

RIGIDITY OF TOPOLOGICAL ENTROPY OF BOUNDARY MAPS ASSOCIATED TO FUCHSIAN GROUPS

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Dedicated to the memory of Anatole Katok

ABSTRACT. Given a closed, orientable surface of constant negative curvature and genus $g \geq 2$, we study a family of generalized Bowen–Series boundary maps and prove the following rigidity result: in this family the topological entropy is constant and depends only on the genus of the surface. We give an explicit formula for this entropy and show that the value of the topological entropy also stays constant in the Teichmüller space of the surface. The proofs use conjugation to maps of constant slope.

1. INTRODUCTION

The notion of topological entropy was introduced by Adler, Konheim, and McAndrew in [5]. Their definition used covers and applied to compact Hausdorff spaces; Dinaburg [11] and Bowen [8] gave definitions involving distance functions and separated sets, which are often more suitable for calculations. While these formulations of topological entropy were originally intended for continuous maps acting on compact spaces, Bowen’s definition can actually be applied to piecewise continuous, piecewise monotone maps on an interval, as explained in [19]. The theory naturally extends to maps of the circle, where piecewise monotonicity is understood to mean local monotonicity or, equivalently, having a piecewise monotone lift to \mathbb{R} .

In [21], following his seminal work [20] on Markov maps, Parry showed that a piecewise monotone, (strongly) transitive interval map with positive topological entropy is conjugate to a constant slope map. In [17], Milnor and Thurston used kneading theory to prove a semi-conjugacy result for continuous, piecewise monotone, but not necessarily transitive, interval maps. In [7], following [6], Alsedà and Misiurewicz give a simpler proof that also generalizes to piecewise continuous, piecewise monotone interval maps.

In this paper we apply the results of [21, 7] to a multi-parameter family of piecewise continuous, piecewise monotone maps of the circle, the so-called “boundary maps” for surfaces of constant negative curvature, as in [15]. Some particular maps in this family—including those considered by Bowen and Series [9] and further studied by Adler and Flatto [4]—are Markov, and the topological entropy can be calculated as the logarithm of the maximal eigenvalue of a transition matrix [22, Theorem 7.13] in these cases. However, not all maps in our family admit a Markov partition, and yet we prove the following rigidity result: in this family, the topological entropy is constant and depends only on the genus of the surface. Therefore, the topological entropy in these non-Markov cases is the

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same logarithmic expression. We also show that topological entropy stays constant in the Teichmüller space of the surface.

Let Γ be a finitely generated cocompact Fuchsian group of the first kind acting freely on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with hyperbolic metric $\frac{2|dz|}{1-|z|^2}$ such that $S = \Gamma \backslash \mathbb{D}$ is a surface of genus $g \geq 2$.

A classical (Ford) fundamental domain for Γ is a $4g$ -sided regular polygon centered at the origin. In [4], Adler and Flatto used another fundamental domain—an $(8g - 4)$ -sided polygon \mathcal{F} —that was much more convenient for their purposes. Its sides are geodesic segments which satisfy the *extension condition*: the geodesic extensions of these segments never intersect the interior of the tiling sets $\gamma\mathcal{F}$, $\gamma \in \Gamma$.

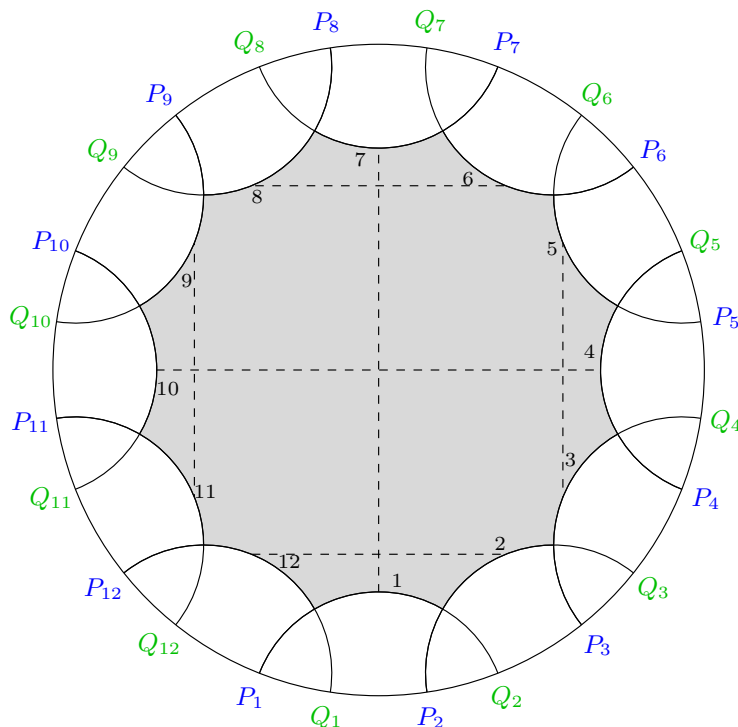


FIGURE 1. Fundamental polygon \mathcal{F} for genus $g = 2$.

We denote the endpoints of the oriented infinite geodesic that extends side k to the circle at infinity $\partial\mathbb{D}$ by P_k and Q_{k+1} , where $1 \leq k \leq 8g - 4$ is considered mod $8g - 4$ throughout this paper (see Figure 1). The counter-clockwise order of endpoints on $\partial\mathbb{D}$ is the following:

$$P_1, Q_1, P_2, Q_2, \dots, Q_{8g-4}.$$

The identification of the sides of \mathcal{F} is given by the side pairing rule

$$\sigma(k) := \begin{cases} 4g - k \bmod (8g - 4) & \text{if } k \text{ is odd} \\ 2 - k \bmod (8g - 4) & \text{if } k \text{ is even.} \end{cases}$$

The generators T_k of Γ associated to this fundamental domain are Möbius transformations satisfying the following properties: denoting $\rho(k) = \sigma(k) + 1$ and with V_k as the vertex

of \mathcal{F} where sides $k-1$ and k meet,

$$T_k(V_k) = V_{\rho(k)}, \quad T_{\sigma(k)}T_k = \text{Id}, \quad T_{\rho^3(k)}T_{\rho^2(k)}T_{\rho(k)}T_k = \text{Id}.$$

Remark. As functions on $\overline{\mathbb{D}} \subset \mathbb{C}$, the generators T_k are Möbius transformations, but restricted to the boundary \mathbb{S} they are real functions of the arguments (but *not* fractional linear transformations of the arguments). To simplify notation we will use “ T_k ” in both situations: $T_k(z)$ with $z \in \partial\mathbb{D}$ for complex (multiplicative) notation and $T_k(x) := \arg(T_k(e^{ix}))$ with $x \in \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ for real (additive) notation. See the left of Figure 5 for a plot of $y = T_k(x)$ with $x, y \in [-\pi, \pi]$.

Notice that in general the polygon \mathcal{F} need not be regular. In fact, one of the definitions of Teichmüller space, used in [3], is the space of all marked canonical hyperbolic $(8g-4)$ -gons in the unit disk \mathbb{D} (up to an isometry of \mathbb{D}) such that side k and side $\sigma(k)$ have equal length and the internal angles at vertices V_k and $V_{\sigma(k)+1}$ sum to π . (The topology on the space of polygons is as follows: $\mathcal{P}_n \rightarrow \mathcal{P}$ if and only if the lengths of all sides converge and the measures of all angles converge.)

If \mathcal{F} is regular, it is the Ford fundamental domain, i.e., the geodesic from P_k to Q_{k+1} (which we denote as just P_kQ_{k+1}) is the isometric circle for T_k , and $T_k(P_kQ_{k+1}) = Q_{\sigma(k)+1}P_{\sigma(k)}$ is the isometric circle for $T_{\sigma(k)}$ so that the inside of the former isometric circle is mapped to the outside of the latter, and all internal angles of \mathcal{F} are equal to $\pi/2$. See [2] for more details and Section 2 for additional properties of the generators T_k .

The object of our study is the family of generalized Bowen–Series boundary maps studied in [15, 2, 1, 3] and defined by the formula

$$f_{\bar{A}}(x) = T_k(x) \quad \text{if } x \in [A_k, A_{k+1}), \quad (1)$$

where

$$\bar{A} = \{A_1, A_2, \dots, A_{8g-4}\} \quad \text{and} \quad A_k \in [P_k, Q_k].$$

When all $A_k = P_k$ we denote the map by $f_{\bar{P}}$ (this map is what Adler and Flatto [4] refer to as “the Bowen–Series boundary map,” although Bowen and Series’ construction [9] used $4g$ -sided polygons). In [3] we analyzed how the measure-theoretic entropy with respect to the smooth invariant measure of maps in this family changes in the Teichmüller space of S and proved a flexibility result: the entropy $h_{\mu_{\bar{A}}}$ takes all values between 0 and a maximum that is achieved on the surface that admits a regular $(8g-4)$ -sided fundamental polygon. In contrast, the main result of this paper is rigidity of topological entropy: its value depends only on the genus of the surface, remains constant in the Teichmüller space $\mathcal{T}(S)$, and does not depend on the (multi-)parameter \bar{A} .

Theorem 1 (Main Theorem). *Let Γ be a cocompact torsion free Fuchsian group such that $S = \Gamma \backslash \mathbb{D}$ is a surface of genus $g \geq 2$. For any $\bar{A} = \{A_1, \dots, A_{8g-4}\}$ with $A_k \in [P_k, Q_k]$, the map $f_{\bar{A}} : \mathbb{S} \rightarrow \mathbb{S}$ has topological entropy $h_{\text{top}}(f_{\bar{A}}) = \log(4g - 3 + \sqrt{(4g - 3)^2 - 1})$.¹*

Remark. Most previous results on boundary maps $f_{\bar{A}} : \mathbb{S} \rightarrow \mathbb{S}$ require the parameters \bar{A} to be in a smaller class: [4] uses only $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$, [1] focuses on extremal parameters and their duals, and [2, 3] require that the parameters have the short cycle property. In this paper Theorem 1 applies to *all* parameters \bar{A} with $A_k \in [P_k, Q_k]$. Although our result shows that all maps $f_{\bar{A}}$ have the same topological entropy for a given genus g , they are not

¹The quantity $\log(4g - 3 + \sqrt{(4g - 3)^2 - 1})$ can also be expressed as $\text{arccosh}(4g - 3)$, but logarithm expressions are more common for entropies in general and especially for shifts, so we use the longer expression.

necessarily topologically conjugate, since, according to [15], the combinatorial structure of the orbits associated to the discontinuity points A_k could differ.

The paper is organized as follows. In Sections 2–4 we restrict ourselves to the case when Γ admits a regular $(8g - 4)$ -sided fundamental polygon. In Section 2 we give the formulas for generators T_k as functions on $\overline{\mathbb{D}} \subset \mathbb{C}$ (Proposition 2) and prove two additional symmetric properties of generators as functions on $\partial\mathbb{D}$. In Section 3 we compute the maximal eigenvalue of the transition matrices for all “extremal” parameters and hence the topological entropy for these Markov cases. In Section 4 we prove some symmetric properties of the map $\psi_{\overline{P}}$ conjugating $f_{\overline{P}}$ to a constant slope map. We conclude that $\psi_{\overline{P}}$ actually conjugates all $f_{\overline{A}}$ to constant slope maps, and in Section 5 we use this to prove Theorem 1, first for Γ admitting regular $(8g - 4)$ -sided fundamental polygons and then in the fully general case. A technical result stated and used in Section 4 is proved in Appendix A.

2. ADDITIONAL PROPERTIES OF GENERATORS

Proposition 2. *If the $(8g - 4)$ -sided fundamental polygon \mathcal{F} is regular, then the generators T_k of the group Γ are given as functions on $\overline{\mathbb{D}} \subset \mathbb{C}$ by*

$$T_k(z) = (-1)^{k+1} \frac{e^{i(1-k)\alpha} z + i\sqrt{\cos \alpha}}{(-i\sqrt{\cos \alpha})z + e^{i(k-1)\alpha}}, \quad \text{where } \alpha := \frac{2\pi}{8g - 4}. \quad (2)$$

Proof. We derive a formula for $T_k(z)$ based on some geometric considerations also presented in [14, Section 4.3] and [15, Appendix].

