a, c\in \mathbf{C}$ and $a\bar{a}-c\bar{c}=1$. The unit circle S^1 is a fixed circle for the group $\Gamma, \Gamma: S^1\to S^1$. All the necessary information about Fuchsian groups can be found in [2].

In this paper we develop a so-called reduction theory for Fuchsian groups Γ with compact quotient $\Gamma\backslash D$. We assume for convenience that 0 is not an elliptic point of Γ, i.e. $c\neq 0$, for $\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}\in\Gamma$. Obviously, every Γ is conjugate to a group satisfying this property. This theory serves the same purpose as Gauss reduction theory for $SL_2(\mathbf{Z})$ based on continued fractions. An important ingredient in the argument is a construction of two expanding maps on the boundary $f_\pm: S^1\to S^1$ associated to the group Γ. This construction is a generalization of that used by Bowen and Series in [1].

1. Construction of the Fundamental Region R_0 and the Special Polygon R

Definition. Let $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$. The circle $J(\gamma)=\{z\in D, |cz+d|=1\}$ is called the *isometric circle* of γ.

Since $\gamma'(z)=(cz+d)^{-2}$, γ expands Euclidean distances within $J(\gamma)$ and contracts outside.

Let R_0 be the intersection of D with the exteriors of all isometric circles $J(\gamma)$, $\gamma \in \Gamma$. The region R_0 constitutes a fundamental region for Γ. Since Γ is finitely generated and contains no parabolic elements, the boundary of R_0 consists of a finite number geodesic arcs with vertices inside S^1 [2, Theorem 15, 16, Sect. 34]. The images of R_0 under Γ exactly fill up D. Each side s of R_0 is identified with another side s' by an element $\gamma(s)$, $s \subset J(\gamma(s))$ and $s' \subset J(\gamma^{-1}(s)) = \gamma(s)(J(\gamma(s)))$. The set $\{\gamma(s), s \text{ is a side of } R_0\}$ forms a set of generators for Γ [2, Sect. 23], and R_0 can be regarded as all that part of D which is exterior to $J(\gamma(s))$, s is a side of R_0. In order to construct the fundamental region R_0, we list elements of the group Γ as follows. Given any $A > 0$ there are only finitely many elements of Γ with $|a| < A$. This follows from the equality $|a|^2 - |c|^2 = 1$ and the discreteness of the group Γ. $\Big($If Γ is an arithmetic group then after a suitable conjugation we will have $|a|^2, |c|^2$ in $\frac{1}{N}\mathbf{Z}$ for some integer N; see the examples in Sect. 5.$\Big)$ We can thus list elements of Γ in increasing order of $|a|$. This list will eventually include all elements of Γ. Taking isometric circles for the elements according to their order we shall obtain the fundamental region R_0 as described above after a finite number of steps. Indeed the distance from the isometric circle $J(\gamma)$, $\gamma = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$ to the center of D is equal to $(|a|-1)/|c|$ which tends to 1 as $|a| \to \infty$. Thus isometric circles with sufficiently large $|a|$ cannot contribute to the boundary of the compact fundamental region R_0. Examples of fundamental regions for some arithmetic groups are given in Sect. 5.

For each geodesic arc $J(\gamma(s))$ we consider the smaller of two arcs of ∂D having the same end points. Since all the vertices of R_0 lie inside ∂D these chosen arcs form a cover of ∂D. We can always choose a subcover of this cover in such a way that no two non-consecutive arcs intersect by deleting some "extra" arcs.

Definition. We shall call a polygon $R \subset D$ a *special polygon* associated to Γ if it satisfies the following properties:

i) R has finite number of vertices and they all lie inside D;

ii) All sides of R belong to isometric circles of some elements of Γ;

iii) Isometric circles containing any two non-consecutive sides of R do not intersect.

Obviously, the polygon formed by the isometric circles corresponding to the arcs of a subcover constructed above is a special polygon.

