LINEAR EXTENSIONS OF DYNAMIC SYSTEMS AND THE REDUCIBILITY PROBLEM

S. B. Katok

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The relation of linear extensions of smooth dynamic systems to cohomologies and to reducibility in the case of flow is investigated. A result is obtained concerning Γ-cohomologies in the neighborhood of a constant cocycle for the case of an arbitrary closed subgroup Γ of the group GL (k, C).

Let X and Y be real-analytic manifolds. We denote by $\mathfrak{A}(X, Y)$ the set of all real-analytic mappings of X into Y. A group $\{T_g, g \in G$, where G is a Lie group$\}$ of diffeomorphisms of the manifold X is said to be a real-analytic dynamic system with time G, and is denoted by (X, T_g) if

$$A(x, g) \in \mathfrak{A}(X \times G, X), \text{where } A(x, g) = T_g x.$$

For the sake of brevity we will use the term dynamic system for a real-analytic dynamic system.

Let Γ be a closed subgroup of $GL(k, \mathbf{C})$. We call the dynamic system $(X \times \mathbf{C}^k, \hat{T}_g)$ a k-dimensional complex linear extension of the dynamic system (X, T_g) with the group Γ, if, for all $g \in G$, $x \in X$, $y \in \mathbf{C}^k$, we have

$$\hat{T}_g(x, y) = (T_g x, h(x, g)y), \tag{1.1}$$

where $h(x, g) \in \mathfrak{A}(X \times G, \Gamma)$. Since $\hat{T}_{g_1}$ is a group, we have

$$h(T_{g_1}x, g_2)h(x, g_1) = h(x, g_1g_2), \quad h(x, e) = E. \tag{1.2}$$

Clearly any function $h(x, g) \in \mathfrak{A}(X \times G, \Gamma)$, satisfying (1.2) determines, through (1.1), a linear extension of the dynamic system (X, T_g). Two linear extensions $(X \times \mathbf{C}^k, \hat{T}_g^1)$ and $(X \times \mathbf{C}^k, \hat{T}_g^2)$ are called isomorphic if there exists a mapping $S: X \times \mathbf{C}^k$, determined by the relation

$$S(x, y) = (x, \varphi(x)y), \quad \varphi(x) \in \mathfrak{A}(X, \Gamma)$$

and such that $S\hat{T}_g^{(1)} = \hat{T}_g^{(2)}S$. If the linear extensions $\hat{T}_g^{(1)}$ and $\hat{T}_g^{(2)}$ are determined by the functions $h_1(x, g)$ and $h_2(x, g)$, respectively, then

$$h_2(x, g) = \varphi(T_g x) \cdot h_1(x, g) \cdot \varphi^{-1}(x). \tag{1.3}$$

By analogy with the known construction of homologies of groups, we sometimes consider (see [1]) various cohomologies (measurable, smooth, analytic) of a dynamic system (X, T_g). For example in the definition of analytic cohomologies, the function $h(x, g) \in \mathfrak{A}(X \times G, \Gamma)$ is called a cocycle if it satisfies (1.2), and two cocycles are called Γ-cohomologies if they are related by (1.3). We use these terms since it is immaterial whether we speak of cohomological cocycles or of isomorphic linear extensions.

In this work, we are interested in the two very simple ("classical") cases $G = \mathbf{Z}$ and $G = \mathbf{R}$.

If $G = \mathbf{Z}$ then (1.2) implies that the cocycle $h(x, n)$ is uniquely generated with respect to the function $h(x, 1) = g(x)$, where (1.3) is equivalent to

$$g_2(x) = \varphi(Tx) \cdot g_1(x) \cdot \varphi^{-1}(x). \tag{1.4}$$

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in which $g_1(x) = h_1(x, 1)$ and $g_2(x) = h_2(x, 1)$. In the sequel we use the term cocycle for the functions $g(x) \in \mathfrak{A}(X, \Gamma)$ themselves (no conditions have yet been imposed on them) and demonstrate their cohomology in the sense (1.4).