Let $T_k(z) = (az + \overline{c})/(cz + \overline{a})$, where $|a|^2 - |c|^2 = 1$. The isometric circle $P_k Q_{k+1}$ of T_k , also denoted $I(T_k)$, is given by the equation $|cz + \overline{a}| = 1$, has center O_k located at $-\overline{a}/c$ with $\arg(-\overline{a}/c) = -\frac{\pi}{2} + (k-1)\alpha$ and radius $R = 1/|c|$.

Let $d = |a|/|c|$ be the distance from the origin O to the center O_k of $I(T_k)$. The following formula for R was obtained in [15, Appendix]:

$$R = \frac{\sqrt{2} \sin(\alpha/2)}{\sqrt{\cos \alpha}} = \frac{\sqrt{1 - \cos \alpha}}{\sqrt{\cos \alpha}}.$$

This implies that

$$|c| = \frac{1}{R} = \frac{\sqrt{\cos \alpha}}{\sqrt{1 - \cos \alpha}} \quad \text{and} \quad |a| = d|c| = \frac{1}{\sqrt{1 - \cos \alpha}}.$$

The isometric circle $I(T_k)$ is mapped by T_k to the isometric circle of $T_k^{-1} = T_{\sigma(k)}$ with center located at a/c . We analyze two cases:

- If k is odd and $k < 4g$, then $T_k^{-1} = T_{4g-k}$. The oriented angle $\angle O_k O O_{4g-k} = (4g - 2k)\alpha = \pi + (2 - 2k)\alpha$, so $a/c = e^{i(\pi + (2-2k)\alpha)}(-\overline{a}/c)$, which implies that $\arg(a) = \pi + (1 - k)\alpha$. From $\arg(-\overline{a}/c) = -\frac{\pi}{2} + (k-1)\alpha$, we get $\arg(c) = \pi/2$. Thus,

$$a = \frac{-e^{i(1-k)\alpha}}{\sqrt{1 - \cos \alpha}} \quad \text{and} \quad c = \frac{i\sqrt{\cos(\alpha)}}{\sqrt{1 - \cos \alpha}},$$

and, after simplifying the common term $-\sqrt{1 - \cos \alpha}$, we get relation (2).

- If k is odd and $k > 4g$, then $\sigma(k) = 4g - k \pmod{8g - 4} = 12g - k - 4$, so $T_k^{-1} = T_{12g-k-4}$. The oriented angle $\angle O_k O O_{12g-k-4} = (12g - 2k - 4)\alpha = 3\pi + (2 - 2k)\alpha$,

so $a/c = e^{i(3\pi+(2-2k)\alpha)}(-\bar{a}/c)$, which implies that $\arg(a) = 2\pi + (1-k)\alpha$. From $\arg(-\bar{a}/c) = -\frac{\pi}{2} + (k-1)\alpha$, we get $\arg(c) = -\pi/2$. Thus

$$a = \frac{e^{i(1-k)\alpha}}{\sqrt{1-\cos\alpha}} \quad \text{and} \quad c = -\frac{i\sqrt{\cos(\alpha)}}{\sqrt{1-\cos\alpha}},$$

and, after simplifying the common term $\sqrt{1-\cos\alpha}$, we get relation (2).

The case when k is even can be treated similarly. \square

Proposition 3. For all $x \in \mathbb{S}$, $T_k(x + \alpha) = T_{k-1}(x) + (4g-3)\alpha$.

Proof. Let $\beta = (4g-3)\alpha$. Then in complex (multiplicative) notation, the claim is

$$T_k(e^{i\alpha}z) = e^{i\beta}T_{k-1}(z).$$

Note that $e^{i\beta} = e^{i(\frac{4g-3}{8g-4})2\pi} = -e^{-i\alpha}$. Then, using (2),

$$\begin{aligned} T_k(e^{i\alpha}z) &= (-1)^{k+1} \frac{(e^{i(1-k)\alpha})(e^{i\alpha}z) + i\sqrt{\cos\alpha}}{(-i\sqrt{\cos\alpha})(e^{i\alpha}z) + e^{i(k-1)\alpha}} \cdot \frac{-e^{i\beta}}{e^{-i\alpha}} \\ &= (-1)^{k+1}(-e^{i\beta}) \frac{(e^{i(1-k)\alpha}e^{i\alpha})z + i\sqrt{\cos\alpha}}{(-i\sqrt{\cos\alpha}) + (e^{i(k-1)\alpha}e^{-i\alpha})} \\ &= (-1)^k(e^{i\beta}) \frac{(e^{i(-k)\alpha})z + i\sqrt{\cos\alpha}}{(-i\sqrt{\cos\alpha})z + e^{i(k-2)\alpha}} \\ &= e^{i\beta} \cdot T_{k-1}(z). \end{aligned} \quad \square$$

A function $r(x)$ is said to be *centrally symmetric around c* if $r(c+x) + r(c-x)$ is constant for all $x \in \mathbb{S}$ (this constant will be $2r(c)$). This property is equivalent to saying that the graph of a lift of r to \mathbb{R} restricted to any rectangle $[c-\delta, c+\delta] \times [r(c-\delta), r(c+\delta)]$ is symmetric under rotation by π around the center of that rectangle. If the circle is modeled as $\partial\mathbb{D} \subset \mathbb{C}$, then the analogous property is that $r(cz) \cdot r(c/z)$ is constant for all $z \in \partial\mathbb{D}$.

Denote by C_k the midpoint of the segment $[P_k, Q_{k+1}]$. The next proposition asserts that the graph of T_k is centrally symmetric around C_k .

Proposition 4. For all $x \in \mathbb{S}$, $T_k(C_k + x) + T_k(C_k - x) = -2C_k$.

Proof. In complex (multiplicative) notation, the claim is that $T_k(C_k z) \cdot T_k(C_k/z) = 1/C_k^2$ for all $z \in \partial\mathbb{D}$. Since $C_k \in \partial\mathbb{D}$ is the midpoint of the counter-clockwise arc of the circle from P_k to Q_{k+1} , it satisfies $C_k^2 = P_k \cdot Q_{k+1}$ as complex numbers. The isometric circle of $T_k(z)$ connects P_k to Q_{k+1} and consists of those $z \in \mathbb{D}$ for which $|T_k'(z)| = 1$, so $P_k, Q_{k+1} \in \partial\mathbb{D}$ are the complex numbers z , $|z| = 1$, satisfying

$$\left| \frac{-i\sqrt{\cos\alpha}}{\sqrt{1-\cos\alpha}} z + \frac{e^{i(k-1)\alpha}}{\sqrt{1-\cos\alpha}} \right| = 1.$$

The solutions to this equation are $-e^{i(k-1)\alpha}w$ and $e^{i(k-1)\alpha}\bar{w}$, where $w = \sqrt{1-\cos\alpha} + i\sqrt{\cos\alpha}$. The product of the two solutions is

$$\begin{aligned} P_k \cdot Q_{k+1} &= e^{i(k-1)\alpha}w \cdot e^{i(k-1)\alpha}(-\bar{w}) = (e^{i(k-1)\alpha})^2(-|w|^2) \\ &= e^{-i\pi}e^{i(2k-2)\alpha} = e^{i((2k-2)\alpha - (4g-2)\alpha)} = (e^{i\alpha})^{2k-4g}. \end{aligned}$$

Since C_k is the midpoint of the smaller of the two arcs comprising $\partial\mathbb{D}\setminus\{P_k, Q_{k+1}\}$, we have that

$$C_k = (e^{i\alpha})^{k-2g}. \quad (3)$$

To prove Proposition 4, we use the alternative form

$$C_k = e^{i(k-2g)\alpha} = e^{-i(2g-1)\alpha} e^{i(k-1)\alpha} = e^{-i(\pi/2)} e^{i(k-1)\alpha} = -ie^{i(k-1)\alpha}$$

to compute

$$\begin{aligned} T_k(C_k z) &= (-1)^{k+1} \frac{e^{i(1-k)\alpha}(-ie^{i(k-1)\alpha}z) + i\sqrt{\cos\alpha}}{(-i\sqrt{\cos\alpha})(-ie^{i(k-1)\alpha}z) + e^{i(k-1)\alpha}} \\ &= \frac{(-1)^{k+1} \cdot i}{e^{i(k-1)\alpha}} \cdot \frac{-z + \sqrt{\cos\alpha}}{(-\sqrt{\cos\alpha})z + 1} = \frac{(-1)^k}{C_k} \cdot \frac{z - \sqrt{\cos\alpha}}{(-\sqrt{\cos\alpha})z + 1} \end{aligned}$$

and then

$$T_k(C_k z) \cdot T_k(C_k/z) = \left(\frac{(-1)^k}{C_k} \frac{z - \sqrt{\cos\alpha}}{(-\sqrt{\cos\alpha})z + 1} \right) \left(\frac{(-1)^k}{C_k} \frac{z^{-1} - \sqrt{\cos\alpha}}{(-\sqrt{\cos\alpha})z^{-1} + 1} \right) = \frac{1}{C_k^2}$$

as claimed. \square

Corollary 5. For all $x \in \mathbb{S}$, $T_k(-x) = -T_{4g-k}(x)$.

Proof. Applying Proposition 3 repeatedly gives

$$T_k(x) = T_{k+n}(x + n\alpha) - n\beta$$

for any $n \in \mathbb{Z}$, where $\beta = (4g - 3)\alpha$. Using $n = 2g - k$ we have

$$\begin{aligned} T_k(-x) &= T_{2g}(-x + (2g - k)\alpha) - (2g - k)\beta \\ &= -T_{2g}(x - (2g - k)\alpha) - (2g - k)\beta \quad \text{by Proposition 4} \\ &= -T_{2g-(k-2g)}(x) + (2g - k)\beta - (2g - k)\beta = -T_{4g-k}(x). \quad \square \end{aligned}$$

3. MARKOV MATRICES FOR EXTREMAL PARAMETERS

Definition 6. A parameter choice $\bar{A} = \{A_1, \dots, A_{8g-4}\}$ with $A_k \in [P_k, Q_k]$ is called *extremal* if for each k either $A_k = P_k$ or $A_k = Q_k$.

Extremal parameters were first introduced in [1], in which several results of [15, 16, 2] for parameters with ‘‘short cycles’’ were extended to extremal parameters. Note that the classical cases $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$ are examples of extremal parameter choices.

Since for all $k = 1, \dots, 8g - 4$, $f_{\bar{A}}(P_k)$ and $f_{\bar{A}}(Q_k)$ belong to the set $\bar{P} \cup \bar{Q}$ (see [15, Proposition 2.2], originally [4, Theorem 3.4]), the partition of \mathbb{S} into intervals I_1, \dots, I_{16g-8} given by

$$I_{2k-1} := [P_k, Q_k], \quad I_{2k} := [Q_k, P_{k+1}], \quad k = 1, \dots, 8g - 4,$$

is a Markov partition for $f_{\bar{A}}$ for every extremal \bar{A} . Each extremal \bar{A} has a transition matrix $M_{\bar{A}} = (m_{i,j})$ with

$$m_{i,j} := \begin{cases} 1 & \text{if } f_{\bar{A}}(I_i) \supset I_j \\ 0 & \text{otherwise,} \end{cases}$$

and an infinite sequence $(\omega_0, \omega_1, \dots)$ or finite sequence $(\omega_0, \omega_1, \dots, \omega_n)$ over the alphabet $\{1, \dots, 16g - 8\}$ is called \bar{A} -admissible if all $m_{\omega_i, \omega_{i+1}} = 1$ for all $i \geq 0$ (and $i < n$ for finite).