2. Construction of the Maps f_+ and f_-

Let R be a special polygon associated to Γ. Its sides and the end points of the corresponding geodesic arcs are labeled in the anticlockwise direction by $s_1, \ldots, s_n$ and $[P_1, Q_1], \ldots, [P_n, Q_n]$ respectively. For each arc $[P_i, Q_i] \subset S^1$ we choose an arc $[P'_i, Q'_i] \subset S^1$ inside $[P_i, Q_i]$ in such a way that the order of the points $P'_1, Q'_n, P'_2, Q'_1, \ldots, P'_n, Q'_{n-1}$, is the same as the order of the points $P_1, Q_n, P_2, Q_1, \ldots, P_n, Q_{n-1}$ (see Fig. 1), so that the arcs $[P'_i, Q'_i]$ still form a cover of S^1.

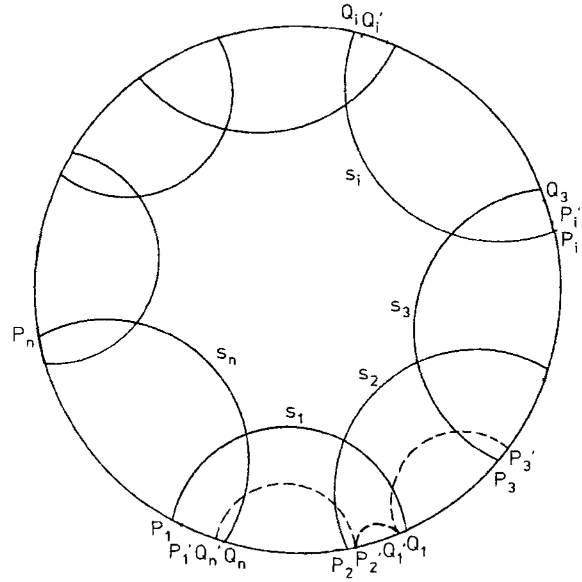

Fig. 1

We define two partitions of S^1:

$$M^+ = \{\omega_i^+\}_{i=1}^n, \qquad \omega_i^+ = [P_i', P_{i+1}'), \qquad i \in \mathbf{Z}/n\mathbf{Z}$$

and

$$M^- = \{\omega_j^-\}_{j=1}^n, \qquad \omega_j^- = [Q_{j-1}', Q_j'), \qquad j \in \mathbf{Z}/n\mathbf{Z},$$

and two piecewise continuous maps $f_+, f_- : S^1 \to S^1$:

$$f_+(x) = \gamma_i(x), \quad \text{if} \quad x \in \omega_i^+;$$
$$f_-(x) = \gamma_j(x), \quad \text{if} \quad x \in \omega_j^-.$$

Since $\omega_i^+ \subset [P_i', Q_i']$, it lies inside the isometric circle $J(\gamma_i)$ and we have for $x \in \omega_i^+$ $|f_+'(x)| = |\gamma_i'(x)| > \lambda_i > 1$. Similarly, for $x \in \omega_j^-$ we have $|f_-'(x)| > \lambda_j > 1$. Taking into account that R has a finite number of sides, we have the following result.

Lemma 1. *The maps f_+ and f_- are expanding, i.e., there exists $\lambda > 1$ such that $|f_+'(x)| > \lambda$ for $x \neq P_i'$, $i = 1, \ldots, n$, and $|f_-'(x)| > \lambda$ for $x \neq Q_i'$, $i = 1, \ldots, n$.*

Lemma 2. *Suppose $[x, y] \subset \omega_i^+$ and let c denote one of the symbols $+$, $-$. Then either $f_+(x)$ and $f_+(y)$ belong to different elements of M^c or, if $f_+(x)$ and $f_+(y)$ belong to the same element ω_j^c of M^c, then $f_+([x, y]) \subset \omega_j^c$.*

Proof. Suppose $f_+(x) \in \omega_j^c$, $f_+(y) \in \omega_j^c$, but $f_+([x, y]) \not\subset \omega_j^c$. For $t \in \omega_i^+$, $f_+(t) = \gamma_i(t)$. We have $\omega_i^+ \subset [P_i, Q_i]$. $\gamma_i([P_i, Q_i])$ is an arc of S^1 lying outside $J(\gamma_i^{-1}) = \gamma_i(J(\gamma_i))$ and therefore does not cover the whole circle S^1. Since $\gamma_i(t)$ is continuous and monotone on the arc $[P_i, Q_i]$, $f_+([x, y])$ does not cover the whole circle S^1, and the

