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Approximate solutions of cohomological equations associated with some Anosov flows

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Abstract. The Livshitz theorem reported in 1971 asserts that any C^1 function having zero integrals over *all* periodic orbits of a topologically transitive Anosov flow is a derivative of another C^1 function in the direction of the flow. Similar results for functions of higher differentiability have also appeared since. In this paper we prove a 'finite version' of the Livshitz theorem for a certain class of Anosov flows on 3-dimensional manifolds which include geodesic flows on negatively curved surfaces as a special case.

1. *Notations and statement of the main result*

Let X be a compact 3-manifold. A flow $\{\psi^t\}(t\in\mathbb{R})$ on X is called *contact* if it preserves a contact form Ω, i.e. a differential 1-form such that $\Omega\wedge d\Omega\neq 0$. In this paper we will be concerned with C^∞ contact Anosov flows. A primary example of such a flow is a geodesic flow on SM, the unit tangent bundle to a compact surface M provided with a Riemannian metric of negative curvature. We shall introduce some notations and list basic facts about contact Anosov flows.

$F1$. A flow $\{\psi^t\}$ is *Anosov*, if there exists a continuous $D\psi^t$-invariant splitting of the tangent bundle to X

$$TX=E^0\oplus E^s\oplus E^u,$$

where E^0, E^s and E^u are one-dimensional distributions spanned by unit vector fields ξ^0, ξ^s and ξ^u, and for any Riemannian metric there exist constants a_1, b_1, $\alpha>0$ such that for all $x\in X$ and any positive real number t

$$\begin{aligned}\|D\psi^t\xi^s(x)\|&\le a_1\,e^{-\alpha t},\\ \|D\psi^t\xi^u(x)\|&\ge b_1\,e^{\alpha t}.\end{aligned}\tag{1.1}$$

Here D denotes the differential of the flow, and the norm of a tangent vector is defined by the Riemannian metric on X. We shall also need the estimates on the other side which hold for any smooth flow: there exist constants a_2, b_2, $\delta>0$ such that for all $x\in X$ and any positive real number t

$$\begin{aligned}\|D\psi^t\xi^s(x)\|&\ge a_2\,e^{-\delta t},\\ \|D\psi^t\xi^u(x)\|&\le b_2\,e^{\delta t}.\end{aligned}\tag{1.2}$$

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Let us denote

$$\chi^s(x,t)=\|D\psi^t\xi^s(x)\|,\quad \chi^u(x,t)=\|D\psi^t\xi^u(x)\|.$$

Let $-\lambda(x,t)$ and $\mu(x,t)$ be the logarithmic derivatives of the functions $\chi^s(x,t)$ and $\chi^u(x,t)$, i.e.

$$\frac{\partial\chi^s(x,t)}{\partial t}=-\lambda(x,t)\chi^s(x,t)$$
$$\frac{\partial\chi^u(x,t)}{\partial t}=\mu(x,t)\chi^u(x,t). \tag{1.3}$$

We have

$$\lambda(x,t)=-\lim_{\tau\to0}\frac{\|D\psi^\tau\xi^s(\psi^t x)\|-1}{\tau}=\Lambda(\psi^t x),$$
$$\mu(x,t)=\lim_{\tau\to0}\frac{\|D\psi^\tau\xi^u(\psi^t x)\|-1}{\tau}=M(\psi^t x).$$

The integral curves of E^0 are the orbits of the flow $\{\psi^t\}$. The integral curves of the distribution E^s (E^u) form the *stable* (*unstable*) foliation which is denoted by σ^s (σ^u). We denote the distance on X by d, and the distance along the leaves of the foliations σ^s and σ^u by d^s and d^u respectively.

*F*2. A contact Anosov flow $\{\psi^t\}$ preserves the measure on X defined by the volume element $\Omega\wedge d\Omega$, which is sometimes called the Liouville measure. We assume that a Riemannian metric on X is chosen in such a way that the Riemannian volume on X coincides with the Liouville measure.

*F*3. A contact Anosov flow $\{\psi^t\}$ is topologically transitive, and each leaf of the foliations σ^s and σ^u is uniformly dense, i.e. for any $\rho>0$ there exists $N>0$ such that for any $x\in X$, any $M\ge N$ and $i\in\{s,u\}$ $D^i_M(x)=\{z\in\sigma^i(x)\,|\,d^i(x,z)<M\}$ is ρ-dense in X, i.e. intersects every ball in X of radius ρ [**1**].

*F*4. The distributions E^s and E^u (and therefore foliations σ^s and σ^u) are of class $C^{2-\varepsilon}$ for any $\varepsilon>0$ [**6**]. In fact, we only use that they are C^1. The latter fact for geodesic flows was known already to Hopf [**4**, § 14], [**5**, § 7]. It follows that $\Lambda(x)$, $M(x)\in C^1(X)$.