For each extremal parameter \bar{A} , we can define the shift space²

$$X_{\bar{A}} = \overline{\{\omega \in \{1, \dots, 16g - 8\}^{\mathbb{N}} : \omega \text{ is admissible}\}}$$

on which we have the left-shift $\sigma_{\bar{A}} : X_{\bar{A}} \rightarrow X_{\bar{A}}$ and the essentially bijective *coding map* $\phi_{\bar{A}} : X_{\bar{A}} \rightarrow \mathbb{S}$ given by

$$\phi_{\bar{A}}(\omega) = \bigcap_{i=0}^{\infty} f_{\bar{A}}^{-i}(I_{\omega_i}) \quad (4)$$

so that the following diagram is commutative:

$$\begin{array}{ccc} X_{\bar{A}} & \xrightarrow{\sigma_{\bar{A}}} & X_{\bar{A}} \\ \phi_{\bar{A}} \downarrow & & \downarrow \phi_{\bar{A}} \\ \mathbb{S} & \xrightarrow{f_{\bar{A}}} & \mathbb{S} \end{array}.$$

In the case where $f_{\bar{A}}$ is Markov, the system $(X_{\bar{A}}, \sigma_{\bar{A}})$ is a topological Markov chain.

The following formulas use [2, Proposition 3.1 and Lemma 3.2]. For odd indices $2k - 1$, depending on whether $A_k = P_k$ or $A_k = Q_k$ we have, respectively, either

$$f_{\bar{A}}(I_{2k-1}) = T_k(I_{2k-1}) = [Q_{\sigma(k)+1}, Q_{\sigma(k)+2}] = I_{2\sigma(k)+2} \cup I_{2\sigma(k)+3} \quad (5)$$

or

$$\begin{aligned} f_{\bar{A}}(I_{2k-1}) &= T_{k-1}(I_{2k-1}) = [P_{\sigma(k-1)-1}, P_{\sigma(k-1)}] = [P_{\sigma(k)+4g-2}, P_{\sigma(k)+4g-1}] \\ &= I_{2\sigma(k)+8g-5} \cup I_{2\sigma(k)+8g-4}. \end{aligned} \quad (6)$$

In either case, $f_{\bar{A}}(I_{2k-1})$ is the union of two consecutive Markov partition elements. For even indices $2k$, we know that for any extremal \bar{A} the image

$$\begin{aligned} f_{\bar{A}}(I_{2k}) &= T_k(I_{2k}) = [Q_{\sigma(k)+2}, P_{\sigma(k)-1}] \\ &= I_{2\sigma(k)+4} \cup I_{2\sigma(k)+5} \cup \dots \cup I_{2\sigma(k)-4} = \mathbb{S} \setminus \bigcup_{\ell=2\sigma(k)-3}^{2\sigma(k)+3} I_{\ell} \end{aligned} \quad (7)$$

is the union of $(16g - 8) - 7 = 16g - 15$ consecutive intervals on the circle. Recall that the indices are mod $16g - 8$; for example, with $g = 2$ and $k = 1$ we get

$$f_{\bar{A}}(I_2) = I_{18} \cup I_{19} \cup \dots \cup I_{10} = I_{18} \cup I_{19} \cup \dots \cup I_{24} \cup I_1 \cup \dots \cup I_{10}.$$

The matrices $M_{\bar{P}}$ and $M_{\bar{Q}}$ for genus 2 are shown in Figure 2.

Proposition 7. *For any extremal \bar{A} , the maximal eigenvalue of $M_{\bar{A}}$ is*

$$\lambda = 4g - 3 + \sqrt{(4g - 3)^2 - 1}.$$

Proof. Gelfand's Formula [13] states that $\lim_{n \rightarrow \infty} \|M_{\bar{A}}^n\|^{1/n}$ equals the maximal eigenvalue of $M_{\bar{A}}$, where $\|\cdot\|$ is any matrix norm. The “entrywise norm” $|M_{\bar{A}}^n|$ given by the sum of (absolute values of) all entries in $M_{\bar{A}}^n$ counts the total number of admissible sequences of length $n + 1$, that is,

$$|M_{\bar{A}}^n| = \#\{(\omega_0, \omega_1, \dots, \omega_n) : m_{\omega_i, \omega_{i+1}} = 1 \text{ for } i = 0, \dots, n - 1\}.$$

²In [2] the notation $X_{\bar{A}}$ is used for a space of sofic sequences in $8g - 4$ symbols. Here we use it for a Markov shift on $16g - 8$ symbols.

has an explicit expression of the form

$$N_n = c_1 \cdot \left(K/2 + \sqrt{(K/2)^2 - 1}\right)^n + c_2 \cdot \left(K/2 + \sqrt{(K/2)^2 - 1}\right)^{-n}$$

for some constants c_1 and c_2 , and therefore $\lim_{n \rightarrow \infty} (N_n)^{1/n} = K/2 + \sqrt{(K/2)^2 - 1}$. For $N_n = |M_{\bar{A}}^n|$ we have exactly this relation with $K = 8g - 6$; therefore the maximal eigenvalue of $M_{\bar{A}}$ is $\lim_{n \rightarrow \infty} |M_{\bar{A}}^n|^{1/n} = 4g - 3 + \sqrt{(4g - 3)^2 - 1}$. \square

Corollary 8. *For any extremal \bar{A} , $h_{\text{top}}(f_{\bar{A}}) = \log \lambda$.*

Proposition 9. *For any extremal \bar{A} , the right eigenvector $v = (v_1, \dots, v_{16g-8})$, corresponding to the maximal eigenvalue λ , normalized so that $\sum v_i = 1$, is given by*

$$v_i = \begin{cases} \frac{1}{\lambda(8g-4)} & \text{if } i \text{ is odd} \\ \frac{\lambda-1}{\lambda(8g-4)} & \text{if } i \text{ is even.} \end{cases}$$

Proof. From the proof of Proposition 7, for each odd i the set $f_{\bar{A}}(I_i)$ is the union of two consecutive Markov partition elements, one with an even index and one with an odd index, and thus

$$\sum_{k=1}^{8g-4} m_{i,2k-1} = 1 \quad \text{and} \quad \sum_{k=1}^{8g-4} m_{i,2k} = 1 \quad \text{for odd } i. \quad (10)$$

Similarly, if i is even then for any extremal \bar{A} we know $f_{\bar{A}}(I_i)$ is the union of $8g - 8$ odd indices and $8g - 7$ even indices, so

$$\sum_{k=1}^{8g-4} m_{i,2k-1} = 8g - 8 \quad \text{and} \quad \sum_{k=1}^{8g-4} m_{i,2k} = 8g - 7 \quad \text{for even } i. \quad (11)$$

We will show that the vector v' given by

$$v'_i = \begin{cases} \lambda - 8g + 7 & \text{if } i \text{ is odd} \\ 8g - 8 & \text{if } i \text{ is even} \end{cases}$$

satisfies $M_{\bar{A}} v' = \lambda v'$ by direct calculation. First, note that $\lambda = 4g - 3 + \sqrt{(4g - 3)^2 - 1}$ is one root of the quadratic equation

$$\lambda(\lambda - 8g + 7) = \lambda - 1. \quad (12)$$

Then we have

$$\begin{aligned} (M_{\bar{A}} v')_i &= \sum_{j=1}^{16g-8} m_{i,j} v'_j = \sum_{k=1}^{8g-4} m_{i,2k-1} v'_{2k-1} + \sum_{k=1}^{8g-4} m_{i,2k} v'_{2k} \\ &= (\lambda - 8g + 7) \left(\sum_{k=1}^{8g-4} m_{i,2k-1} \right) + (8g - 8) \left(\sum_{k=1}^{8g-4} m_{i,2k} \right) \\ &= \begin{cases} (\lambda - 8g + 7)(1) + (8g - 8)(1) & \text{if } i \text{ is odd} \\ (\lambda - 8g + 7)(8g - 8) + (8g - 8)(8g - 7) & \text{if } i \text{ is even} \end{cases} \quad \text{by (10), (11)} \\ &= \begin{cases} \lambda - 1 & \text{if } i \text{ is odd} \\ \lambda(8g - 8) & \text{if } i \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \lambda(\lambda - 8g + 7) & \text{if } i \text{ is odd} \\ \lambda(8g - 8) & \text{if } i \text{ is even} \end{cases} \quad \text{by (12)} \\
&= \lambda v'_i.
\end{aligned}$$

The normalized eigenvector v is then obtained by dividing v' by

$$\sum_{i=1}^{16g-8} v'_i = (8g-4)(\lambda-8g+7) + (8g-4)(8g-8) = (8g-4)(\lambda-1).$$

From (12), we have that $\lambda(8g-4) = (\lambda+1)^2$ and so the coordinates of v are

$$v_i = \frac{\lambda - 8g + 7}{(8g-4)(\lambda-1)} = \frac{(\lambda-1)/\lambda}{(8g-4)(\lambda-1)} = \frac{1}{\lambda(8g-4)}$$

for odd i and

$$v_i = \frac{(8g-7) - 1}{(8g-4)(\lambda-1)} = \frac{(\lambda - \frac{\lambda-1}{\lambda}) - 1}{(8g-4)(\lambda-1)} = \frac{1 - \frac{1}{\lambda}}{8g-4} = \frac{\lambda-1}{\lambda(8g-4)}$$

for even i . □

4. CONJUGACY TO CONSTANT-SLOPE MAP

We begin by stating a theorem combining several results of [21, 7], stated here for circle maps instead of interval maps (as in [18]):

Theorem 10. *Given a piecewise monotone, piecewise continuous, topologically transitive map $f : \mathbb{S} \rightarrow \mathbb{S}$ of positive topological entropy $h > 0$, there exists a unique (up to rotation of \mathbb{S}) increasing homeomorphism $\psi : \mathbb{S} \rightarrow \mathbb{S}$ conjugating f to a piecewise continuous map with constant slope e^h .*

Existence follows from [7, Corollary B], and uniqueness follows from [7, Lemma 8.1, Theorem 8.2, Corollary 1], where the continuity assumption is replaced by piecewise continuity (as in [12, 10]).