Let $G = \mathbf{R}$. A flow $\{S_t\}$ on a real-analytic manifold X can be specified in each local system of coordinates by a system of differential equations

$$\dot{x} = \omega(x).$$

Consider the flow $\{\hat{S}_t\}$ in $X \times C^k$, defined by the differential-equation system

$$\left.\begin{aligned} \dot{y} &= A(x)\, y, \\ \dot{x} &= \omega(x), \end{aligned}\right\} \qquad (1.5)$$

where $A(x) \in \mathfrak{A}(X, A_\Gamma)$; here A_Γ denotes the Lie algebra of the group Γ. The following proposition is easily proved.

PROPOSITION 1.1. The flow $\{\hat{S}_t\}$ is a linear extension of the flow $\{S_t\}$ and, conversely, any linear extension of the flow $\{S_t\}$ is determined by a system of the form (1.5).

The role of the function h(x, t) will be played by $Y_x(t)$, which is the solution of the matrix system corresponding to (1.5) with initial conditions

$$Y(0) = E, \quad x(0) = x.$$

We say that two systems of differential equations

$$\text{(I)} \begin{cases} \dot{y} = A(x)\, y, \\ \dot{x} = \omega(x) \end{cases} \text{ and } \text{(II)} \begin{cases} \dot{z} = B(x)\, z, \\ \dot{x} = \omega(x) \end{cases}$$

are mutually reducible, if there is a function $C(x) \in \mathfrak{A}(X, \Gamma)$ such that the change of variables $z = C(x)\, y$ transforms the system (II) into the system (I).

PROPOSITION 1.2. Linear extensions are isomorphic if and only if the corresponding differential-equation systems are reducible.

We note that in this case the cocycle h(x, t) is uniquely generated with respect to the function A(x) from (1.5), where relation (1.3) is equivalent to

$$A(x) = C^{-1}(x) \cdot B(x) \cdot C(x) - C^{-1}(x)\dot{C}(x), \qquad (1.6)$$

where$C(x) \in \mathfrak{A}(X, \Gamma)$, and C(x) is its derivative in the direction of the vector $\omega(x)$. [Here A(x) and B(x) are the right sides of (I) and (II) respectively.]

2. THEOREM 2.1. Let the dynamic system (X, T_n) with time Z have no analytic characteristic functions. Then the cohomology of constant cocycles Λ and M, reducible to diagonal form, is equivalent to the similarity of the matrices of Λ and M.

Proof. We write diag(a, b, . . .), for a diagonal matrix with diagonal elements a, b, ... Λ is similar to $\mathrm{diag}(\lambda_1, \lambda_2, \ldots)$, and M is similar to $\mathrm{diag}(\mu_1, \mu_2, \ldots)$. Plainly, with no loss of generality, we may assume that Λ and M are themselves diagonal. We assume that there is $\varphi(x) \in \mathfrak{A}(X, \Gamma)$ such that $\varphi(Tx)\Lambda\varphi^{-1}(x) = M$. Let $\varphi(x) = \|\varphi^{ij}(x)\|$. The condition implies that there are i and j such that $\varphi^{ij}(x)$ is not constant. Then $\varphi^{ij}(Tx\lambda_j = \mu_i\varphi^{ij}(x)$, and $\varphi^{ij}(Tx) = (\mu_i/\lambda_j)\varphi^{ij}(x)$, i.e., $\varphi^{ij}(x)$ is a characteristic function, which contradicts the condition. This proves the theorem.

We note that, for $\Gamma = GL(k, C)$, the absence of any analytic characteristic functions is also necessary: If there is a characteristic function $\psi(x)$ with characteristic value e^{id}, then there are cohomological constant cocycles which are not adjoint as matrices; for example,

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \text{ and } M = \mathrm{diag}(\lambda_1 e^{i\alpha}, \lambda_2, \ldots, \lambda_k)$$

are cohomological: $\varphi(Tx)\Lambda\varphi^{-1}(x) = M$, where

$$\varphi(x) = \mathrm{diag}(\psi(x), 1, \ldots, 1).$$

3. Let T^m be an m-dimensional torus

$$T^m = \{x = (x_1, \ldots, x_m), \ x_i \in \mathbf{R}/2\pi\mathbf{Z}, \ i = 1, \ldots, m\},$$

let $G = \mathbf{Z}$ with the group translations of the torus

$$Tx = T^{(\alpha)}x = x + \alpha \pmod{2\pi}, \quad \alpha \in \mathbf{T}^m,$$

as generating diffeomorphism, and let Γ be a closed subgroup of $GL(d, \mathbf{C})$. We propose to study in $\mathfrak{A}(\mathbf{T}^m, \Gamma)$, the neighborhood of a constant cocycle reducible to diagonal form.