We denote the operators of differentiation in the directions of ξ^0, ξ^s and ξ^u by $\mathcal{D}=\mathcal{D}_0$, $\mathcal{D}_s$ and $\mathcal{D}_u$ respectively. Let α^0, α^s and α^u be differential 1-forms dual to the vector fields ξ^0, ξ^s and ξ^u:

$$\alpha^i(\xi^j)=\delta_{ij},\quad \text{for } i,j\in\{0,s,u\}.$$

Hereafter C and K with various subscripts will denote positive constants which may depend on the manifold X. The dependence on a parameter, if any, will be specified.

THEOREM 1.1. (Finite Livshitz Theorem.) *Let X be a compact 3-manifold, $\{\psi^t\}$ be a contact Anosov flow on X, and $T>0$. Then for any λ, $0<\lambda<\alpha/\delta$ there exists a constant $C(\lambda)$ such that if $f\in C^2(X)$, $\|f\|_{C^2}=1$, and $\int_{[o]}f\,dt=0$ for all periodic orbits $[o]$ of $\{\psi^t\}$ of length $\le T$, then there exist F, $h\in C^{1+\lambda}(X)$ such that $f=\mathcal{D}F+h$, and $\|h\|_{C^1}\le C(\lambda)T^{-\lambda/(3-\lambda)}$.*

Remarks. 1. Notice that a weak form of Theorem 1.1 (without an explicit estimate of how $\|h\|_{C^1}$ tends to 0 as $T\to\infty$) follows immediately from the Livshitz† theorem [**7**] and the fact that the unit sphere in C^2 is compact in C^1 (the Ascoli–Arzelà Theorem).

2. For results similar to the Livshitz theorem for functions of higher differentiability see [**3**], [**8**] and [**6**].

2. *Construction of a Hölder continuous differential form on X*

Let us fix a Riemannian metric on X as in (F2), and define the following functions

$$k^0(x)=f(x),\ k^s(x)=-\int_0^\infty \mathcal{D}_s f(\psi^t x)\chi^s(x,t)\,dt,\ k^u(x)=-\int_0^{-\infty}\mathcal{D}_u f(\psi^t x)\chi^u(x,t)\,dt.$$

It follows from (1.1) and (1.2) that these integrals converge. We define a differential 1-form $\omega_f=\omega$ associated to the function f by the formula $\omega=\omega^0+\omega^s+\omega^u$, where $\omega^0=k^0(x)\alpha^0$, $\omega^s=k^s(x)\alpha^s$, $\omega^u=k^u(x)\alpha^u$. For notational simplicity in most cases we will suppress the dependence ω_f on f. The following theorem holds for all contact Anosov flows.

THEOREM 2.1. *The differential form ω_f satisfies a Hölder condition of order λ for any λ, $0<\lambda<1$.*

Proof. In view of (F4), it is sufficient to prove that each form ω^0, ω^s and ω^u satisfy a Hölder condition, i.e. that for any λ, $0<\lambda<1$, there exists $C_0(\lambda)>0$ such that for i, $j\in\{0,s,u\}$ and $x'\in\sigma^j(x)$ $|k^i(x)-k^i(x')|\le C_0(\lambda)d^j(x,x')^\lambda$. We shall make calculations for $i=s$ and leave the other cases to the reader. Let $d^j(x,x')=d$. If $j=0$, s we choose $T=T(x)$ such that

$$\chi^s(x,T(x))=d. \tag{2.1}$$

If $j=u$ we choose $T=T(x)$ such that

$$d^u(\psi^T x,\psi^T x')=1. \tag{2.2}$$

Let us parametrize the piece of the leaf $\sigma^u(x)$ between x and x' by a parameter u as follows: $u(x)=0$, $u(x'')=d^u(x,x'')$, $x''\in[x,x']$, $x''\in\sigma^u(x)$, and let $l(u,t)=d^u(\psi^t x,\psi^t x'')$.

It follows from (1.1) and (1.2) that

$$a_2\,e^{-\delta(T-t)}\le d^u(\psi^t x,\psi^t x'')\le a_1\,e^{-\alpha(T-t)}.$$

It follows from (1.3) that for any $x''\in[x,x']$

$$\chi^s(x'',t)=\exp\int_0^t-\lambda(x'',\tau)\,d\tau. \tag{2.3}$$

Therefore

$$\frac{\chi^s(x'',t)}{\chi^s(x,t)}=\exp\int_0^t(\lambda(x,\tau)-\lambda(x'',\tau))\,d\tau\le\exp C_1\int_0^t d^u(\psi^\tau x,\psi^\tau x'')\,d\tau.$$

For our choice of T, (2.2), we have for $0\le t\le T$:

$$\int_0^t d^u(\psi^\tau x,\psi^\tau x'')\,d\tau\le C_2,$$

and therefore $\chi^s(x'',t)/\chi^s(x,t)\le C_3$. The same argument shows that

† A phoenetic transliteration of his name, Livčic, appears in the translations of his papers from Russian into English.