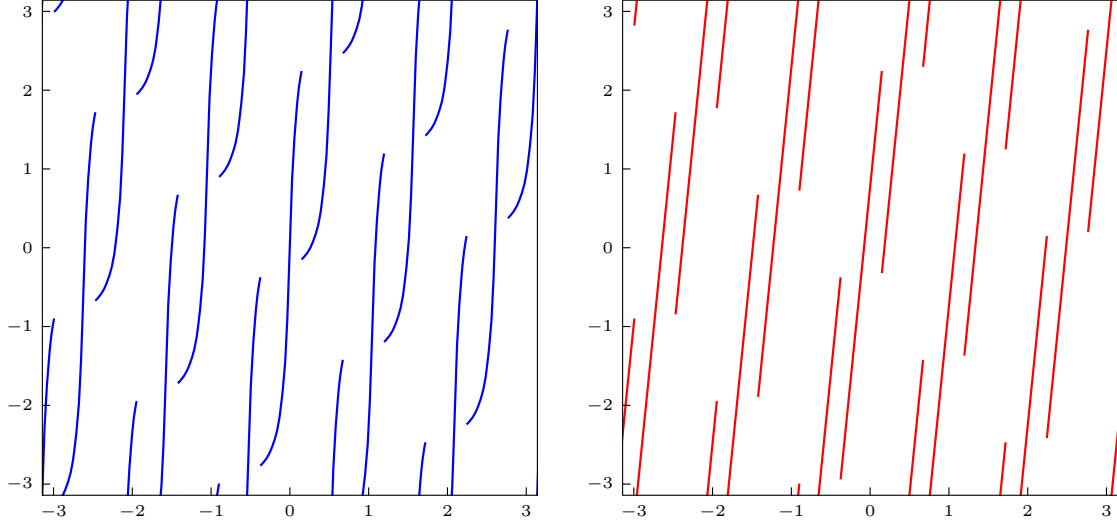
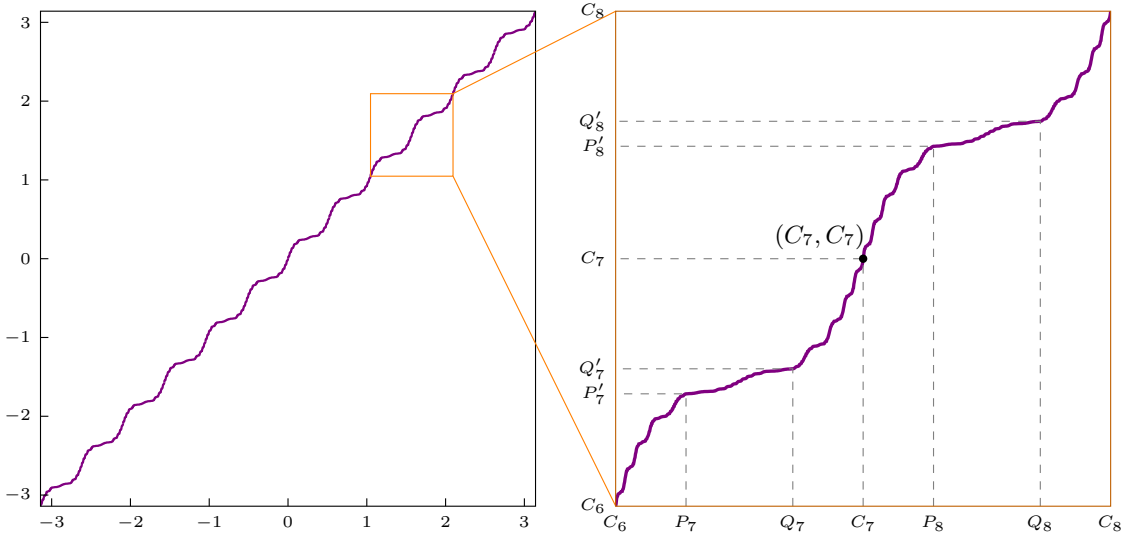
The map $f_{\bar{P}} : \mathbb{S} \rightarrow \mathbb{S}$ is piecewise monotone, piecewise continuous, topologically transitive (see [9, Lemma 2.5]), and with positive topological entropy (see Corollary 8), so by Theorem 10 there exists an increasing homeomorphism $\psi_{\bar{P}} : \mathbb{S} \rightarrow \mathbb{S}$ conjugating it to a map

$$\ell_{\bar{P}} := \psi_{\bar{P}} \circ f_{\bar{P}} \circ \psi_{\bar{P}}^{-1}$$

with constant slope, see Figures 3 and 4. The map $\psi_{\bar{P}}$ is unique up to rotation of \mathbb{S} , and the slope of $\ell_{\bar{P}}$ is exactly $\lambda = e^{h_{\text{top}}(f_{\bar{P}})}$. Although the existence of a conjugacy to a constant-slope map holds for $f_{\bar{P}}$ associated to irregular fundamental polygons as well as regular, we will assume that \mathcal{F} is regular for the remainder of this section.

The map $f_{\bar{Q}} : \mathbb{S} \rightarrow \mathbb{S}$, just like $f_{\bar{P}}$, is piecewise monotone, piecewise continuous, topologically transitive, and with positive topological entropy, so by Theorem 10 there exists an increasing homeomorphism $\psi_{\bar{Q}} : \mathbb{S} \rightarrow \mathbb{S}$ conjugating it to a map $\ell_{\bar{Q}}$ of constant slope, unique up to rotation of \mathbb{S} . By Corollary 8, both $\ell_{\bar{P}}$ and $\ell_{\bar{Q}}$ have the same slope.

Because $f_{\bar{P}}$ and $f_{\bar{Q}}$ are Markov maps, the conjugacies $\psi_{\bar{P}}$ and $\psi_{\bar{Q}}$ follow the classical construction due to Parry [20, 21] and used in the proof of [7, Lemma 5.1]. For each extremal parameter \bar{A} , we define the probability measure $\rho_{\bar{A}}$ on $X_{\bar{A}}$ as follows: let λ, v be


 FIGURE 3. Plots of $f_{\bar{P}}(x)$ (left) and $\ell_{\bar{P}}(x)$ (right) for $g = 2$.

 FIGURE 4. Plot of $\psi_{\bar{P}}(x)$ for $g = 2$.

the maximal eigenpair for the transition matrix $M_{\bar{A}}$; for an \bar{A} -admissible finite sequence $(\omega_0, \dots, \omega_n)$, we denote the symbolic cylinder

$$C_{\bar{A}}^{(\omega_0, \dots, \omega_n)} := \{ \omega' \in X_{\bar{A}} : \omega'_i = \omega_i \ \forall 0 \leq i \leq n \}$$

and define the measure $\rho_{\bar{A}}$ of this cylinder as

$$\rho_{\bar{A}}(C_{\bar{A}}^{(\omega_0, \dots, \omega_n)}) = \frac{v_{\omega_n}}{\lambda^n}.$$

The measure $\rho_{\bar{A}}$ is equivalent to the shift-invariant ‘‘Parry measure’’ (the measure of maximal entropy; see [20, 21]). The measure $\rho_{\bar{A}}$ is not shift-invariant but has the ‘‘expanding

property”

$$\rho_{\bar{A}}(\sigma_{\bar{A}}(C)) = \lambda \rho_A(C)$$

for all cylinders C on $(X_{\bar{A}}, \sigma_{\bar{A}})$.

Using the measure $\rho_{\bar{A}}$, one constructs the push-forward measure $\rho'_{\bar{A}}$ on \mathbb{S} given by

$$\rho'_{\bar{A}}(E) = \rho_{\bar{A}}(\phi_{\bar{A}}^{-1}(E)) \quad \text{for a Borel set } E,$$

where $\phi_{\bar{A}} : X_{\bar{A}} \rightarrow \mathbb{S}$ is the symbolic coding map (4). With the convention that $\psi_{\bar{A}}(0) = 0$, the conjugacy map $\psi_{\bar{A}} : \mathbb{S} \rightarrow \mathbb{S}$ is given by

$$\psi_{\bar{A}}(x) := 2\pi \cdot \begin{cases} \rho'_{\bar{A}}([0, x]) & \text{if } x \geq 0 \\ -\rho'_{\bar{A}}([x, 0]) & \text{if } x < 0 \end{cases} \quad (13)$$

(the 2π appears because of our convention that the circle is $\mathbb{S} = [-\pi, \pi]$).

It turns out that the maps $\psi_{\bar{P}}$ and $\psi_{\bar{Q}}$ thus constructed coincide:

Theorem 11. *For all $x \in \mathbb{S}$, $\psi_{\bar{P}}(x) = \psi_{\bar{Q}}(x)$.*

To prove this, we need to connect the cylinder intervals of the two circle maps $f_{\bar{P}}$ and $f_{\bar{Q}}$. Given an \bar{A} -admissible sequence $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ with $\omega_i \in \{1, \dots, 16g - 8\}$, we define the corresponding \bar{A} -cylinder interval

$$I_{\bar{A}}^{(\omega_0, \omega_1, \dots, \omega_n)} := I_{\omega_0} \cap f_{\bar{A}}^{-1}(I_{\omega_1}) \cap \dots \cap f_{\bar{A}}^{-n}(I_{\omega_n}). \quad (14)$$

Theorem 12. *Let $\omega = (\omega_0, \dots, \omega_n)$ be \bar{P} -admissible. There exists a \bar{Q} -admissible sequence $(\eta_0, \dots, \eta_n, \eta_{n+1})$ such that $\eta_0 = \omega_0$, η_{n+1} is odd, and*

(i) *if ω_n is odd then*

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1})} \cup I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1} + 1)}. \quad (15)$$

(ii) *if ω_n is even then either*

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(\eta_0, \dots, \eta_m)} = \bigcup_{i=0}^{16g-14} I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, 2\sigma(\eta_n/2) + 4 + i)} \quad (16)$$

or

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1})} \cup I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1} + 1)} \cup \bigcup_{i=0}^{16g-16} I_{\bar{Q}}^{(\eta_0, \dots, \eta_n + 1, 2\sigma(\frac{\eta_n+1}{2}) + 6 + i)}. \quad (17)$$

The proof of Theorem 12, as well as the distinction between the two forms (16) and (17), is rather technical and is left for Appendix A.

Proof of Theorem 11. Recall from Proposition 9 that for both $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$ the right-eigenvector v of $M_{\bar{A}}$ corresponding to eigenvalue λ is

$$v = c \cdot (1, \lambda - 1, 1, \lambda - 1, \dots, 1, \lambda - 1),$$

where $c = 1/(\lambda(8g - 4))$ corresponds to $\rho'_{\bar{A}}(\mathbb{S}) = 1$. We prove $\psi_{\bar{P}} = \psi_{\bar{Q}}$ by showing that $\rho'_{\bar{P}}(I_{\bar{P}}^\omega) = \rho'_{\bar{Q}}(I_{\bar{Q}}^\omega)$ for all finite \bar{P} -admissible sequences ω . Note that, because $\phi_{\bar{A}}^{-1}$ maps a cylinder interval to a symbolic cylinder, we have

$$\rho'_{\bar{P}}(I_{\bar{P}}^{(\omega_0, \dots, \omega_n)}) = \frac{v_{\omega_n}}{\lambda^n} \quad \text{and} \quad \rho'_{\bar{Q}}(I_{\bar{Q}}^{(\eta_0, \dots, \eta_n)}) = \frac{v_{\eta_n}}{\lambda^n},$$

where $(\omega_0, \dots, \omega_n)$ is \bar{P} -admissible and (η_0, \dots, η_n) is \bar{Q} -admissible.

Let $\omega = (\omega_0, \dots, \omega_n)$ be \bar{P} -admissible, and suppose ω_n is odd. Then $v_{\omega_n} = c$, and so

$$\rho'_{\bar{P}}(I_{\bar{P}}^\omega) = \frac{v_{\omega_n}}{\lambda^n} = \frac{c}{\lambda^n}.$$

By Theorem 12,

$$I_{\bar{P}}^\omega = I_Q^\eta \cup I_Q^{\eta'}$$

for some $\eta = (\eta_0, \dots, \eta_{n+1})$ and $\eta' = (\eta_0, \dots, \eta_n, \eta_{n+1} + 1)$. Since η_{n+1} and η'_{n+1} have different parities, we know

$$v_{\eta_{n+1}} + v_{\eta'_{n+1}} = c + (\lambda - 1)c = \lambda c,$$

and can compute

$$\begin{aligned} \rho'_{\bar{Q}}(I_{\bar{P}}^\omega) &= \rho'_{\bar{Q}}(I_Q^\eta \cup I_Q^{\eta'}) = \frac{v_{\eta_{n+1}}}{\lambda^{n+1}} + \frac{v_{\eta'_{n+1}}}{\lambda^{n+1}} = \frac{1}{\lambda^{n+1}}(v_{\eta_{n+1}} + v_{\eta'_{n+1}}) \\ &= \frac{1}{\lambda^{n+1}}(\lambda c) = \frac{c}{\lambda^n} = \rho'_{\bar{P}}(I_{\bar{P}}^\omega). \end{aligned}$$

If instead ω_n is even, then Theorem 12 gives $I_{\bar{P}}^\omega = I_Q^{\eta^{(1)}} \cup \dots \cup I_Q^{\eta^{(16g-15)}}$ with exactly $8g - 7$ of the final symbols $\eta_{n+1}^{(i)}$ being even (so $8g - 8$ are odd). Therefore

$$\begin{aligned} \rho'_{\bar{Q}}(I_{\bar{P}}^\omega) &= \rho'_{\bar{Q}}(I_Q^{\eta^{(1)}} \cup \dots \cup I_Q^{\eta^{(16g-15)}}) = \frac{1}{\lambda^{n+1}}(v_{\eta_{n+1}^{(1)}} + \dots + v_{\eta_{n+1}^{(16g-15)}}) \\ &= \frac{1}{\lambda^{n+1}}((8g - 7)(\lambda - 1)c + (8g - 8)c) \\ &= \frac{c}{\lambda^{n+1}}\lambda(\lambda - 1) \quad \text{by (12)} \\ &= \frac{(\lambda - 1)c}{\lambda^n} = \frac{v_{\omega_n}}{\lambda^n} = \rho'_{\bar{P}}(I_{\bar{P}}^\omega), \end{aligned}$$

where $v_{\omega_n} = (\lambda - 1)c$ because ω_n is even.