All functions (scalar-, vector-, or matrix-valued) which we consider are analytic on the torus. This means that every such function, considered as a periodic function on $\mathbf{R}^m$, can, for some $r > 0$, be continued analytically into the region $|\operatorname{Im} z_i| \le r$, $i = 1, \ldots, m$. We use the notation $\|f\|_r = \max_{|\operatorname{Im} z_i| \le r} |f(z)|$ (the concrete choice of norms for vectors and matrices is immaterial). In this section we prove

THEOREM 3.1. Let Λ be similar to $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, and, for $\nu \ne 0$ ($\nu = (\nu_1, \ldots, \nu_m)$), $\nu_i \in \mathbf{Z}$, $i = 1, \ldots, m$) let

$$|\lambda_i - \lambda_j e^{i(\nu\alpha)}| > C|\nu|^{-s}, \; C > 0, \; s > 0. \tag{3.1.1}$$

For any $r_0 > 0$, there exists $\varepsilon_0 > 0$ such that, if $\|f - \Lambda\|_{r_0} < \varepsilon_0$, for the cocycle $f(x) \in \mathfrak{A}(\mathbf{T}^m, \Gamma)$, then there is a matrix $P \in \Gamma$, differing only slightly from E, for which the cocycle $f(x) \cdot P$ is Γ-cohomological to the constant cocycle Λ.

The method of proof of Theorem 3.1 can be made to apply to the case of continuous time. We have

THEOREM 3.2. Consider the differential-equation system

$$\text{(III)} \quad \begin{cases} \dot{y} = Ay, \\ \dot{x} = \omega, \end{cases}$$

where the matrix $A \in A_\Gamma$ is similar to $\operatorname{diag}(a_1, \ldots, a_k)$ and, for $\nu \ne 0$, we have

$$|a_i - a_j - i(\nu, \omega)| > C|\nu|^{-s}, \; C > 0, \; s > 0. \tag{3.2.1}$$

For any $r_0 > 0$ there exists $\varepsilon_0 > 0$ such that, if $\|B(x) - A\|_{r_0} < \varepsilon_0$ for $B(x) \in \mathfrak{A}(\mathbf{T}^m, \Gamma)$, then there is a constant matrix $Q \in A_\Gamma$, for which the system

$$\begin{cases} \dot{y} = (B(x) + Q)y, \\ \dot{x} = \omega \end{cases}$$

is reducible to the system (III) by a transformation $C(x) \in \mathfrak{A}(\mathbf{T}^m, \Gamma)$.

The problem of the reducibility of systems of linear differential equations with periodic coefficients has been studied by many authors (see A. E. Gel'man [3, 4], L. Ya. Adrianova [5], I. N. Blinov [6], and O. B. Lykova [7]). Yu. A. Mitropol'skii and A. M. Samoilenko [8] used Newton's method (the method of accelerated convergence) in their investigation of the reducibility of systems with right-hand sides differing only slightly from a constant. The actual results obtained refer to that formulation of the problem corresponding, in our notation, to the case $\Gamma = GL(k, \mathbf{C})$, which, to the author's knowledge, was first considered by V. I. Arnol'd ([9], p. 181), although some of the results appeared before [9]. The authors usually formally consider reducibility in the classical sense ([2], p. 251). In this case it is a question of the mutual reducibility of one system $\dot{y} = A(t)y$, $t \in \mathbf{R}$, to another $\dot{z} = B(t)z$, so that in (I) and (II) we have families of systems

$$(\mathrm{I}_x)\; \dot{y} = A(S_t x)y, \qquad (\mathrm{II}_x)\; \dot{z} = B(S_t x)z,$$

depending on $x \in X$, and our definition requires not only the reducibility of each of the systems (I_x) to (II_x), but also imposes limitations on the dependence of the coefficients of the reducing transformation with respect to x. Reducibility in the sense under consideration is clearly stronger than classical reducibility of a system of the corresponding family (and irreducibility is weaker).