$\chi^s(x, t)/\chi^s(x'', t) \le C_3$. Hence

$$C_3^{-1} \le \frac{\chi^s(x'', t)}{\chi^s(x, t)} \le C_3 \tag{2.4}$$

for $0 \le t \le T$ and any $x'' \in [x, x']$. Since the flow $\{\psi^t\}$ preserves the Liouville measure (F2), we have $C_4^{-1} \le \chi^s(x'', t)\chi^u(x'', t) \le C_4$, and therefore

$$C_5^{-1} \le \frac{\chi^u(x'', T)}{\chi^u(x, T)} \le C_5.$$

We have $\partial l(u(x''), t)/\partial u = \|D\psi^t \xi^u(x'')\| = \chi^u(x'', t)$, and

$$1 = d^u(\psi^T x, \psi^T x') = l(d, T)$$

$$= \int_0^d \frac{\partial l}{\partial u}(u, T)\, du \le C_6 d\chi^u(x, T) = C_7 d(\chi^s(x, T)^{-1}, \tag{2.5}$$

which implies

$$\chi^s(x, T) \le C_7 d, \chi^s(x', T) \le C_8 d. \tag{2.6}$$

We have

$$|k^s(x) - k^s(x')| = \left| \int_0^\infty (\mathscr{D}_s(f(\psi^t x))\chi^s(x, t) - (\mathscr{D}_s(f(\psi^t x'))\chi^s(x', t))\, dt \right|$$

$$\le \int_0^\infty |\mathscr{D}_s(f(\psi^t x)) - \mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$+ \int_0^\infty |\mathscr{D}_s(f(\psi^t x'))| \cdot |\chi^s(x, t) - \chi^s(x', t)|\, dt$$

$$\le \int_0^T |\mathscr{D}_s(f(\psi^t x)) - \mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$+ \int_0^T |\mathscr{D}_s(f(\psi^t x'))| \cdot |\chi^s(x, t) - \chi^s(x', t)|\, dt$$

$$+ \int_T^\infty |\mathscr{D}_s(f(\psi^t x)) - \mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$+ \int_T^\infty |\mathscr{D}_s(f(\psi^t x'))| \cdot |\chi^s(x, t) - \chi^s(x', t)|\, dt$$

$$\le \int_0^T |\mathscr{D}_s(f(\psi^t x)) - \mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$+ \int_0^T |\mathscr{D}_s(f(\psi^t x'))| \cdot |\chi^s(x, t) - \chi^s(x', t)|\, dt$$

$$+ \int_T^\infty |\mathscr{D}_s(f(\psi^t x))|\chi^s(x, t)\, dt + 2\int_T^\infty |\mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$+ \int_T^\infty |\mathscr{D}_s(f(\psi^t x'))|\chi^s(x', t)\, dt.$$

We claim that $\chi^j(x', T) \le C_9 d$. For $j = u$ this was proved above (2.6). For $j = 0$, s this follows from the choice of T, (2.1), and the inequalities

$$d^j(\psi^t x, \psi^t x') \le C_{10} d, \tag{2.7}$$

and

$$|\chi^s(x, T) - \chi^s(x', T)| \le C_{11} d^j(\psi^T x, \psi^T x') \le C_{12} d.$$

It follows from (1.1) that in both cases

$$T \le C_{13}|\ln d|. \tag{2.8}$$

Thus, (1.3), (2.4), (2.6) and $\|f\|_{C^2} = 1$ imply that each of the last three integrals is estimated from above by $C_{14}\int_T^\infty \chi^s(x, t)\, dt \le C_{15}\chi^s(x, T) \le C_{16} d$.

We estimate now the first two integrals. There exists $\theta(t) \in [x, x']$ such that

$$\int_0^T |\mathscr{D}_s(f(\psi^t x)) - \mathscr{D}_s(f(\psi^t x'))|\chi^s(x, t)\, dt$$

$$= \int_0^T |\mathscr{D}_s\mathscr{D}_j(f(\psi^t\theta(t)))| d^j(\psi^t x, \psi^t x')\chi^s(x, t)\, dt.$$