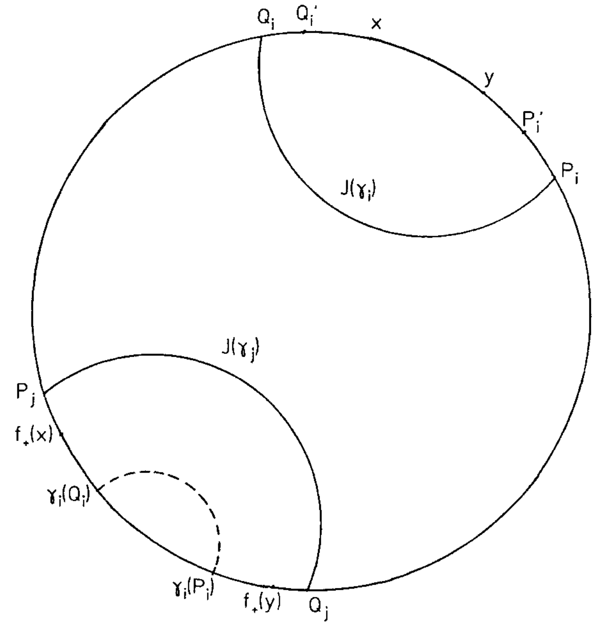

Fig. 2

assumption forces $\gamma_i(P_i)$ and $\gamma_i(Q_i)$ to lie inside ω_j^c (see Fig. 2). Since $\omega_j^+ \subset [P_j, Q_j]$ and $\omega_j^- \subset [P_j, Q_j]$, we have $J(\gamma_i^{-1})$ lies within $J(\gamma_j)$, which contradicts the properties of the fundamental region R_0, described in Sect. 1. □

Theorem. *Let $x, y \in \omega_i^+$. There exists a finite sequence $n_1, n_2, \dots, n_k$ of positive integers such that $f_\pm^{n_k}, \dots, f_-^{n_2} f_+^{n_1}(x)$ and $f_\pm^{n_k}, \dots, f_-^{n_2} f_+^{n_1}(y)$ belong to different elements of both partitions M^+ and M^-.*

Proof. There exists an integer n_1 such that for $i < n_1$ $f_+^i([x, y]) \subset \omega_{k_i}^+$ and $f_+^{n_1}(x)$ and $f_+^{n_1}(y)$ belong to different elements of the partition M^+. Otherwise, according to Lemma 2, we have $f_+^n([x, y]) \subset \omega_{k_n}^+$ for all $n > 0$, which contradicts the fact that f_+ is expanding (Lemma 1). Let $f_+^{n_1}(x) = x_1$, $f_+^{n_1}(y) = y_1$. Suppose $x_1, y_1 \subset \omega_j^-$. According to Lemma 2 $f_+^{n_1}([x, y]) \subset \omega_j^-$, i.e. $f_+^{n_1}([x, y]) = [x_1, y_1]$. Applying the same argument to the arc $[x_1, y_1]$ we find $n_2 > 0$ such that for $i < n_2$ $f_-^i([x_1, y_1]) \subset \omega_{k_i}^-$ and $f_-^{n_2}(x_1)$ and $f_-^{n_2}(y_1)$ belong to different elements of the partition M^-. If $f_-^{n_2}(x_1) = x_2$ and $f_-^{n_2}(y_1) = y_2$ belong to the same element of M^+, so does the arc $f_-^{n_2}([x_1, y_1]) = [x_2, x_2]$. Since both f_+ and f_- are expanding (Lemma 1), the length of $[x_2, y_2]$ is at least λ times the length of $[x, y]$ with a fixed $\lambda > 1$. This consideration shows that after a finite number of steps we obtain a pair of points x_k, y_k belonging to different elements of both M^+ and M^-. □