In both cases we have $\rho'_{\bar{Q}}(I_{\bar{P}}^\omega) = \rho'_{\bar{P}}(I_{\bar{P}}^\omega)$, and since $\{I_{\bar{P}}^\omega : \omega \text{ is } \bar{P}\text{-admissible}\}$ generates all Borel sets in \mathbb{S} , the two measures $\rho'_{\bar{P}}$ and $\rho'_{\bar{Q}}$ on \mathbb{S} are identical. From (13), this implies that $\psi_{\bar{P}} = \psi_{\bar{Q}}$. \square

For the remainder of Section 4, we deal almost exclusively with $\psi_{\bar{P}}$, although we will briefly invoke Theorem 11. We now show that $\psi_{\bar{P}}$ has translational (Proposition 13) and central (Proposition 14) symmetry. Both of these properties can be seen in Figure 4.

Proposition 13. *For all $x \in \mathbb{S}$, $\psi_{\bar{P}}(x + \alpha) = \psi_{\bar{P}}(x) + \alpha$.*

Proof. Define $\phi : \mathbb{S} \rightarrow \mathbb{S}$ recursively by

$$\phi(x) := \begin{cases} \psi_{\bar{P}}(x) & \text{if } x \in [P_1, P_2] \\ \phi(x - \alpha) + \alpha & \text{otherwise.} \end{cases}$$

Thus $\phi(x + \alpha) = \phi(x) + \alpha$ for all x by design, and since ϕ is increasing and continuous, we also have $\phi^{-1}(x + \alpha) = \phi^{-1}(x) + \alpha$ for all x .

Denote $x' = \phi(x)$. By induction on $1 \leq k \leq 8g - 4$, we will prove that there exists b_k such that $\phi \circ T_k \circ \phi^{-1}(x') = \lambda x' + b_k$ for $x' \in [P'_k, P'_{k+1}]$. By construction the claim is true for $k = 1$ since $\phi|_{[P_1, P_2]} = \psi_{\bar{P}}|_{[P_1, P_2]}$. Now assume it is true for k . Then by Proposition 3, writing $\beta = (4g - 3)\alpha$, we have

$$\phi \circ T_{k+1} \circ \phi^{-1}(x') = \phi(T_{k+1}(x)) = \phi(T_k(x - \alpha) + \beta) = \phi(T_k(x - \alpha)) + \beta$$

$$\begin{aligned}
&= \phi(T_k(\phi^{-1}(x' - \alpha))) + \beta = \lambda(x' - \alpha) + b_k + \beta \\
&= \lambda x' + (b_k + \beta - \lambda\alpha)
\end{aligned}$$

and so the claim holds for $k + 1$ with $b_{k+1} = b_k + \beta - \lambda\alpha$.

Thus our map ϕ , which satisfies $\phi(x + \alpha) = \phi(x) + \alpha$ for all x by construction, conjugates $f_{\bar{P}}$ to a constant-slope map on \mathbb{S} . By the uniqueness of $\psi_{\bar{P}}$ (Theorem 10), $\phi = \psi_{\bar{P}}$. \square

Notice that $\psi_{\bar{P}}(x + n\alpha) = \psi_{\bar{P}}(x) + n\alpha$ for any integer n , as well as $\psi_{\bar{P}}^{-1}(x + n\alpha) = \psi_{\bar{P}}^{-1}(x) + n\alpha$. Since 2π is an integer multiple of α , $\psi_{\bar{P}}$ is well defined on \mathbb{S} , and we can choose the point where it is equal to 0 at our convenience. We will assume that $\psi_{\bar{P}}$ fixes the point $C_{2g} = 0$. Then Proposition 13 implies that $\psi_{\bar{P}}(C_k) = C_k$ for all k .

Proposition 14. *For all $x \in \mathbb{S}$, $\psi_{\bar{P}}(C_k + x) + \psi_{\bar{P}}(C_k - x) = 2C_k$.*

Proof. First, we prove that

$$\psi_{\bar{Q}}(x) = -\psi_{\bar{P}}(-x). \quad (18)$$

By Corollary 5 we have $T_k(-x) = -T_{4g-k}(x)$. Since $f_{\bar{P}}$ acts by the generator T_k on $x \in [P_k, P_{k+1}]$ and $f_{\bar{Q}}$ acts by the generator T_{4g-k} on the reflected interval $-[P_k, P_{k+1}] = [Q_{4g-k}, Q_{4g-k+1}]$, we have as a result that

$$f_{\bar{Q}}(x) = -f_{\bar{P}}(-x).$$

To prove (18), set $\tilde{x} = \xi(x) := -\psi_{\bar{P}}(-x)$. Then $-\tilde{x} = \psi_{\bar{P}}(-x)$ and $-x = \psi_{\bar{P}}^{-1}(-\tilde{x})$, and then

$$\begin{aligned}
\xi \circ f_{\bar{Q}} \circ \xi^{-1}(\tilde{x}) &= \xi(f_{\bar{Q}}(x)) = \xi(-f_{\bar{P}}(-x)) = -\psi_{\bar{P}}(f_{\bar{P}}(-x)) \\
&= -\psi_{\bar{P}} \circ f_{\bar{P}} \circ \psi_{\bar{P}}^{-1}(-\tilde{x}) = -\ell_{\bar{P}}(-\tilde{x}).
\end{aligned}$$

Since this is a function with constant slope λ , the claim (18) follows by uniqueness of the conjugacy.

Combing (18) with Theorem 11, we obtain $\psi_{\bar{P}}(-x) = -\psi_{\bar{P}}(x)$. Because $C_k = (k - 2g)\alpha$, Proposition 13 implies

$$\psi_{\bar{P}}(C_k + x) = C_k + \psi_{\bar{P}}(x)$$

for all k . Therefore we compute that

$$\psi_{\bar{P}}(C_k + x) + \psi_{\bar{P}}(C_k - x) = C_k + \psi_{\bar{P}}(x) + C_k + \psi_{\bar{P}}(-x) = 2C_k. \quad \square$$

A crucial observation for the proof of Theorem 1 is that $\psi_{\bar{P}}$ conjugates each T_k as a function on the circle \mathbb{S} , and the resulting function

$$S_k := \psi_{\bar{P}} \circ T_k \circ \psi_{\bar{P}}^{-1}$$

consists of two linear pieces, one with slope λ and the other with slope λ^{-1} . See Figure 5, where

$$P'_k := \psi_{\bar{P}}(P_k) \quad \text{and} \quad Q'_k := \psi_{\bar{P}}(Q_k).$$

Lemma 15. *The function $S_k : \mathbb{S} \rightarrow \mathbb{S}$ can be fully described as follows:*

- (a) S_k is linear on $[P'_k, Q'_{k+1}]$ with slope λ ;
- (b) S_k is linear on $[Q'_{k+1}, P'_k]$ with slope λ^{-1} .

Proof. (a) By construction, S_k is linear on $[P'_k, P'_{k+1}]$ since $[P_k, P_{k+1}]$ is the interval where $f_{\bar{P}}$ acts as T_k . Given the central symmetry of T_k (Proposition 4) and $\psi_{\bar{P}}$ (Proposition 14) around C_k , the composition $S_k = \psi_{\bar{P}} \circ T_k \circ \psi_{\bar{P}}^{-1}$ must also be symmetric around C_k . The image of $[P'_k, P'_{k+1}]$ under the symmetry $C_k + x \mapsto C_k - x$ is $[Q'_k, Q'_{k+1}]$, and thus S_k is linear on $[Q'_k, Q'_{k+1}]$ with the same slope. Since the intervals of linearity $[P'_k, Q'_k]$ and $[Q'_k, Q'_{k+1}]$ overlap, there is no jump within their union, which is $[P'_k, Q'_{k+1}]$. We can in fact calculate

$$\begin{aligned} S_k(P'_k) &= \psi_{\bar{P}}(T_k(P_k)) = \psi_{\bar{P}}(Q_{\sigma(k)+1}) = Q'_{\sigma(k)+1} \\ S_k(Q'_{k+1}) &= \psi_{\bar{P}}(T_k(Q_{k+1})) = \psi_{\bar{P}}(P_{\sigma(k)}) = P'_{\sigma(k)} \end{aligned}$$

directly using [15, Proposition 2.2].

(b) Because part (a) holds for all k , we know $S_{\sigma(k)}$ maps $[P'_{\sigma(k)}, Q'_{\sigma(k)+1}]$ linearly to $[Q'_{\sigma(k)+1}, P'_{\sigma(k)}] = [Q'_{k+1}, P'_k]$ with slope λ , and therefore $S_{\sigma(k)}^{-1}$ maps $[Q'_{k+1}, P'_k]$ linearly to $[P'_{\sigma(k)}, Q'_{\sigma(k)+1}]$ with slope $1/\lambda$. But

$$S_{\sigma(k)}^{-1} = (\psi_{\bar{P}} \circ T_{\sigma(k)} \circ \psi_{\bar{P}}^{-1})^{-1} = \psi_{\bar{P}} \circ T_{\sigma(k)}^{-1} \circ \psi_{\bar{P}}^{-1} = \psi_{\bar{P}} \circ T_k \circ \psi_{\bar{P}}^{-1}$$

is exactly S_k . □

5. PROOF OF THEOREM 1

We can now prove the rigidity of topological entropy, that is, $h_{\text{top}}(f_{\bar{A}})$ is the same for all parameters \bar{A} and for all fundamental polygons \mathcal{F} .

Regular polygon. First we prove Theorem 1 in the case where $f_{\bar{A}}$ is associated to a regular $(8g-4)$ -gon. Let $\bar{A} = \{A_1, \dots, A_{8g-4}\}$ consist of any points satisfying $A_k \in [P_k, Q_k]$. Because $S_k = \psi_{\bar{P}} \circ T_k \circ \psi_{\bar{P}}^{-1}$ is linear on all of $[P'_k, Q'_{k+1}]$ with slope λ , the function $\psi_{\bar{P}} \circ f_{\bar{A}} \circ \psi_{\bar{P}}^{-1}$ (note the use of $f_{\bar{A}}$ with $\psi_{\bar{P}}$) is piecewise affine with constant slope λ , and so, by [18, Theorem 3'] applied to such maps, the topological entropy of $f_{\bar{A}}$ is $\log \lambda$.