Let $j = 0$, s. The inequalities (1.1) and (1.2) imply that

$$\int_0^T \chi^s(x, t)\, dt \le C_{17},$$

and using (2.7) we estimate the first integral from above by $C_{18} d$. The second integral in this case is estimated as follows:

$$\int_0^T |\mathscr{D}_s(f(\psi^t x'))| \cdot |\chi^s(x, t) - \chi^s(x', t)|\, dt \le C_{19}\int_0^T d^j(\psi^t x, \psi^t x')\, dt \le C_{20}\, dT$$

$$\le C_{21} d \cdot |\ln d| \le C_{21}C_{22}(\lambda) d^\lambda$$

for any λ, $0 < \lambda < 1$. The last two inequalities follow from (2.8) and the fact that for any λ, $0 < \lambda < 1$ there exists $C_{22}(\lambda)$ such that $d \cdot |\ln d| \le C_{22}(\lambda) d^\lambda$. Now let $j = u$. A calculation similar to (2.5) gives us

$$d^u(\psi^t x, \psi^t x') = d \cdot \chi^u(x_0, t) \le C_4 d \cdot (\chi^s(x_0, t))^{-1},$$

where $x_0 \in [x, x']$, $x_0 \in \sigma^u(x)$. This equality, together with (2.4) and (2.8), implies that the first integral in this case is estimated from above by

$$C_{23}\int_0^T |d^u(\psi^t x, \psi^t x')\chi^s(x, t)\, dt \le C_{24}\, dT \le C_{25} d \cdot |\ln d| \le C_{26}(\lambda) d^\lambda$$

for any λ, $0 < \lambda < 1$. The second integral is estimated from above by $C_{27}\int_0^T |\chi^s(x, t) - \chi^s(x', t)|\, dt$. We use (1.3) and (2.3) to estimate the integrand

$$|\chi^s(x, t) - \chi^s(x', t)| = \left| \exp\int_0^t -\lambda(x, \tau)\, d\tau - \exp\int_0^t -\lambda(x', \tau)\, d\tau \right|$$

$$= \left(\exp\int_0^t -\lambda(x, \tau)\, d\tau \right) \cdot \left| 1 - \exp\int_0^t (\lambda(x, \tau) - \lambda(x', \tau))\, d\tau \right|$$

$$\le C_{28}\chi^s(x, t) \cdot \left| \int_0^t (\lambda(x, \tau) - \lambda(x', \tau))\, d\tau \right|$$

$$\le C_{29}\chi^s(x, t)\int_0^t d^u(\psi^\tau x, \psi^\tau x')\, d\tau$$

$$\le C_{30}\chi^s(x, t)\int_0^t d \cdot (\chi^s(x, \tau))^{-1}\, d\tau$$

$$\le C_{31}\chi^s(x, t)(\chi^s(x, t))^{-1} \cdot d \le C_{32} d.$$

Thus the second integral is estimated by $C_{32}\,dT \le C_{33}d\cdot|\ln d| \le C_{34}(\lambda)d^{\lambda}$ for any λ, $0<\lambda<1$, and the theorem follows. □

3. *Construction of an ε-dense orbit for the flow $\{\psi^t\}$*

THEOREM 3.1. *Given $\varepsilon>0$ sufficiently small, there exists an ε-dense piece of orbit of the flow $\{\psi^t\}$ of length T:*

$$\mathcal{O}=\{x\in X, x=\psi^t x_0, 0\le t\le T\}$$

with $T=C\ln\varepsilon^{-1}/\varepsilon^2$ where the constant $C>0$ depends only on the Riemannian metric on the manifold X and the flow $\{\psi^t\}$.

Proof. We prove first that any two points $x, y\in X$ can be 'ε-joined' by a piece of orbit of length $C_1\ln\varepsilon^{-1}$, i.e. there exist $x', y'\in X$ and a constant $C_1>0$ such that $d(x,x')<\varepsilon$, $d(y,y')<\varepsilon$, and $y'=\psi^T x'$ for $T=C_1\ln\varepsilon^{-1}$. Let ρ be a sufficiently small number which will be specified below. We assume that $\varepsilon<\rho$. By (F3) one can choose a constant $N>0$ such that any piece of the leaf of the foliation σ^u of size $M\ge N$ is $\rho/2$-dense. For $T_1=\alpha^{-1}\ln(N/b_1\varepsilon)$ the piece of the leaf $\psi^{T_1}(D^u_\varepsilon y)=D^u_M(\psi^{T_1}y)$ is of size $M>N$, and therefore is $\rho/2$-dense. Let $T_0=2\alpha^{-1}\ln(a_1\rho/\varepsilon)$, and $x_0=\psi^{-T_0}x$. For small enough ρ there exist $y_0\in D^u_M(\psi^{T_1}y)$ and $z=\psi^t y_0$ with $|t|<\rho$ such that $z\in D^s_\rho(x_0)$. We also have $d(\psi^{T_0}z, \psi^{T_0}x_0)=d(\psi^{T_0}z, x)\le a_1\rho\, e^{-\alpha T_0}<\varepsilon$. Thus we obtained two points $x'=\psi^{T_0}z$ and $y'=\psi^{-T_1}y_0$ such that $d(x,x')<\varepsilon$, $d(y,y')<\varepsilon$, and $y'=\psi^T x'$. If $\varepsilon<\min(1/a_1\rho, b_1/N, e^{-\rho}, \rho)$ then for some constant $C_1>0$ $T=T_0+T_1+d(y_0,z)<C_1\ln\varepsilon^{-1}$.