Yu. Mozer [11] considered the group aspect of the problem. He studied a more general problem with nonlinear terms. Our Theorem 3.2 for the case $\Gamma = GL(k, \mathbf{C})$ is a consequence of his Theorem 1 and also a consequence of Theorem 2 of [8].

Proof of Theorem 3.1. Our proof uses Newton's method in the form employed by Yu. Mozer in [10].

We introduce the operator

$$\mathfrak{F}\colon \mathfrak{A}(\mathbf{T}^m, \Gamma) \times \mathfrak{A}(\mathbf{T}^m, \Gamma) \to \mathfrak{A}(\mathbf{T}^m, \Gamma)$$

by means of the relation*

$$(\mathfrak{F}(g\, x),\quad \varphi(x)) = \varphi(T^{(\alpha)}x) \cdot g(x) \cdot \varphi^{-1}(x).$$

We must find $\varphi(x) \in \mathfrak{A}(T^m, \Gamma)$ and $P \in \Gamma$, such that

$$\mathfrak{F}\,(f(x) \cdot P,\ \varphi(x)) = \Lambda.$$

We seek $\varphi(x)$ in the form of a limit $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$, and $f(x) \cdot P$ in the form of a limit $f(x) \cdot P = \lim_{n\to\infty} f^{(n)}(x)$ where $f^{(n)}(x), \in \mathfrak{A}(T^m, \Gamma)$ and $f^{(n+1)} = f^{(n)} \cdot K_n$, with K_n a constant cocycle.

Suppose that $\varphi_1(x), \ldots, \varphi_n(x)$ have already been found [we assume that $\varphi_1(x) = E$]. We express $\varphi_{n+1}(x)$ in the form $v_n(x)\varphi_n(x)$. Then $\mathfrak{F}\,(\bar{f}_{n+1}\ (x),\ v_n(x)\ \varphi_n\ (x)) = \mathfrak{F}\ (\mathfrak{F}(f^{(n+1)},\ \varphi_n),\ v_n) = \mathfrak{F}\ (f_{n+1},\ v_n)$ where $f_{n+1} = \mathfrak{F}\ (f^{(n+1)},\ \varphi_n)$.

Determine $v_n(x)$ from the equation

$$\mathfrak{L}_{(\Lambda, E)}(\mathfrak{F}\,(\bar{f}_{n+1}, v_n)) = \Lambda, \tag{3.1}$$

where the operator $\mathfrak{L}_{(\Lambda, E)}$ denotes the selection of the linear terms in the formal expansion of $\mathfrak{F}$ in a Taylor series (a linearization) at the point (Λ, E), i.e.,

$$\mathfrak{L}_{(\Lambda, E)}\,(\mathfrak{F}\,(\bar{f}_{n+1}, v_n)) = \Lambda + \mathfrak{F}_f'\,(\Lambda,\ E)\,\hat{f}_{n+1} + \mathfrak{F}_\varphi'\,(\Lambda,\ E)\,\hat{v}_n,$$

where $\hat{v}_n(x)$ and $\Lambda^{-1}\hat{f}_{n+1}(x)$ are in $\mathfrak{A}(T^m, A_\Gamma)$, and $\exp \hat{v}_n = v_n$, $\Lambda(\exp \Lambda^{-1}\hat{f}_{n+1}) = \bar{f}_{n+1}$. Since

$$\mathfrak{F}_f'\,(\Lambda, E)\,\hat{f}_{n+1} = \lim_{t\to 0} \frac{1}{t}(\mathfrak{F}\,(\Lambda + t\hat{f}_{n+1}, E) - \mathfrak{F}\,(\Lambda, E)) = \hat{f}_{n+1},$$

we have

$$\mathfrak{L}_{(\Lambda, E)}\,(\mathfrak{F}\,(\bar{f}_{n+1}, v_n)) = \Lambda + \hat{f}_{n+1} + \mathfrak{F}_\varphi'\,(\Lambda, E)\,\hat{v}_n,\ \ \mathfrak{F}_\varphi'\,(\Lambda, E)\,\hat{v}_n$$
$$= \lim_{t\to 0}\frac{1}{t}\ (\mathfrak{F}(\Lambda,(E + t\hat{v}_n)\,(x)) - \mathfrak{F}\,(\Lambda, E)) = \hat{v}_n\,(Tx)\,\Lambda - \Lambda\hat{v}_n\,(x),$$

and Eq. (3.1) becomes

$$\hat{f}_{n+1}\,(x) + \hat{v}_n\,(Tx)\,\Lambda - \Lambda\hat{v}_n\,(x) = 0. \tag{3.2}$$

For the proof we demonstrate the induction lemmas 1_n and 2_n, assuming that these lemmas have already been proved for $k < n$ with ε_0 sufficiently small. To state the lemmas we introduce the following notation.