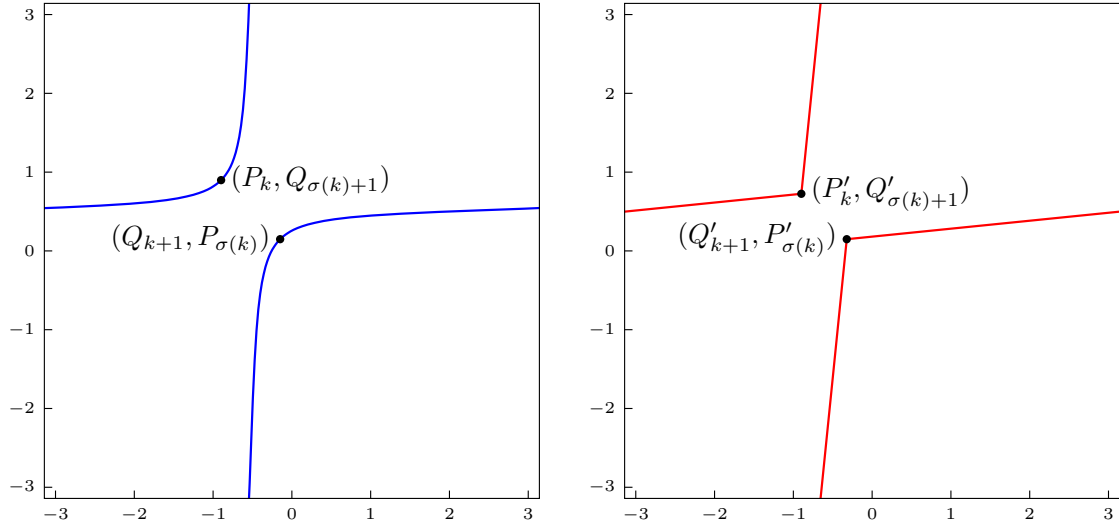


FIGURE 5. Plots of $T_k(x)$ (left) and $S_k(x)$ (right) with $k = 3$ and $g = 2$.

Teichmüller space. As explained in [3, Introduction], the Teichmüller space of a compact surface of genus g may be viewed as the space of marked $(8g - 4)$ -fundamental polygons, and the partitions of the boundary \mathbb{S} for various polygons are related via a homeomorphism of \mathbb{S} by Fenchel–Nielsen Theorem.

Let $\tilde{\Gamma}$ be a Fuchsian group such that $\tilde{\Gamma} \backslash \mathbb{D}$ is a compact surface of genus g whose fundamental $(8g - 4)$ -gon $\tilde{\mathcal{F}}$ is not regular. As explained in [16], there is a Fuchsian group Γ having a regular fundamental $(8g - 4)$ -gon \mathcal{F} and an orientation-preserving homeomorphism $h : \mathbb{D} \rightarrow \mathbb{D}$ such that $\tilde{\Gamma} = h \circ \Gamma \circ h^{-1}$. Side k of $\tilde{\mathcal{F}}$ extends to a geodesic $\tilde{P}_k \tilde{Q}_{k+1}$ and is glued to side $\sigma(k)$ by the map $\tilde{T}_k = h \circ T_k \circ h^{-1}$, where $\{T_k\}$ are generators of Γ identifying the sides of \mathcal{F} .

For any $\tilde{A} = \{\tilde{A}_1, \dots, \tilde{A}_{8g-4}\}$ with $\tilde{A}_k \in [\tilde{P}_k, \tilde{Q}_k]$, we define

$$\tilde{f}_{\tilde{A}}(x) := \tilde{T}_k(x) \quad \text{if } x \in [\tilde{A}_k, \tilde{A}_{k+1}).$$

Then the map $f_{\tilde{A}}$ with $\bar{A} = \{h^{-1}(\tilde{A}_1), \dots, h^{-1}(\tilde{A}_{8g-4})\}$ is associated to the regular fundamental polygon, and (correcting a typo in [16])

$$\tilde{f}_{\tilde{A}} = h \circ f_{\bar{A}} \circ h^{-1}.$$

Since $\tilde{f}_{\tilde{A}}$ is conjugate to $f_{\bar{A}}$, we conclude that $h_{\text{top}}(\tilde{f}_{\tilde{A}}) = h_{\text{top}}(f_{\bar{A}}) = \log \lambda$, and this completes the proof of the main theorem. (In fact, the map $\psi \circ h^{-1}$ with $\psi = \psi_{\bar{P}}$ from Section 4 will conjugate $\tilde{f}_{\tilde{A}}$ to a map of constant slope λ .)

APPENDIX A. PROOF OF THEOREM 12

We now prove Theorem 12, that is, that each cylinder interval $I_{\bar{P}}^{\omega}$ can be written as unions of cylinder intervals $I_{\bar{Q}}^{\eta}$ (see Figure 6 for the decompositions of $I_{\bar{P}}^{(1,16)}$ and $I_{\bar{P}}^{(1,17)}$). For the remainder of the appendix, we use the term “cylinder” (specifically, “ \bar{P} -cylinder” and “ \bar{Q} -cylinder”) instead of “cylinder interval” for brevity.

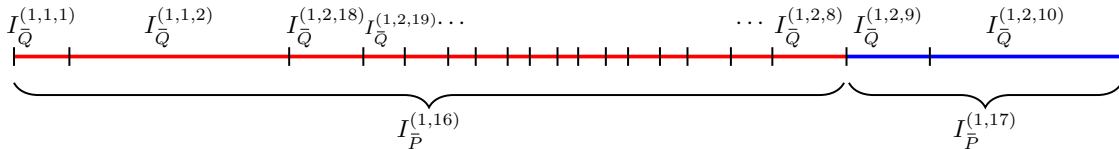


FIGURE 6. For $g = 2$, $I_{\bar{P}}^{(1,16)}$ as a union of 17 \bar{Q} -cylinders (left) and $I_{\bar{P}}^{(1,17)}$ as a union of two \bar{Q} -cylinders (right).

From (14), we derive a common recursive description of cylinders:

$$I_{\bar{A}}^{(\omega_0, \omega_1, \dots, \omega_n)} = I_{\omega_0} \cap f_{\bar{A}}^{-1}(I_{\bar{A}}^{(\omega_1, \dots, \omega_n)}).$$

For our particular boundary maps, we have an alternative recursive relation: using the fact that each T_k is bijective on all of \mathbb{S} , we can compute

$$I_{\bar{P}}^{(\omega_0, \omega_1, \dots, \omega_n)} = T_{[\omega_0/2]}^{-1}(I_{\bar{P}}^{(\omega_1, \dots, \omega_n)})$$

for \bar{P} -admissible ω and

$$I_{\bar{Q}}^{(\omega_0, \omega_1, \dots, \omega_n)} = T_{[\omega_0/2]}^{-1}(I_{\bar{Q}}^{(\omega_1, \dots, \omega_n)})$$

for \bar{Q} -admissible ω without the need for an intersection. This is because $T_{[\omega_0/2]}^{-1}$ is contracting on $I_{\bar{P}}^{(\omega_1, \dots, \omega_n)}$ and therefore $T_{[\omega_0/2]}^{-1}(I_{\bar{P}}^{(\omega_1, \dots, \omega_n)})$ is already contained in I_{ω_0} . Note the use of ceiling $\lceil \cdot \rceil$ for \bar{P} and floor $\lfloor \cdot \rfloor$ for \bar{Q} , owing to the fact that $f_{\bar{P}}$ acts by T_k on I_{2k-1} and I_{2k} while $f_{\bar{Q}}$ acts by T_k on I_{2k} and I_{2k+1} , and $k = \lceil \frac{2k-1}{2} \rceil = \lceil \frac{2k}{2} \rceil = \lfloor \frac{2k}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor$. The formulas above can be extended recursively to

$$\begin{aligned} I_{\bar{P}}^\omega &= T_{[\omega_0/2]}^{-1} \circ T_{[\omega_1/2]}^{-1} \circ \cdots \circ T_{[\omega_{n-1}/2]}^{-1}(I_{\omega_n}) \quad \text{if } \omega \text{ is } \bar{P}\text{-admissible} \\ I_{\bar{Q}}^\omega &= T_{[\omega_0/2]}^{-1} \circ T_{[\omega_1/2]}^{-1} \circ \cdots \circ T_{[\omega_{n-1}/2]}^{-1}(I_{\omega_n}) \quad \text{if } \omega \text{ is } \bar{Q}\text{-admissible.} \end{aligned}$$

Equations (5), (6), and (7) can be interpreted as statements about admissible pairs of symbols:

- In a \bar{P} -admissible sequence, an odd symbol $2k-1$ can only be followed by $2\sigma(k)+2$ or $2\sigma(k)+3$.
- In a \bar{Q} -admissible sequence, an odd symbol $2k-1$ can only be followed by $2\sigma(k)+8g-5$ or $2\sigma(k)+8g-4$.
- In a \bar{P} - or \bar{Q} -admissible sequence, an even symbol $2k$ can only be followed by a symbol from $\{2\sigma(k)+4, 2\sigma(k)+5, \dots, 2\sigma(k)-4\}$.

In the final item above, and in Lemma 16 below, recall that these values are mod $16g-8$; see the explanation after (7) on page 7. The next lemma expands on the admissible pairs above and lists some longer admissible words used explicitly in the proof of Theorem 12.

Lemma 16.

- (a) For all k , if $\ell \in \{2k+8g, 2k+8g+1, \dots, 2k+8g-8\}$ then $(2k-1, 2\sigma(k)+8g-4, \ell)$ is \bar{Q} -admissible.
- (b) For all k , if $\ell \in \{2\sigma(k)+4, 2\sigma(k)+5, \dots, 2\sigma(k)-4\}$ then $(2k-1, 2\sigma(k)+8g-5, 2k, \ell)$ is \bar{Q} -admissible.
- (c) For all k , if $\ell \in \{2k+8g-9, 2k+8g-8\}$ then $(2k-1, 2\sigma(k)+8g-5, 2k-1, 2\sigma(k)+8g-4, \ell)$ is \bar{Q} -admissible.

The proof of Lemma 16 consists of careful analysis of the transition matrix $M_{\bar{Q}}$ along with the useful identities

$$\sigma(k-1) = \sigma(k) - 4g + 3 \quad \text{and} \quad \sigma(k-2) = \sigma(k) + 2,$$

which follow by direct verification (see also [2, Lemma 3.2]).

The following two lemmas establish some relations among the generators $\{T_k\}$ which will be used in the proof of Theorem 12. We omit the composition notation (writing, e.g., $T_k^{-1}T_{\sigma(k)+1}^{-1}$, instead of $T_k^{-1} \circ T_{\sigma(k)+1}^{-1}$).

Lemma 17 ([15, Lemma 3.2]). $T_k^{-1}T_{\sigma(k)+1}^{-1} = T_{k-1}^{-1}T_{\sigma(k)+4g-2}^{-1}$.

Lemma 18. For $m \geq 1$, $T_k^{-1}(T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1})^m = (T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1})^m T_k^{-1}$.

Proof. The base case, $m = 1$, is proven using Lemma 17 twice, the second time for index $\sigma(k) + 4g - 2$:

$$T_k^{-1}T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1} = T_{k-1}^{-1}T_{\sigma(k)+4g-2}^{-1}T_{k+4g-1}^{-1} = T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1}T_k^{-1}.$$

Then $m > 1$ follows by induction:

$$\begin{aligned}
T_k^{-1}(T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1})^m &= T_k^{-1}(T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1})^{m-1}T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1} \\
&= (T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1})^{m-1}T_k^{-1}T_{\sigma(k)+1}^{-1}T_{k+4g-1}^{-1} \\
&= (T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1})^{m-1}T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1}T_k^{-1} \\
&= (T_{k-1}^{-1}T_{\sigma(k)+4g-3}^{-1})^m T_k^{-1}. \quad \square
\end{aligned}$$

We are now ready to proceed with an inductive proof of Theorem 12, with the following refinement of part (ii):

- (a) If all ω_k are even, or if (ω_{n-1}, ω_n) are even but not of the form $(2m, 2\sigma(m) + 4)$ for any m , then $I_{\bar{P}}^\omega = I_{\bar{Q}}^{(\eta_0, \dots, \eta_n)}$ with η_n even, and therefore

$$I_{\bar{P}}^\omega = \bigcup_{i=0}^{16g-14} I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, 2\sigma(\eta_n/2) + 4 + i)}.$$

- (b) If ω_n is even and either ω_{n-1} is odd or $(\omega_{n-1}, \omega_n) = (2m, 2\sigma(m) + 4)$ for some m (but not all ω_i are even), then η_n is odd and

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1})} \cup I_{\bar{Q}}^{(\eta_0, \dots, \eta_n, \eta_{n+1} + 1)} \cup \bigcup_{i=0}^{16g-16} I_{\bar{Q}}^{(\eta_0, \dots, \eta_n + 1, 2\sigma(\frac{\eta_n+1}{2}) + 6 + i)},$$

where $\eta_{n+1} = 2\sigma(\lfloor \eta_n/2 \rfloor) + 8g - 5$.