3. Reduction Theory and Coding of Geodesics

Any hyperbolic element $\gamma \in \Gamma(|\mathrm{tr}\,\gamma| > 2)$ has two hyperbolic fixed points on S^1, one repulsive and one attractive. We shall denote them by w_1 and w_2. Let $C(\gamma)$ be the oriented geodesic on D from w_1 to w_2. It is called the *axis* of γ and clearly it is

$\langle\gamma\rangle$-invariant. We shall call a hyperbolic element γ *reduced* if $C(\gamma)$ intersects the given fundamental region R_0 (see Sect. 1). $C(\gamma)$ becomes a closed geodesic in $\Gamma\backslash D$ and it can be coded according to the order it intersects the sides of R_0. This idea goes back to Morse [4] (see also [5, p. 104]). More precisely, suppose $C(\gamma)$ enters R_0 through the side s_i and leaves it through the side s_j. The side s_j is identified in R_0 with the side s_j' by the transformation γ_j, therefore the transformation $\gamma_j\gamma\gamma_j^{-1}=\gamma'$ will have its axis entering R_0 through s_j'. The first symbol in the code will be j, and we continue the process with $C(\gamma')$ instead of $C(\gamma)$. Since the geodesic is closed, the code will be periodic. If two elements γ_0 and γ_1 are conjugate in Γ, i.e., $\gamma_1=\gamma\gamma_0\gamma^{-1}$ for some $\gamma\in\Gamma$, then $C(\gamma_1)=\gamma C(\gamma_0)$. There is 1–1 correspondence between conjugacy classes of hyperbolic elements in Γ and closed geodesics in $\Gamma\backslash D$. If two hyperbolic elements are conjugate in Γ, their closed geodesics in R_0 coincide and therefore their codes differ by a cyclic permutation. Conversely, if two closed geodesics have the same code, we can make a homotopy between them along each pair of corresponding sides. Therefore the code determines the free homotopy class of a closed geodesic, and since there is only one closed geodesic in each free homotopy class, those geodesics coincide. If two hyperbolic elements having the same trace have the same code (up to a cyclic permutation), they correspond to the same closed geodesic and therefore are conjugate in Γ. If two hyperbolic elements of different trace have the same code, then both are conjugate to powers of the same primitive hyperbolic element having the same code. We know that each hyperbolic element is conjugate in Γ to a reduced one. The goal of a reduction theory is to give an algorithm producing this conjugation.

We shall define a compact region D_0 associated to the special polygon R described in Sect. 1. Consider $2n$ geodesics connecting each pair of the consecutive points P_i', Q_{i-1}' and Q_i', P_{i+2}', $i\in\mathbf{Z}/n\mathbf{Z}$ (see Fig. 1). Let D_0 be the smallest circle concentric with the principal circle S^1 which intersects all those $2n$ geodesic circles. D_0 is completely covered by R_0 and a finite number of its images under elements of Γ. We shall call a hyperbolic element γ *almost reduced* if $C(\gamma)$ intersects the compact region D_0. The reduction algorithm we are about to describe assumes that we know the following data about the group Γ:

1. a list of elements of Γ, the fundamental region R_0 as in Sect. 1 and generators of the group Γ;
2. the special polygon R described in Sect. 1;
3. the compact region D_0 as above and a finite set of elements $\gamma_{(i)}\in\Gamma$, $i=1,\dots,N$ such that $\bigcup_{i=1}^{N}\gamma_{(i)}R_0\supset D_0$.

We shall describe now the reduction algorithm, starting from a hyperbolic element γ:

Step 1. If the end points w_1, w_2 of $C(\gamma)$ (i.e., the hyperbolic fixed points of γ) belong to the same element $\omega_i^+\in M^+$, then conjugate γ by $\gamma_i=f_+$. This replaces γ by a new element with the fixed points f_+w_1, f_+w_2. If these points belong to the same element of M^+, repeat Step 1; otherwise proceed to Step 2.

Step 2. If the end points w_1, w_2 of $C(\gamma)$ belong to the same element $\omega_j^-\in M^-$, then conjugate γ by $\gamma_j=f_-$. If the fixed points f_-w_1, f_-w_2 of a new element belong to the same element of M^-, repeat Step 2, if they belong to the same element of M^+, go to Step 1; otherwise proceed to Step 3.