Let $f_n = \mathfrak{F}\ (f^{(n)},\ \varphi_n) = \mathfrak{F}\ (\bar{f}_n,\ v_{n-1})$; and $\Phi_n = f_n - \Lambda$. Define r_n by the relation† $r_{n+1} = r_n - \alpha_n$, where $\alpha_n = d\varepsilon_n^{1/4(m+s)}$, and $\varepsilon_n = \|\Phi_n\|_{r_n}$ (the value of the constant d will be determined in the process of the proof).

<u>LEMMA 1_n.</u> The constant cocycle $K_n \in \Gamma$ can be chosen so that

$$\int_{T^m} \hat{f}_{n+1}\,(x)\,dx = 0 \tag{3.3}$$

and $|K_n - E| \le l_1\varepsilon_n$.

<u>LEMMA 2_n.</u> If $\bar{f}_{n+1}$ satisfies (3.3), then Eq. (3.2) has a solution $\hat{v}_n\ (x) \in \mathfrak{A}\ (T^m, A_\Gamma)$ analytic in the region $\|\mathrm{Im}\ z_i\| < r_{n+1}$, where $\|\hat{v}_n\ (x)\|_{r_{n+1}} \le \varepsilon_n^{3/4}$ and $\|\Phi_{n+1}\ (x)\|_{r_{n+1}}\, l_2\varepsilon_n^{3/2}$.

<u>Proof of the Theorem and the Lemmas.</u> Lemma 1_n implies that K_n, and so $f^{(n+1)} = f^{(n)}K_n$, can be constructed and relation (3.3) will hold. Lemma 2_n is then used to find $\hat{v}_n(x)$, and this means that $v_n(x)$ and $\varphi_{n+1} = v_n(x)\,\varphi_n(x)$ have been determined. Repeating the same process with n replaced by n + 1, etc., we obtain infinite sequences $\{\varphi_n(x)\}$ and $\{K_n\}$. The sequence $\{\varphi_n(x)\}$ converges in $|\mathrm{Im}\ z_i| < r_\infty$, where

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*Clearly the mapping $\mathfrak{F}$ can be extended, by the same formula, to the mapping

$$\mathfrak{A}\,(T^m, GL\,(k, C)) \times \mathfrak{A}\,(T^m, GL\,(k, C)) \to \mathfrak{A}\,(T^m, GL\,(k, C)),$$

which we use below in the calculation of $\mathfrak{F}_f'$ and $\mathfrak{F}_\varphi'$.

†Constants denoted by small latin letters d, l, and c with various indices depend only on Λ, C, s, and m.

$r_\infty = \lim_{n\to\infty} r_n$ (if ε_0 is sufficiently small we have $r_\infty > 0$). In fact let $m > n$. Then

$$\|\varphi_m(x) - \varphi_n(x)\|_{r_\infty} \leqslant \|\varphi_m(x) - \varphi_n(x)\|_{r_m} = \left\|\prod_{i=1}^n v_i(x)\left(\prod_{j=n}^m v_j(x) - E\right)\right\|_{r_m}$$

$$\leqslant c' \left\|\prod_{i=1}^n \exp \hat{v}_i(x)\right\|_{r_m} \left\|\prod_{j=n+1}^m \exp v_j(x) - E\right\|_{r_m} \leqslant c'' \prod_{i=1}^n (1 + \varepsilon_i^{3/4}) \left[\prod_{j=n+1}^m (1 + \varepsilon_j^{3/4}) - 1\right].$$

Since, for sufficiently small ε_0, the product

$$\prod_{n=1}^\infty (1 + \varepsilon_n^{3/4})$$

is convergent, we have

$$\|\varphi_m(x) - \varphi_n(x)\|_{r_\infty} \underset{n,m\to\infty}{\longrightarrow} 0.$$

It is proved similarly that $f^{(n)}(x)$ converges to $f(x)\cdot P$ in the region $|\mathrm{Im}\, z_i| \leq r_\infty$, where $P = \Pi_{n=1}^\infty K_n$. Further, since $\|\Phi_n(x)\|_{r_n} \to 0$ and the operator $\mathfrak{F}$ is continuous, we have $\mathfrak{F}(f(x)\cdot P, \varphi(x)) = \Lambda$, and this proves the theorem.