We begin with the base case $n = 0$ for all parts. The original Markov partition sets I_i are both \bar{P} - and \bar{Q} -cylinders:

$$I_{\bar{P}}^{(\omega_0)} = I_{\omega_0} = I_{\bar{Q}}^{(\omega_0)}.$$

For $\omega_0 = 2k - 1$ odd,

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(2k-1)} = I_{\bar{Q}}^{(2k-1, 2\sigma(k) + 8g - 5)} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k) + 8g - 4)},$$

and for $\omega_0 = 2k$ even,

$$I_{\bar{P}}^\omega = I_{\bar{Q}}^{(2k)} = I_{\bar{Q}}^{(2k, 2\sigma(k) + 4)} \cup I_{\bar{Q}}^{(2k, 2\sigma(k) + 5)} \cup \dots \cup I_{\bar{Q}}^{(2k, 2\sigma(k) - 5)} \cup I_{\bar{Q}}^{(2k, 2\sigma(k) - 4)}$$

by (6) and (7) with $f_{\bar{A}} = f_{\bar{Q}}$.

For some parts of the proof, $n = 0$ is a sufficient base case, but we do at times implicitly assume $n \geq 1$, so we also provide here a ‘‘base case’’ with $n = 1$. If ω_0 is even, then

$$I_{\bar{P}}^{(\omega_0, \omega_1)} = I_{\bar{Q}}^{(\omega_0, \omega_1)},$$

and equations (15) and (16) follow immediately when ω_1 is odd, or, respectively, even. If $\omega_0 = 2k - 1$ is odd, then ω_1 can be either $2\sigma(k) + 2$ or $2\sigma(k) + 3$. We investigate the interval $I_{\bar{P}}^{(2k-1, 2\sigma(k)+3)}$. For that, notice that $I_{\bar{P}}^{(2k-1, 2\sigma(k)+3)} = T_k^{-1}(I_{2\sigma(k)+3})$. From relation (6) written for index $\sigma(k) + 2$, the interval $I_{2\sigma(k)+3}$ itself can be expressed as

$$\begin{aligned}
I_{2\sigma(k)+3} &= T_{\sigma(k)+1}^{-1}([P_{\sigma(\sigma(k)+1)-1}, P_{\sigma(\sigma(k)+1)}]) = T_{\sigma(k)+1}^{-1}[P_{(k-2)+4g-2}, P_{(k-2)+4g-1}] \\
&= T_{\sigma(k)+1}^{-1}(I_{2(k-2)+8g-5} \cup I_{2(k-2)+8g-4}) = T_{\sigma(k)+1}^{-1}(I_{2k+8g-9} \cup I_{2k+8g-8}).
\end{aligned}$$

Now we use Lemma 17 to write

$$\begin{aligned}
I_{\bar{P}}^{(2k-1, 2\sigma(k)+3)} &= T_k^{-1} T_{\sigma(k)+1}^{-1} (I_{2k+8g-9} \cup I_{2k+8g-8}) \\
&= T_{k-1}^{-1} T_{\sigma(k-1)-1}^{-1} (I_{2k+8g-9} \cup I_{2k+8g-8}) \\
&= T_{k-1}^{-1} T_{\sigma(k)+4g-2}^{-1} (I_{2k+8g-9} \cup I_{2k+8g-8}) \\
&= T_{k-1}^{-1} (I_{\bar{Q}}^{(2\sigma(k)+8g-4, 2k+8g-9)} \cup I_{\bar{Q}}^{(2\sigma(k)+8g-4, 2k+8g-8)}) \\
&= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, 2k+8g-9)} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, 2k+8g-8)}
\end{aligned}$$

which proves (15), that is, part (i), for $n = 1$.

The other \bar{P} -cylinder interval $I_{\bar{P}}^{(2k-1, 2\sigma(k)+2)} = I_{\bar{P}}^{(2k-1)} \setminus I_{\bar{P}}^{(2k-1, 2\sigma(k)+3)}$. Since

$$\begin{aligned}
I_{\bar{P}}^{(2k-1)} &= I_{\bar{Q}}^{(2k-1)} = I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5)} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4)} \\
&= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k-1)} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k)} \\
&\quad \cup \bigcup_{\ell=2k+8g}^{2k+8g-8} I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, \ell)}
\end{aligned}$$

by Lemma 16(a) and, rewriting $I_{\bar{P}}^{(2k-1, 2\sigma(k)+3)}$ as the union of two \bar{Q} -cylinders above, we have

$$\begin{aligned}
I_{\bar{P}}^{(2k-1, 2\sigma(k)+2)} &= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k-1)} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k)} \\
&\quad \cup \bigcup_{\ell=2k+8g}^{2k+8g-10} I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, \ell)},
\end{aligned}$$

proving (17) for $n = 1$.

We proceed now with induction for $n \geq 2$. We say that a cylinder $I_{\bar{A}}^{(\omega_0, \dots, \omega_n)}$ has *rank* $n + 1$. Assume $I_{\bar{P}}^\omega$ of rank $\leq n$ is a union of \bar{Q} -cylinders as desired; we want $I_{\bar{P}}^{(\omega_0, \dots, \omega_n)}$ to be a union of \bar{Q} -cylinders of rank $n + 2$.

When ω_0 is even, the induction argument is straightforward for both parts. We demonstrate it for part (i), that is, when ω_n is odd. Using the induction hypothesis for $I_{\bar{P}}^{(\omega_1, \dots, \omega_n)}$, we have

$$\begin{aligned}
I_{\bar{P}}^{(\omega_0, \omega_1, \dots, \omega_n)} &= T_{\omega_0/2}^{-1} (I_{\bar{P}}^{(\omega_1, \dots, \omega_n)}) \\
&= T_{\omega_0/2}^{-1} (I_{\bar{Q}}^{(\eta_1, \dots, \eta_n, \eta_{n+1})} \cup I_{\bar{Q}}^{(\eta_1, \dots, \eta_n, \eta_{n+1} + 1)}) \text{ by induction} \\
&= T_{\omega_0/2}^{-1} I_{\bar{Q}}^{(\eta_1, \dots, \eta_n, \eta_{n+1})} \cup T_{\omega_0/2}^{-1} I_{\bar{Q}}^{(\eta_1, \dots, \eta_n, \eta_{n+1} + 1)} \\
&= I_{\bar{Q}}^{(\omega_0, \eta_1, \dots, \eta_n, \eta_{n+1})} \cup I_{\bar{Q}}^{(\omega_0, \eta_1, \dots, \eta_n, \eta_{n+1} + 1)},
\end{aligned}$$

where the final substitution uses the fact that $\eta_1 = \omega_1$ (from induction) and that the pair $(\omega_0, \eta_1) = (\omega_0, \omega_1)$ is \bar{P} -admissible if and only if it is \bar{Q} -admissible (because ω_0 is even, and the even rows of $M_{\bar{P}}$ and $M_{\bar{Q}}$ are identical). Part (ii) can be treated similarly.

We now prove parts (i) and (ii) separately when ω_0 is odd.

(i) From the induction hypothesis,

$$I_{\bar{P}}^{(\omega_1, \omega_2, \dots, \omega_n)} = I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_n, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_n, \xi_{n+1} + 1)}$$

with $\xi_1 = \omega_1$ and ξ_{n+1} odd. (We use ξ here instead of η because the terms ξ_i will not necessarily be η_i for $I_{\bar{P}}^{(\omega_0, \dots, \omega_n)}$ from the statement of Theorem 12.) Thus

$$\begin{aligned} I_{\bar{P}}^\omega &= T_{[\omega_0/2]}^{-1} \left(I_{\bar{P}}^{(\omega_1, \omega_2, \omega_3, \dots, \omega_n)} \right) \\ &= T_{[\omega_0/2]}^{-1} \left(I_{\bar{Q}}^{(\omega_1, \xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\omega_1, \xi_2, \xi_3, \dots, \xi_{n+1} + 1)} \right) \\ &= T_{[\omega_0/2]}^{-1} T_{[\omega_1/2]}^{-1} \left(I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1} + 1)} \right). \end{aligned}$$

Let $\omega_0 = 2k - 1$. Then ω_1 must be $2\sigma(k) + 2$ or $2\sigma(k) + 3$, and either way $[\omega_1/2] = \sigma(k) + 1$, giving

$$I_{\bar{P}}^\omega = T_k^{-1} T_{\sigma(k)+1}^{-1} \left(I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1} + 1)} \right). \quad (19)$$

There are now several cases and sub-cases to consider; these are summarized in Figure 7.

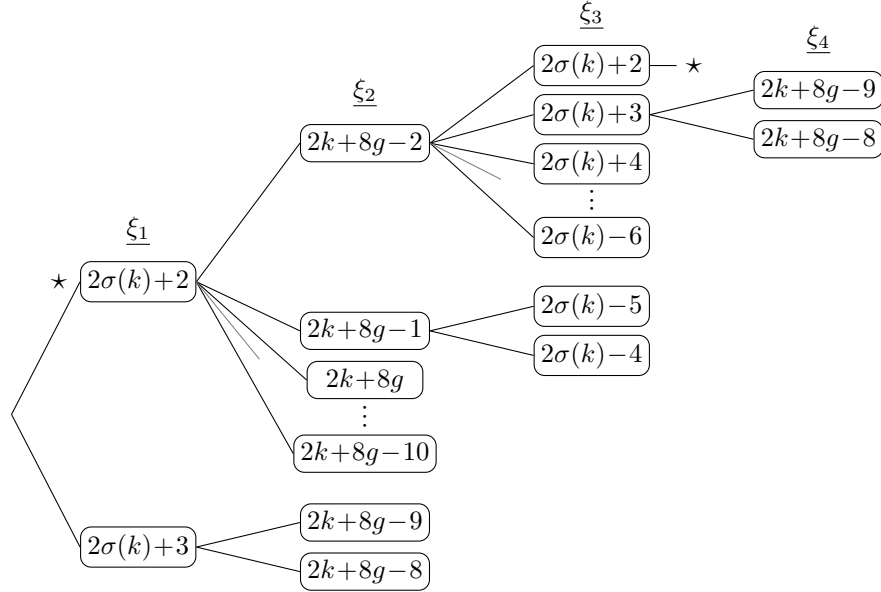


FIGURE 7. Relevant cases when $\omega_0 = 2k - 1$ is odd (all indices mod $16g - 8$).