For each point $x_0\in X$ we define the following cylinder sets:

$$C_\rho(x_0)=\{x=(z,t)\,|\,z\in S_\rho(x_0), -\rho<t<\rho\}, \tag{3.1}$$

where $S_\rho(x_0)$ is a 2-dimensional smooth local cross-section transversal to the flow $\{\psi^t\}$ passing through x_0, and $x=\psi^t z$. To complete the proof of the theorem we choose a cover of X by a finite number of cylinders $C_\rho(x_i)$, $i=0,1,2,\ldots,N$. Let us choose a smooth coordinate system in each local cross-section $S_\rho(x_i)$. Then it makes sense to talk about square lattices in $S_\rho(x_i)$ relative to this coordinate system.

Definition. We call a set of points in $S_\rho(x_i)$ *ε-regular* if it is an ε^2-perturbation of some square lattice in $S_\rho(x_i)$ of size $\varepsilon/2$.

For each $i=0,1,2,\ldots,N$ we choose an ε-regular set $E_i\subset S_\rho(x_i)$, and let $\Lambda_i=\{x\in C_\rho(x_i)\,|\,z\in E_i\}$. The number of pieces in $\bigcup_{i=0}^N\Lambda_i$ is C_2/ε^2. We can 'ε^2-join' them together using the estimate in the beginning of the proof. An application of the Bowen specification theorem [2] gives us a desired ε-dense piece of orbit of length $C\ln\varepsilon^{-1}/\varepsilon^2$ which we denote by $\mathcal{O}$. □

Remark. By the Bowen specification theorem [2] for each $i=0,1,\ldots,N$, $\mathcal{O}$ contains a subset $\bar\Lambda_i$ consisting of a number of pieces approximating pieces of orbits constituting Λ_i. Let $\mathcal{S}=\bigcup_{i=0}^N\bar\Lambda_i$. Notice that $\mathcal{S}\cap S_\rho(x_i)$ is also an ε-regular set for each $i=0,\ldots,N$.

4. *The proof of Theorem* 1.1

Definition. Let r be the injectivity radius of X. We say that a function F defined on $\mathscr{E}$, a subset of X, is of class $C^{1+\lambda}_K(0<\lambda<1, K>0)$ if there exists a family of linear functions $l_x(v)$ for $x\in X$, $v\in T_xX$ such that for any $x,y\in\mathscr{E}$, $d(x,y)<r$,

$$|F(y)-F(x)-l_x(v_{xy})|\le Kd(x,y)^{1+\lambda},$$

where $v_{xy}\in T_xX$ is a tangent vector to the geodesic from x to y (on X) of length $d(x,y)$.

LEMMA 4.1. *Let $\mathcal{O}$ be a piece of orbit of the flow $\{\psi^t\}$ of length T, and $f\in C^2(X)$ is such that $\|f\|_{C^2}=1$ and $\int_{[o]}f\,dt=0$ for all periodic orbits $[o]$ of $\{\psi^t\}$ of length $\le T$. We define the following function on $\mathcal{O}$: for $x=\psi^t x_0$, $0\le t\le T$*

$$F(x)=\int_0^t f(\psi^s x_0)\,ds. \tag{4.1}$$

Then for any λ, $0<\lambda<\alpha/\delta$ there exists $K_0(\lambda)$ such that $F(x)$ is of class $C^{1+\lambda}_{K_0(\lambda)}$ on $\mathcal{O}$.

Proof. We shall show that the role of the linear function l_x is played by the differential form ω_f introduced in § 2. For $j=s,u$ we define $\sigma^{0,j}(x)=\{y=\psi^t z, -\infty<t<\infty$; for some $z\in\sigma^j(x)\}$; they are called leaves of the *weak stable* (for $j=s$) and the *weak unstable* (for $j=u$) foliations. Let $d^{0,j}$ denote the distance on $\sigma^{0,j}$, ρ be as in § 3, and $D^{0,j}_\rho(x)=\{y\in\sigma^{0,j}(x)\,|\,d^{0,j}(x,y)<\rho\}$. Suppose $x_0=x$, $y_0=y\in\mathcal{O}$, $y_0=\psi^{T'}x_0$, $T'\le T$, $d(x_0,y_0)=d<r$. Let $z_0=z=D^s_\rho(x_0)\cap D^{0,u}_\rho(y_0)$, and $y_{00}=\psi^{T_0}x_0$, $|T_0-T'|<d$, $y_{00}\in D^u_\rho(z_0)$. We denote the arc of $\sigma^s(x)$ between x and z by $\sigma^s(x,z)$, the arc of $\sigma^u(y_{00})$ between y_{00} and z by $\sigma^u(y_{00},z)$, and the piece of orbit $\mathcal{O}$ between y and y_{00} by $\mathcal{O}(y,y_{00})$. Notice that

$$\begin{aligned}\int_{\sigma^s(x,z)}\omega_f&=\int_0^\infty(f(\psi^t x)-f(\psi^t z))\,dt,\\ \int_{\sigma^u(y_{00},z)}\omega_f&=\int_0^{-\infty}(f(\psi^t y_{00})-f(\psi^t z))\,dt,\\ \int_{\mathcal{O}(y,y_{00})}\omega_f&=\int_0^{T_0-T}f(\psi^t y)\,dt.\end{aligned}\tag{4.2}$$