Proof of Lemma 1_n. Consider the mapping $\psi^{(n)}\colon A_\Gamma \to \Lambda A_\Gamma$ and, for $P \in A_\Gamma$, let

$$\psi^{(n)}(P) = \int_{T^m} \Lambda \ln(\Lambda^{-1}\mathfrak{F}(f^{(n)}(x)\cdot \exp P, \varphi_n(x))\, dx.$$

For $1/2 < |\exp P| < 2$ and sufficiently small ε_0, we prove that

$$|\psi^{(n)}(P) - \Lambda\cdot P| < c_1\varepsilon_0^{3/4}. \tag{3.4}$$

For a fixed point $x \in T^m$ we have

$$\begin{aligned}&|\Lambda \ln(\Lambda^{-1}\mathfrak{F}(f^{(n)}(x)\cdot\exp P, \varphi_n(x)) - \Lambda\cdot P|\\ &\quad\leqslant c_2|\Lambda|\cdot|\ln(\Lambda^{-1}\mathfrak{F}(f^{(n)}(x)\exp P, \varphi_n(x)) - P|\\ &\quad\leqslant c_2|\Lambda|\cdot|\Lambda^{-1}\mathfrak{F}(f^{(n)}(x)\cdot\exp P, \varphi_n(x)) - \exp P|\cdot\beta(x)^{-1},\end{aligned}$$

where $\beta(x) \geq \min(|\exp P|, |\Lambda^{-1}\mathfrak{F}(f^{(n)}\exp P, \varphi_n(x))|)$. This inequality is obtained by applying the mean-value theorem to the matrix-valued function $\ln Z$.

Repeated use of the triangle inequality yields

$$\begin{aligned}&|\Lambda^{-1}\varphi_n(Tx)f^{(n)}(x)\ \exp P\cdot\varphi_n^{-1}(x) - \exp P| \leqslant c_3|\Lambda^{-1}|\\ &\times|\exp P|(|\varphi_n(Tx)|\cdot|f^{(n)}(x)|\cdot|\varphi_n^{-1}(x) - E| + |\varphi_n(Tx)|\\ &\times|f^{(n)}(x) - \Lambda| + |\varphi_n(Tx) - E|\cdot|\Lambda|) \leqslant c_4\varepsilon_0^{3/4}|\exp P|.\end{aligned}$$

The last inequality is a consequence of Lemmas 1_k and 2_k for $k < n$. For sufficiently small ε_0 and $|\exp P| > 1/2$, we have $\beta(x)^{-1} \leq 4$, and (3.4) is proved. But (3.4) implies that there exists P_n satisfying the inequality $|P_n| \leq c_5\varepsilon_0^{3/4}$ and such that $\psi^{(n)}(P_n) = 0$.

Since

$$\psi^{(n)}(P_n) - \psi^{(n)}(0) - (d\psi_0^{(n)})(P_n)| \leqslant c_6|P_n|^2$$

and

$$|\psi^{(n)}(0)| \leqslant c_7\|\Phi_n(x)\|_{r_n} = c_7\varepsilon_n,$$

we have

$$|(d\psi_0^{(n)})P_n| - c_6|P_n|^2 \leqslant c_7\varepsilon_n.$$

If ε_0 is sufficiently small, the operator $\psi^{(n)}$ together with its differential, differs only slightly from the operation of multiplication by Λ in the $c_5\varepsilon_0^{3/4}$-neighborhood of the origin. Hence $|P_n|\cdot(c_8 - c_6|P_n|) \leq c_7\varepsilon_n$ for such ε_0.

If ε_0 is such that $c_6|P_n| \leq (1/2)c_8$, then $|P_n| \leq (2c_7/c_8)\varepsilon_n = c_9\varepsilon_n$, i.e., for $K_n = \exp P_n$ relation (3.3) holds and $|K_n - E| \leq l_1\varepsilon_n$.