If $\xi_1 = 2\sigma(k) + 2$ then $\xi_2 \in \{2k + 8g - 2, 2k + 8g - 1, \dots, 2k + 8g - 10\}$, and if $\xi_1 = 2\sigma(k) + 3$ then ξ_2 is $2k + 8g - 9$ or $2k + 8g - 8$. Other than when $\xi_2 \in \{2k + 8g - 2, 2k + 8g - 1\}$, we can apply Lemma 17 to (19) to get

$$I_{\bar{P}}^\omega = T_{k-1}^{-1} T_{\sigma(k)+4g-2}^{-1} \left(I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1} + 1)} \right),$$

and then Lemma 16(a) implies

$$\begin{aligned} I_{\bar{P}}^\omega &= T_{\lfloor \frac{2k-1}{2} \rfloor}^{-1} T_{\lfloor \frac{2\sigma(k)+8g-4}{2} \rfloor}^{-1} \left(I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_2, \xi_3, \dots, \xi_{n+1}+1)} \right) \\ &= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, \xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-4, \xi_2, \xi_3, \dots, \xi_{n+1}+1)}. \end{aligned}$$

We are left with analyzing the cases $\xi_2 = 2k+8g-2$ and $\xi_2 = 2k+8g-1$, with $\xi_1 = 2\sigma(k)+2$. Here $\lfloor \xi_2/2 \rfloor = k+4g-1$ and so we proceed from (19) as

$$\begin{aligned} I_{\bar{P}}^\omega &= T_k^{-1} T_{\sigma(k)+1}^{-1} T_{k+4g-1}^{-1} \left(I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1}+1)} \right) \\ &= T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1} T_k^{-1} \left(I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1}+1)} \right) \end{aligned}$$

using Lemma 18 with $m = 1$. If $\xi_2 = 2k+8g-2$, then $\xi_3 \in \{2\sigma(k)+2, 2\sigma(k)+3, \dots, 2\sigma(k)-6\}$, and if $\xi_2 = 2k+8g-1$, then ξ_3 is $2\sigma(k)-5$ or $2\sigma(k)-4$. For all possible pairs (ξ_2, ξ_3) *except* $(\xi_2, \xi_3) = (2k+8g-2, 2\sigma(k)+2)$ and $(\xi_2, \xi_3) = (2k+8g-2, 2\sigma(k)+3)$, Lemma 16(b) implies precisely that

$$\begin{aligned} I_{\bar{P}}^\omega &= T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1} T_k^{-1} \left(I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1}+1)} \right) \\ &= T_{\lfloor \frac{2k-1}{2} \rfloor}^{-1} T_{\lfloor \frac{2\sigma(k)+8g-5}{2} \rfloor}^{-1} T_{\lfloor \frac{2k}{2} \rfloor}^{-1} \left(I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1}+1)} \right) \\ &= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(2k-1, \dots, \xi_{n+1}+1)}. \end{aligned}$$

Now we only need to analyze the cases $\xi_3 = 2\sigma(k)+2$ and $\xi_3 = 2\sigma(k)+3$, where we have already set $\xi_2 = 2k+8g-2$ and $\xi_1 = 2\sigma(k)+2$.

If $\xi_3 = 2\sigma(k)+3$, then ξ_4 is either $2k+8g-8$ or $2k+8g-9$, and so

$$\begin{aligned} I_{\bar{P}}^\omega &= T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1} T_k^{-1} \left(I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_3, \dots, \xi_{n+1}+1)} \right) \\ &= T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1} T_k^{-1} T_{\sigma(k)+1}^{-1} \left(I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1}+1)} \right) \\ &= T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1} T_{k-1}^{-1} T_{\sigma(k)+4g-2}^{-1} \left(I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1}+1)} \right) \quad \text{by Lemma 17} \\ &= T_{\lfloor \frac{2k-1}{2} \rfloor}^{-1} T_{\lfloor \frac{2\sigma(k)+8g-5}{2} \rfloor}^{-1} T_{\lfloor \frac{2k-1}{2} \rfloor}^{-1} T_{\lfloor \frac{2\sigma(k)+8g-4}{2} \rfloor}^{-1} \left(I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_4, \dots, \xi_{n+1}+1)} \right) \\ &= I_{\bar{Q}}^{(2k-1, 2\sigma(k)+8g-5, 2k-1, 2\sigma(k)+8g-4, \xi_4, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(2k-1, \dots, \xi_{n+1}+1)} \end{aligned}$$

by Lemma 16(c).

If $\xi_3 = 2\sigma(k)+2$, notice that $\xi_3 = \xi_1$, so we now analyze the situation when the sequence $(\xi_1, \dots, \xi_{n+1})$ consists of several alternating entries $(2\sigma(k)+2, 2k+8g-2)$ until some $\xi_j \notin \{2\sigma(k)+2, 2k+8g-2\}$ (this situation is denoted by \star in Figure 7). Notice that $j < n+1$: otherwise, all ξ_1, \dots, ξ_n would be even, and then

$$I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n, \xi_{n+1})} \subset I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n)} = I_{\bar{P}}^{(\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n)}$$

would imply $I_{\bar{P}}^{(\xi_1, \xi_2, \dots, \xi_n)} = I_{\bar{P}}^{(\omega_1, \omega_2, \dots, \omega_n)}$, which is not possible since ω_n is odd.

We assume j is odd (the case of even j can be treated similarly). Then

$$(\xi_1, \xi_2, \dots, \xi_{n+1}) = (2\sigma(k)+2, 2k+8g-2, \dots, 2\sigma(k)+2, 2k+8g-2, \xi_j, \dots, \xi_{n+1}),$$

where ξ_j is one of $\{2\sigma(k) + 3, \dots, 2\sigma(k) - 6\}$. Thus

$$\begin{aligned} I_{\bar{P}}^\omega &= T_k^{-1} \left(I_{\bar{Q}}^{(\xi_1, \xi_2, \xi_3, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_1, \dots, \xi_{n+1} + 1)} \right) \\ &= T_k^{-1} (T_{\sigma(k)+1}^{-1} T_{k+4g-1}^{-1})^{(j-1)/2} \left(I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \\ &= (T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1})^{(j-1)/2} T_k^{-1} \left(I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \end{aligned} \quad (20)$$

by Lemma 18. For $\xi_j \neq 2\sigma(k) + 3$, Lemma 16(b) implies that

$$\begin{aligned} I_{\bar{P}}^\omega &= \left(T_{\lfloor \frac{2k-1}{2} \rfloor}^{-1} T_{\lfloor \frac{2\sigma(k)+8g-5}{2} \rfloor}^{-1} \right)^{(j-1)/2} T_{\lfloor \frac{2k}{2} \rfloor}^{-1} \left(I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \\ &= I_{\bar{Q}}^{(2k-1, 2\sigma(k) + 8g - 5, \dots, 2k-1, 2\sigma(k) + 8g - 5, 2k, \xi_j, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(2k-1, \dots, \xi_{n+1} + 1)}, \end{aligned}$$

and for $\xi_j = 2\sigma(k) + 3$ we proceed from (20) with

$$\begin{aligned} I_{\bar{P}}^\omega &= (T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1})^{(j-1)/2} T_k^{-1} \left(I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \\ &= (T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1})^{(j-1)/2} T_k^{-1} T_{\sigma(k)+1}^{-1} \left(I_{\bar{Q}}^{(\xi_{j+1}, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \\ &= (T_{k-1}^{-1} T_{\sigma(k)+4g-3}^{-1})^{(j-1)/2} T_{k-1}^{-1} T_{\sigma(k)+4g-2}^{-1} \left(I_{\bar{Q}}^{(\xi_{j+1}, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_j, \dots, \xi_{n+1} + 1)} \right) \\ &= I_{\bar{Q}}^{(2k-1, 2\sigma(k) + 8g - 5, \dots, 2k-1, 2\sigma(k) + 8g - 4, \xi_{j+1}, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(2k-1, \dots, \xi_{n+1} + 1)}, \end{aligned}$$

using Lemma 17 for a substitution and then Lemma 16(a) for the final line since, following $\xi_j = 2\sigma(k) + 3$, we know ξ_{j+1} is either $2k - 8g - 9$ or $2k - 8g - 8$. Having followed all paths in Figure 7, this completes the proof of part (i).

(ii) When ω_n is even, there are two possible structures, (16) and (17), for the decomposition of $I_{\bar{P}}^\omega$, corresponding to the two cases (a) and (b) on page 17.

(a) First, if all ω_i are even then $I_{\bar{P}}^\omega = I_{\bar{Q}}^\omega$ because the even rows of $M_{\bar{P}}$ and $M_{\bar{Q}}$ coincide. Then $I_{\bar{P}}^\omega$ can be trivially decomposed into $16g - 15$ cylinders of higher rank as in (16).

If (ω_{n-1}, ω_n) are both even and not of the form $(2m, 2\sigma(m) + 4)$ for any m , then, from the induction hypothesis for case (a), there exists a \bar{Q} -admissible sequence $(\xi_1, \xi_2, \dots, \xi_n)$ such that

$$I_{\bar{P}}^{(\omega_1, \omega_2, \dots, \omega_n)} = I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_n)},$$

with ξ_n even. Now the analogue of relation (19) is

$$I_{\bar{P}}^\omega = T_k^{-1} T_{\sigma(k)+1}^{-1} (I_{\bar{Q}}^{(\xi_2, \dots, \xi_n)}).$$

The sequence $(\xi_1, \xi_2, \dots, \xi_n)$ cannot consist entirely of alternating even entries $(2\sigma(k) + 2, 2k + 8g - 2)$: if this were the case, then

$$I_{\bar{Q}}^{(\xi_1, \xi_2, \dots, \xi_n)} = I_{\bar{P}}^{(\xi_1, \xi_2, \dots, \xi_n)}, \text{ so } (\omega_1, \omega_2, \dots, \omega_n) = (\xi_1, \xi_2, \dots, \xi_n),$$

which is impossible because the last two entries (ω_{n-1}, ω_n) are not of the form $(2m, 2\sigma(m) + 4)$. One can then proceed as in case (i) and express $I_{\bar{P}}^\omega$ as a single \bar{Q} -cylinder of rank $n + 1$, which is then a union of $16g - 15$ \bar{Q} -cylinders of rank $n + 2$ as desired.

(b) If ω_{n-1} is odd or the final pair $(\omega_{n-1}, \omega_n) = (2m, 2\sigma(m) + 4)$ for some m , then we have (17), as will we now show.

We follow the proof of (i), where a stricter key step $j < n$ will now follow from the new assumptions. Indeed, from the induction hypothesis for case (b),

$$I_{\bar{P}}^{(\omega_1, \omega_2, \dots, \omega_n)} = I_{\bar{Q}}^{(\xi_1, \dots, \xi_n, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_1, \dots, \xi_n, \xi_{n+1} + 1)} \cup \bigcup_{i=0}^{16g-16} I_{\bar{Q}}^{(\xi_1, \dots, \xi_n + 1, 2\sigma(\frac{\xi_{n+1}}{2}) + 6 + i)},$$

where $\xi_1 = \omega_1$, ξ_n is odd, and $\xi_{n+1} = 2\sigma(\lfloor \xi_n/2 \rfloor) + 8g - 5$. The analogous statement to (19) is now

$$I_{\bar{P}}^\omega = T_k^{-1} T_{\sigma(k)+1}^{-1} \left(I_{\bar{Q}}^{(\xi_2, \dots, \xi_{n+1})} \cup I_{\bar{Q}}^{(\xi_2, \dots, \xi_{n+1} + 1)} \cup \bigcup_{i=0}^{16g-16} I_{\bar{Q}}^{(\xi_2, \dots, \xi_n + 1, 2\sigma(\frac{\xi_{n+1}}{2}) + 6 + i)} \right).$$

Notice that it is not possible for the sequence $(\xi_1, \xi_2, \dots, \xi_n)$ to consist entirely of alternating entries $(2\sigma(k) + 2, 2k + 8g - 2)$ because ξ_n is odd. Nor can the sequence $(\xi_1, \xi_2, \dots, \xi_n + 1)$ consist entirely of such alternating even entries because then (ξ_{n-1}, ξ_n) would not be \bar{Q} -admissible.

Therefore, there exists $j < n$ such that the sequence (ξ_1, \dots, ξ_j) stops alternating between $2\sigma(k) + 2$ and $2k + 8g - 2$. We can then express each $T_k^{-1} T_{\sigma(k)+1}^{-1} (I_{\bar{Q}}^{(\xi_2, \dots)})$ above as a \bar{Q} -cylinder of rank $n + 2$, thus making $I_{\bar{P}}^\omega$ a union of $2 + (16g - 17) = 16g - 15$ \bar{Q} -cylinders of rank $n + 2$. This does not affect the last two entries of the \bar{Q} -cylinders from the induction hypothesis, so the structure of the decomposition is as needed.

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