Step 3. Now the fixed points of γ belong to different elements of both M^+ and M^-. Therefore $C(\gamma)$ intersects the compact region D_0 and γ is almost reduced. Since $D_0 \subset \bigcup_{i=1}^{N} \gamma_{(i)} R_0$, $C(\gamma)$ intersects $\gamma(i) R_0$, for some i. Conjugate γ by $\gamma_{(i)}^{-1}$ and obtain a reduced element.

4. Remarks

The algorithm described in Sect. 3 depends only on the geodesic $C(\gamma)$ and can be applied to any (not necessarily closed) geodesic in D. It allows us to reduce such a geodesic to obtain its code with respect to the given fundamental region R_0. The code will not be periodic unless the geodesic is closed, i.e., is an axis of a hyperbolic element in Γ.

Consider the arithmetic case where Γ is contained in some quaternion algebra H over $\mathbf{Q}$ with $H \otimes_{\mathbf{Q}} \mathbf{R} = M_2(\mathbf{R})$. Then to each $\alpha \in H$ which is hyperbolic [i.e., $(\operatorname{tr}\alpha)^2 - 4\det\alpha > 0$] corresponds a geodesic $C(\alpha)$ whose image in $\Gamma \backslash D$ is closed since there exists a hyperbolic $\gamma \in \Gamma$ with $C(\gamma) = C(\alpha)$ (the centralizer of α in H is a real quadratic field and γ corresponds to a non-trivial unit). Note that any element $\lambda\alpha + \mu$ ($\lambda \in \mathbf{Q}^*$, $\mu \in \mathbf{Q}$) has the same geodesic. Consider the set of $\alpha \in H$ modulo the equivalence relation $\alpha \sim \lambda\alpha + \mu$. Choosing λ suitably we can assume that $\alpha \in \mathcal{O}$, a given order of H, containing Γ, and that α is primitive (not divisible by an integer bigger than 1) in $\mathcal{Q} = \mathcal{O}/\mathbf{Z} \simeq \mathbf{Z}^3$. The group Γ acts on $\mathcal{Q}$ by conjugations. What our reduction algorithm does is to pick out of each Γ-equivalence class of hyperbolic elements of $\mathcal{Q}$ a canonical (finite and non-empty) set of representatives which form a cycle in a natural way. In the classical case $\Gamma = SL_2(\mathbf{Z})$, $H = M_2(\mathbf{Q})$, $\mathcal{O} = M_2(\mathbf{Z})$ the space $\mathcal{Q}$ is the space of all binary quadratic forms with integer coefficients, and the analog of our theory is Gauss reduction theory of indefinite binary quadratic forms (as described, for instance, in [6, Chap. 13]).

5. Examples

1. The following example illustrates the algorithm of the construction of the fundamental region R_0 given in Sect. 1 for a special arithmetic group Γ. We begin from a subgroup of $PSL_2(\mathbf{R})$

$$\Gamma_{15} = \left\{ \gamma = \pm \begin{pmatrix} \frac{l + m\sqrt{3}}{2} & \sqrt{5}\left(\frac{w - u\sqrt{3}}{2}\right) \\ \sqrt{5}\left(\frac{w + u\sqrt{3}}{2}\right) & \frac{l - m\sqrt{3}}{2} \end{pmatrix}, \right.$$

where

$$\left. (l, m, u, w) \in \mathbf{Z}^4,\ l \equiv w (\operatorname{mod} 2),\ m \equiv u (\operatorname{mod} 2) \text{ and } \det\gamma = 1, \quad \text{i.e. } l^2 - 3m^2 - 5w^2 + 15u^2 = 4 \right\}.$$