LEMMA A. Let V be a finite-dimensional vector space over $\mathbf{R}$, let $M \in GL(V)$ be reduced over $\mathbf{C}$ to the form $\mathrm{diag}(\mu_1, \mu_2, \ldots)$, and for $\nu \neq 0$

$$|\mu_j e^{i(\nu\alpha)} - 1| > b|\nu|^{-s}, \qquad b > 0, s > 0. \tag{A.1}$$

Let the function $\varphi(x) \in \mathfrak{A}(T^m, V)$ be analytic for $|\mathrm{Im}\, z_i| \leq r$ and let

$$\int_{T_m} \varphi(x)\,dx = 0. \tag{A.2}$$

Then the equation

$$v(x) = \varphi(x) + Mv(T^x x) \tag{A.3}$$

has a solution $v(x) \in \mathfrak{A}(T^m, V)$ analytic for $|\mathrm{Im}\, z_i| \leq r$, and $0 < \alpha < 1$

$$\|v\|_{r-\alpha} \leqslant a \frac{\|\varphi\|_r}{\alpha^{m+s}},$$

where a depends only on M, b, s, and m.

Proof. We first find and estimate the magnitude of the solution w(x) of Eq. (A.3) taking a value in the complex vector hull V^C of the space V. In some basis of the space V^C we have $M = \mathrm{diag}(\mu_1, \mu_2, \ldots)$. In this basis, the coordinates $w_j(x)$ and $\varphi_j(x)$ of the vector-valued functions w(x) and $\varphi(x)$ satisfy a system of equations

$$w_j(x) = \varphi_j(x) + \mu_j w_j(Tx),$$

equivalent to (A.3). Hence the Fourier coefficients $w_{j,\nu}$ and $\varphi_{j,\nu}$ of these functions are related by the equation $w_{j,\nu} = \frac{\varphi_{j,\nu}}{1-\mu_j e^{i(\nu\alpha)}}$ for $\nu \neq 0$, while $w_{j,0}$ is arbitrary [note that (A.2) implies that $\varphi_{j,0} = 0$]; we set $w_{j,0} = 0$. We now find a bound for w. Cauchy's formula yields $\|\varphi_{j,\nu}\| \leq e^{-|\nu|r}\|\varphi_j\|_r$ (we use the integral torus $\mathrm{Im}\, z_i = -r\,\mathrm{sign}\,\nu_i$). The formula for $w_{j,\nu}$ implies that

$$|w_{j,\nu}| < a_1 e^{-|\nu|r}|\nu|^s\|\varphi_j\|_r$$

and so

$$\|w\|_{r-\alpha} \leqslant a_1\|\varphi\|_r \sum_{\nu\neq 0}|\nu|^s e^{-|\nu|r}\cdot e^{|\nu|(r-\alpha)}$$

$$\leqslant a_2\|\varphi\|_r\sum_{k=0}^{\infty} k^{s+m-1}e^{-k\alpha} \leqslant a_2\|\varphi\|_r(s+m-1)! \times \sum_{k=0}^{\infty}\frac{(k+1)\ldots(k+s+m-1)}{(s+m-1)!}e^{-k\alpha}$$

$$= a_3\|\varphi\|_r\sum_{k=0}^{\infty}\binom{-m-s}{k}(e^{-\alpha})^k = a_3\|\varphi\|_r(1-e^{-\alpha})^{-m-s} \leqslant a\|\varphi\|_r\alpha^{-m-s}.$$

We have thus obtained a bound for $\|w\|_{r-\alpha}$. To prove the lemma it only remains to set $v(x) = (1/2)\, w(x) + (1/2)\, w(\bar{x})$.

Proof of Lemma 2_n. The fact that Eq. (3.2) has a solution $\hat{v}_n(x) \in \mathfrak{A}(T^m, A_\Gamma)$, analytic in the region $|\mathrm{Im}\, z_i| \leq r_{n+1}$, follows directly from Lemma A. In fact let $V = A_\Gamma$, $M = \mathrm{Ad}\,\Lambda|A_\Gamma$: $v \to \Lambda^{-1}v\Lambda$, $\varphi(x) = \Lambda^{-1}f_{n+1}(x)$. This converts Eq. (3.2) into Eq. (A.3).