The fact that ω_f satisfies a Hölder condition of order λ (Theorem 2.1) and (4.2) imply that Lemma 4.1 follows from the following statement: given λ, $0<\lambda<\alpha/\delta$, there exists a constant $K(\lambda)$ such that for any $x,y\in\mathcal{O}$, $y=\psi^{T_0}x$, $T_0\le T$, $d(x,y)=d<r$ with the property that $D^s_\rho(x_0)\cap D^u_\rho(y_0)\ne\varnothing$, there exists a constant $K(\lambda)$ such that for $D^s_\rho(x_0)\cap D^u_\rho(y_0)=z$

$$\left|(F(y)-F(x))-\left(\int_0^\infty(f(\psi^t x)-f(\psi^t z))\,dt+\int_0^{-\infty}(f(\psi^t z)-f(\psi^t y))\,dt\right)\right|\le K(\lambda)\,d(x,y)^{1+\lambda}.$$

We notice that it is sufficient to prove the above statement for sufficiently small d. Without loss of generality we may assume that $x=x_0$ and therefore $F(x)=0$. We construct five sequences of points $\{x_j\}$, $\{y_j\}$, $\{y_{jj}\}$, $\{z_j\}$ and $\{u_j\}$, and a sequence of numbers $\{T_j\}$ $(j=0,1,2,\ldots)$ inductively as follows:

$$\begin{aligned}&x_0=x,\quad y_0=y_{00}=y=\psi^{T_0}x_0,\quad z_0=z=D^s_\rho(x_0)\cap D^{0,u}_\rho(y_0),\\ &y_{00}\in D^u_\rho(z_0),\quad u_0\in D^u_\rho(x_0),\quad \psi^{T_0}u_0=z_0,\\ &x_j=D^s_\rho(u_{j-1})\cap D^{0,u}_\rho(y_{j-1,j-1})=D^{0,u}_\rho(z_{j-1}),\quad y_j=\psi^{T_{j-1}}x_j,\\ &z_j=D^s_\rho(x_j)\cap D^{0,u}_\rho(y_j),\quad y_{jj}=\psi^{T_j}x_j,\\ &y_{jj}\in D^u_\rho(z_j),\quad u_j\in D^u_\rho(x_j),\quad \psi^{T_j}u_j=z_j\end{aligned}\tag{4.3}$$

Let $x \in X$, $S_\rho(x)$ be as in § 3, and $\varphi : S_\rho(x) \to S_\rho(x)$ be a return map for the flow $\{\psi^t\}$. For any $y \in S_\rho(x)$ we let $P^j_\rho(y) = \sigma^{0,j}(y) \cap S_\rho(x)$ for $j = s, u$. The local foliations P^j_ρ are stable and unstable foliations for the return map φ. They inherit the smoothness of the weak foliations $\sigma^{0,j}$ which is not less than the smoothness of foliations σ^j. Let l^j be the distance on the leaves of the local foliation P^j_ρ $(j = s, u)$. We assume that ρ is chosen such that there exists a constant $C_1 > 1$ such that for any $x, y \in P^j_\rho$

$$C_1^{-1} < \frac{l^j(x, y)}{d(x, y)} < C_1.$$

Thus it follows from (F4) that there exists a constant $C_2 > 1$ such that for any 'quadrangle' $[x_1, x_2, x_3, x_4]$ such that

$$x_2 \in P^u_\rho(x_1),\ x_3 \in P^s_\rho(x_2),\ x_4 \in P^u_\rho(x_3),\ x_1 \in P^s_\rho(x_4), \tag{4.4}$$

the following inequalities hold:

$$C_2^{-1} \le \frac{d(x_1, x_2)}{d(x_4, x_3)} \le C_2, \quad C_2^{-1} \le \frac{d(x_2, x_3)}{d(x_1, x_4)} \le C_2. \tag{4.5}$$

Let $\pi : C_\rho(x_0) \to S_\rho(x_0)$ be defined by the formula $\pi(z, t) = z$, and $S_j = T_0 + \cdots + T_{j-1}$. By properties (1.1) and (1.2), for some constants C_3, C_4, C_5, $C_6 > 0$, and $j = 0, 1, \ldots$ we have

$$\begin{gathered} C_3\, e^{-\delta S_j} d \le d(\pi x_j, \pi y_{jj}) \le C_4\, e^{-\alpha S_j} d, \\ d(\pi x_j, x_0) < C_5 d,\ d(\pi y_j, x_0) < C_5, \\ |T_j - T_0| \le C_6 d. \end{gathered} \tag{4.6}$$