This group is an embedding of the group of units of a maximal order of the quaternion algebra over Q with discriminant 15 [7, p. 123]. The group $\Gamma = R\Gamma_{15}R^{-1}$, where $R = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ acts on the unit disc D. Let us denote $R\gamma R^{-1} = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$. Then $a = \frac{l - iu\sqrt{15}}{2}$, $c = \frac{w\sqrt{5} - im\sqrt{3}}{2}$, $|a|^2 = \frac{r}{4} + 1$, $|c|^2 = \frac{r}{4}$, so $|a|^2, |c|^2 \in \frac{1}{4}\mathbf{Z}$. We can therefore list all elements $\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$ of Γ in increasing order of $|a|$ by solving the equations $l^2 + 15u^2 = r + 4$, $5w^2 + 3m^2 = r$, $l \equiv w (\mathrm{mod}\, 2)$, $m \equiv u (\mathrm{mod}\, 2)$ for $r = 0, 1, 2 \ldots$.

Table 1

r	ℓ	m	u	w	x	y	R
5	3	0	0	-1	1.342	0.000	0.894
5	3	0	0	1	-1.342	0.000	0.894
12	4	-2	0	0	0.000	1.155	0.577
12	4	2	0	0	0.000	-1.155	0.577
27	4	-3	-1	0	0.745	0.770	0.385
27	4	3	-1	0	-0.745	-0.770	0.385
27	4	-3	1	0	-0.745	0.770	0.385
27	4	3	1	0	0.745	-0.770	0.385
32	6	-2	0	-2	0.839	0.650	0.354
32	6	-2	0	2	-0.839	0.650	0.354
32	6	2	0	-2	0.839	-0.650	0.354
32	6	2	0	2	-0.839	-0.650	0.354
45	7	0	0	-3	1.043	0.000	0.298
45	7	0	0	3	-1.043	0.000	0.298
47	6	-3	-1	-2	0.999	0.295	0.292
47	6	-3	-1	2	-0.143	1.032	0.292
47	6	3	-1	-2	0.143	-1.032	0.292
47	6	3	-1	2	-0.999	-0.295	0.292
47	6	-3	1	-2	0.143	1.032	0.292
47	6	-3	1	2	-0.999	0.295	0.292
47	6	3	1	-2	0.999	-0.295	0.292
47	6	3	1	2	-0.143	-1.032	0.292
57	1	-2	-2	-3	0.588	-0.851	0.265
57	1	-2	-2	3	0.353	0.972	0.265
57	1	2	-2	-3	-0.353	-0.972	0.265
57	1	2	-2	3	-0.588	0.851	0.265
57	1	-2	2	-3	-0.353	0.972	0.265
57	1	-2	2	3	-0.588	-0.851	0.265
57	1	2	2	-3	0.588	0.851	0.265
57	1	2	2	3	0.353	-0.972	0.265
75	8	-5	-1	0	0.447	0.924	0.231
75	8	5	-1	0	-0.447	-0.924	0.231
75	8	-5	1	0	-0.447	0.924	0.231
75	8	5	1	0	0.447	-0.924	0.231
92	6	-2	-2	-4	0.875	-0.527	0.209
92	6	-2	-2	4	-0.292	0.979	0.209
92	6	2	-2	-4	0.292	-0.979	0.209
92	6	2	-2	4	-0.875	0.527	0.209
92	6	-2	2	-4	0.292	0.979	0.209
92	6	-2	2	4	-0.875	-0.527	0.209
92	6	2	2	-4	0.875	0.527	0.209
92	6	2	2	4	-0.292	-0.979	0.209
137	9	-2	-2	-5	0.930	-0.405	0.171
137	9	-2	-2	5	-0.539	0.860	0.171
137	9	2	-2	-5	0.539	-0.860	0.171
137	9	2	-2	5	-0.930	0.405	0.171
137	9	-2	2	-5	0.539	0.860	0.171
137	9	-2	2	5	-0.930	-0.405	0.171
137	9	2	2	-5	0.930	0.405	0.171
137	9	2	2	5	-0.539	-0.860	0.171

In Table 1 we give the beginning of this list. (For $r=0$ we get the identity element which we do not include in the table.) Columns 1–5 give values of r, l, m, u, w. Columns 6–8 give the coordinates (x, y) of the center of the corresponding isometric circle and its radius R. The isometric circles of the first 8 elements form a boundary of the fundamental region R_0 (see Fig. 3), and therefore those elements can be chosen as generators of the group Γ. The genus of $\Gamma \backslash D$ is 1 and the number of non-equivalent in Γ elliptic points of order 3 equals 2. The special