Since the matrix Λ is similar to a diagonal matrix, M is reduced over C to diagonal form. The characteristic values of M form a subset of the set $(\ldots, \lambda_i/\lambda_j, \ldots, \ldots, \bar{\lambda}_i/\bar{\lambda}_j, \ldots)$ which, if it contains λ_i/λ_j, also contains the conjugate number. Condition (A.1) holds because (3.1.1) is satisfied not only for any pair λ_i, λ_j, but also for the pair of complex conjugate numbers; condition (A.2) follows from (3.3).

We now estimate $\|\hat{f}_{n+1}\|_{r_n}$:

$$\hat{f}_{n+1} = \Lambda\ln(\Lambda^{-1}\bar{f}_{n+1}) = \Lambda\ln(\Lambda^{-1}(f_n + \bar{\varepsilon}_n)) = \Lambda\ln(E + \Lambda^{-1}(\Phi_n + \varepsilon_n)),$$

where $\bar{\varepsilon}_n = \bar{f}_{n+1} - f_n = \mathfrak{F}(f^{(n)}(K_n - E), \varphi_n)$. Lemma 1_n implies $\|\varepsilon_n\|_{r_n} \leq c_{10}\varepsilon_n$ and $\|\Phi_n\|_{r_n} = \varepsilon_n$. Hence

$$\|\hat{f}_{n+1}\|_{r_n} \leqslant c_{11}\varepsilon_n. \tag{3.5}$$

Applying (3.5) and Lemma A, we obtain

$$\|\hat{v}_n\|_{r_n-\alpha_n} \leqslant a\frac{\|\hat{f}_{n+1}\|_{r_n}}{\alpha_n^{m+s}} \leqslant c_{12}\frac{\varepsilon_n}{\alpha_n^{m+s}}. \tag{3.6}$$

If $\alpha_n^{m+s} \geq c_{12}^{-1}\varepsilon_n^{1/4}$, i.e., $\alpha_n \geq c_{12}^{-(m+s)-1}\varepsilon_n^{1/4(m+s)}$, then (3.6) implies that $\|\hat{v}_n\|_{r_n-\alpha_n} \leq \varepsilon_n^{3/4}$. Setting $d = c_{12}^{-(m+s)-1}$ in the definition of α_n, we obtain an estimate of $\|\hat{v}_n\|_{r_{n+1}}$. It remains to estimate $\|\Phi_{n+1}\|_{r_{n+1}}$:

$$\Phi_{n+1}(x) = \mathfrak{F}(\bar{f}_{n+1}, v_n) - \Lambda,$$

but

$$\Lambda = \mathfrak{L}_{(\Lambda,E)}(\mathfrak{F}(\bar{f}_{n+1}, v_n)).$$

It follows that we must estimate*

$$\|\mathfrak{F}(\bar{f}_{n+1}(x),\ v_n(x)) - \mathfrak{L}_{(\Lambda,E)}(F(\bar{f}_{n+1}(x),\ v_n(x))\ \|_{r_{n+1}}. \tag{3.7}$$

Since $(\bar{f}_{n+1}, v_n)$ belongs, for all n, to a fixed neighborhood (Λ, E) in $\mathfrak{A}(T^m, \Gamma) \times \mathfrak{A}(T^m, \Gamma)$, in which the operator $\mathfrak{F}$ is infinitely differentiable, the expression (3.7) does not exceed

$$c_{13}(\|\bar{f}_{n+1} - \Lambda\|_{r_{n+1}}^2 + \|v_n - E\|_{r_{n+1}}^2) \leqslant c_{13}(\|f_n - \Lambda\|_{r_{n+1}} + \|\mathfrak{F}(f^{(n)}(K_n - E), \varphi_n)\|_{r_{n+1}})^2 + c_{14}\|\hat{v}_n\|_{r_{n+1}}^2 \leqslant l_2\varepsilon_n^{3/2}.$$

In the last inequalities we have used Lemma 1_n, the bound for $\|\hat{v}_n\|_{r_{n+1}}$, obtained above, and the boundedness of the operator $\mathfrak{F}$.

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*The expression whose norm is taken belongs to $\mathfrak{A}(T^m, GL(k, C))$.