Therefore the sequences $\{\pi x_j\}$, $\{\pi y_j\}$, $\{\pi y_{jj}\}$, $\{\pi z_j\}$ and $\{\pi u_j\}$ converge in $S_\rho(x_0)$ to a fixed point of the continuous map $\varphi : S_\rho(x) \to S_\rho(x)$. The orbit of the flow $\{\psi^t\}$ passing through this point is a periodic orbit. This completes a well-known proof of the Anosov closing lemma. For our purposes, however, we have to look closely at the rate of convergence of this process. Let k be an integer, $k \ge 1$ (it will be chosen later, in (4.7)). We have

$$F(y) = \int_0^{T_0} f(\psi^t x)\, dt = \sum_{j=0}^{k-1} \left(\int_0^{T_0} (f(\psi^t x_j) - f(\psi^t z_j))\, dt + \int_0^{T_0} (f(\psi^t z_j) - f(\psi^t x_{j+1}))\, dt \right) + \int_0^{T_0} f(\psi^t x_k)\, dt.$$

On the other hand,

$$\int_0^\infty (f(\psi^t x) - f(\psi^t z))\, dt = \sum_{j=0}^{k-1} \int_{S_j}^{S_{j+1}} (f(\psi^t x) - f(\psi^t z))\, dt + \int_{S_k}^\infty (f(\psi^t x) - f(\psi^t z))\, dt,$$

and

$$\int_0^{-\infty} (f(\psi^t z) - f(\psi^t y))\, dt = \sum_{j=0}^{k-1} \int_{-S_j}^{-S_{j+1}} (f(\psi^t z) - f(\psi^t y))\, dt + \int_{-S_k}^{-\infty} (f(\psi^t z) - f(\psi^t y))\, dt.$$

An easy calculation gives us the following estimate:

$$\left| F(y) - \left(\int_0^\infty (f(\psi^t x) - f(\psi^t z))\, dt + \int_0^{-\infty} (f(\psi^t z) - f(\psi^t y))\, dt \right) \right| \le \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + I_1 + I_2 + I_3,$$

where

$$\Sigma_1 = \left| \sum_{j=1}^{k-1} \int_0^{T_j} (f(\psi^t x_j) - f(\psi^t z_j)) - (f(\psi^{S_j+t} x) - f(\psi^{S_j+t} z))\, dt \right|$$

$$\Sigma_2 = \left| \sum_{j=0}^{k-1} \int_0^{T_j} (f(\psi^t z_j) - f(\psi^t x_{j+1})) - (f(\psi^{t-S_{j+1}} y) - f(\psi^{t-S_{j+1}} z))\, dt \right|,$$

$$\Sigma_3 = \left| \sum_{j=0}^{k-1} \int_{T_0}^{T_j} (f(\psi^t x_j) - f(\psi^t z_j))\, dt \right|, \quad \Sigma_4 = \left| \sum_{j=0}^{k-1} \int_{T_0}^{T_j} (f(\psi^t z_j) - f(\psi^t x_{j+1}))\, dt \right|,$$

$$I_1 = \left| \int_0^{T_0} f(\psi^t x_k)\, dt \right|, \quad I_2 = \left| \int_{S_k}^\infty (f(\psi^t x) - f(\psi^t z))\, dt \right|,$$

$$I_3 = \left| \int_{-S_k}^{-\infty} (f(\psi^t z) - f(\psi^t y))\, dt \right|.$$

It follows from (4.3) that for $j \ge 0$

$$\pi\psi^{S_j} x \in P^u_\rho(\pi x_j), \quad \pi\psi^{S_j} z \in P^u_\rho(\pi z_j), \quad \pi\psi^{S_j} z \in P^s_\rho(\pi\psi^{S_j} x), \quad \text{and}$$
$$\pi\psi^{-S_{j+1}} y \in P^s_\rho(\pi z_j), \quad \pi\psi^{-S_{j+1}} z \in P^s_\rho(\pi x_{j+1}), \quad \pi\psi^{-S_{j+1}} y \in P^u_\rho(\pi\psi^{-S_{j+1}} z).$$