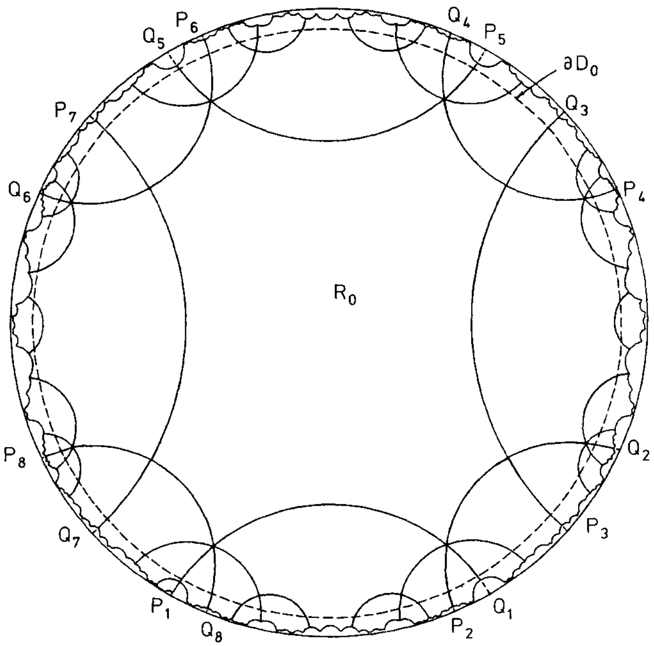

Fig. 3

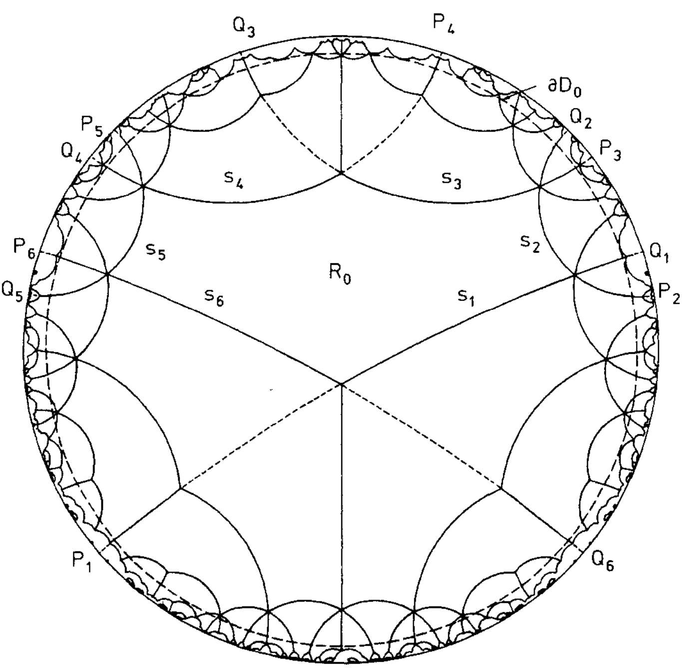

Fig. 4

polygon R in this example coincides with the fundamental region R_0. The compact region D_0 described in Sect. 3 and a finite part of the tesselation of D by the images of R_0 is given in Fig. 3.

2. In Fig. 4 we give the fundamental region R_0, the tesselation of D and the compact region D_0 for $\Gamma = R\Gamma_{10}R^{-1}$, where Γ_{10} is the following arithmetic

Table 2

Side of R_0	ℓ	m	u	w	x	y	R
s_1	1	0	-1	1	2.581	-4.581	5.162
s_2	6	0	4	-16	0.930	0.481	0.310
s_3	1	0	3	-9	0.352	1.172	0.705
s_4	1	0	-3	9	-0.352	1.172	0.705
s_5	6	0	-4	16	-0.930	0.481	0.310
s_6	1	0	1	-1	-2.581	-4.581	5.162

Table 3

TRACE	CODE					
4	5	3	5	1		
4	6	2	4	2		
4	1	3				
4	4	6				
6	2					
6	5					
7	4	1				
7	6	3				
9	6	3	2			
9	5	1	4			
9	4	1	5			
9	3	6	2			
9	2	4	1			
9	6	5	3			
9	5	6	3			
9	4	2	1			
11	4	6	4	1		
11	6	3	1	3		
11	6	2	4	2	6	3
11	4	1	5	3	5	1
11	6	3	3	6	2	
11	5	1	4	4	1	
11	4	1	5	1	4	
11	3	6	2	6	3	
11	3	1	4	1		
11	6	4	6	3		
11	5	1	5	3	6	3
11	4	2	6	2	4	1
11	1	4	2	4	1	
11	6	6	3	5	3	
11	5	3	6	6	3	
11	4	2	4	1	1	
12	3	6	4	1		
12	5	1	4	2	6	3
13	4	4	2	4	1	
13	6	3	3	5	3	
13	6	2	6	6	3	
13	4	1	5	1	1	
13	1	5	1	4	1	
13	6	6	2	6	3	
13	5	3	3	6	3	
13	4	2	4	4	1	

TRACE	CODE						
14	5	3	5	1			
14	6	2	4	2			
14	1	3					
14	4	6					
16	5	3	5	5	1		
16	6	2	4	2	2		
16	1	2	3				
16	4	6	5				
16	3	6	2	4	1		
16	6	4	1	5	3		
16	5	1	4	6	3		
16	4	2	6	3	1		
16	6	3	1	4	2		
16	5	1	4	6	3		
16	6	2	2	4	2		
16	5	1	5	5	3		
16	4	5	6				
16	3	2	1				
17	3	6	3	5	1		
17	6	4	1	4	2		
17	5	1	4	1	3		
17	4	2	6	3	6		
19	6	4	1	2			
19	5	1	3	6			
19	4	2	6	3	5		
19	3	5	1	4	2		
19	4	2	3	5	1		
19	6	3	5	1	2		
19	5	1	4	2	6		
19	4	1	3	5			
19	3	6	4	2			
19	4	2	6	5	1		
21	6	3	6	3	2		
21	5	1	4	1	4		
21	4	1	4	1	5		
21	3	6	3	6	2		
24	5	1	4	1	4	4	1
24	4	1	4	1	5	1	4
24	3	6	3	6	2	6	3
24	6	3	3	6	2	6	3

subgroup of $PSL_2(\mathbf{R})$:

$$\Gamma_{10}=\left\{\gamma=\pm\begin{pmatrix}\frac{l+m\sqrt{10}}{2} & \frac{w-u\sqrt{10}}{6}\\ \frac{w+u\sqrt{10}}{2} & \frac{l-m\sqrt{10}}{2}\end{pmatrix},\right.$$

where

$$\left.\begin{array}{c}(l,m,u,w)\in Z^4,\ 3|(u+w),\ l-n\equiv\frac{u+w}{3}\equiv m(\mathrm{mod}\,2),\ \text{and}\ \det\gamma=1,\\ \text{i.e. } 3l^2-30m^2+10u^2-w^2=12\end{array}\right\}.$$

The set of generators of Γ whose isometric circles form the boundary of the fundamental region R_0 is given in Table 2.

Table 3 gives codes of all elements of Γ with traces up to 24 with respect to the fundamental region R_0, as explained at the beginning of Sect. 3. This group has been studied in great detail in [3, Chap. 4]. Here we give several examples of elements of trace 4. Elements $\gamma_1=(4,0,8,-26)$ and $\gamma_2=(4,1,43,-136)$ have different codes: *code* $\gamma_1=(13)$, *code* $\gamma_2=(6242)$ and therefore they are not conjugate in Γ and represent different closed geodesics. Elements $\gamma_3=(4,1,1,-4)$ and $\gamma_4=(4,0,-8,26)$ have the same code (46) and therefore are conjugate in Γ. For trace 4 there are four different closed geodesics, (5351) and (6242), (13) and (46), the geodesics in each pair differ only in orientation.

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