Therefore the two 'quadrangles' $Q_1(t) = [\pi\psi^t x_j, \pi\psi^t z_j, \pi\psi^{S_j+t} z, \pi\psi^{S_j+t} x]$ and $Q_2(t) = [\pi\psi^t z_j, \pi\psi^t x_{j+1}, \pi\psi^{t-S_{j+1}} z, \pi\psi^{t-S_{j+1}} y]$ satisfy (4.4), and we obtain the following estimates for the lengths of their sides:

$$\begin{gathered} d(\pi\psi^t x_j, \pi\psi^{S_j+t} x) \le C_7 d(\pi\psi^{T_j} x_j, \pi\psi^{S_j}\psi^{T_j} x) \le C_8\, e^{\delta S_j} d \le C_8\, e^{\delta S_{k-1}} e, \\ d(\pi\psi^t x_j, \pi\psi^t z_j) \le C_9 d(\pi x_j, \pi z_j) \le C_{10}\, e^{-\alpha S_j} d, \\ d(\pi\psi^t z_j, \pi\psi^{t-S_{j+1}} y) \le C_{11} d(\pi z_j, \pi\psi^{-S_{j+1}} y) \le C_{12}\, e^{\delta S_j} d \le C_{12}\, e^{\delta S_{k-1}} d, \end{gathered}$$

and

$$d(\pi\psi^t z_j, \pi\psi^t x_{j+1}) \le C_{13} d(\pi\psi^{T_j} z_j, \pi\psi^{T_j} x_{j+1}) \le C_{14} d(\varphi\pi z_j, \varphi\pi x_{j+1}) \le C_{15}\, e^{-\alpha S_j} d.$$

If d is sufficiently small, we can choose $k \ge 1$ satisfying the following inequalities

$$(\max C_{12}, C_8)^{-1} \rho\, e^{-\delta S_k} \le d < e^{-\delta S_{k-1}} \rho (\max C_{12}, C_8)^{-1}, \tag{4.7}$$

it will follow that $Q_1(t), Q_2(t) \subset S_\rho(x)$, and therefore (4.5) applies. We have $e^{S_{k-1}} \le C_{16} d^{-1/\delta}$, $S_{k-1} \le C_{17} \ln d^{-1}$. Thus, for some point $\theta(t) \in Q_1(t)$ we have

$$\begin{aligned} &\left| \sum_{j=1}^{k-1} \int_0^{T_j} (f(\pi\psi^t x_j) - f(\pi\psi^t z_j)) - (f(\pi\psi^{S_j+t} x) - f(\pi\psi^{S_j+t} z))\, dt \right| \\ &\quad \le C_{18} \sum_{j=1}^{k-1} \left| \int_0^{T_j} \mathscr{D}_u \mathscr{D}_s (f(\theta(t))) \cdot d(\pi\psi^t x_j, \pi\psi^t z_j) \cdot d(\pi\psi^t x_j, \pi\psi^{S_j+t} x)\, dt \right| \\ &\quad \le C_{19}(k-1) T d^2\, e^{S_{k-1}(\delta-\alpha)} \le C_{20} d^{\alpha/\delta} d \cdot \ln d^{-1} \le C_{21}(\lambda) d^{1+\lambda}. \end{aligned}$$

and therefore $\Sigma_1 \le C_{22}(\lambda) d^{1+\lambda}$ for any λ, $0<\lambda<\alpha/\delta$. Similarly we obtain the estimate $\Sigma_2 \le C_{23}(\lambda) d^{1+\lambda}$ for any λ, $0<\lambda<\alpha/\delta$.

In order to estimate I_1 we let

$$\mathcal{O}(x_k)=\{x\in X, x=\psi^t x_k, 0\le t\le T_0\}.$$

By the Anosov closing lemma, using the fact that $d(x_k, \psi^{T_k}x_k)=d(x_k, y_{kk})\le C_1 e^{-\alpha S_k}d$, and that the integral of $f(x)$ over any closed orbit of length $\le T$ is equal to zero, we obtain the following estimate:

$$I_1=\left|\int_0^{T_0} f(\psi^t x_k)\,dt\right| \le C_{24}\, e^{-\alpha S_k}\cdot d \le C_{25} d^{1+(\alpha/\delta)}.$$

The last inequality follows from (4.7). Following the previous argument we conclude that

$$I_2=\left|\int_{S_k}^{\infty} (f(\psi^t x)-f(\psi^t z))\,dt\right| \le C_{26}\int_{S_k}^{\infty} e^{-\alpha t}\,dt\cdot d=\frac{C_{27}}{\alpha}e^{-\alpha S_k}\cdot d\le C_{28}d^{1+(\alpha/\delta)}.$$

Similarly,

$$I_3=\left|\int_{-S_k}^{-\infty} (f(\psi^t z)-f(\psi^t y))\,dt\right| \le C_{29}\left|\int_{-S_k}^{-\infty} e^{\delta t}\,dt\right| =\frac{C_{30}}{\delta}e^{-\delta S_k}\cdot d\le C_{31}d^2.$$

Using (4.6) we obtain the following estimates for Σ_3 and Σ_4:

$$\Sigma_3=\left|\sum_{j=0}^{k-1}\int_{T_0}^{T_j} (f(\psi^t x_j)-f(\psi^t z_j))\,dt\right| \le C_{32}d\sum_{j=0}^{k-1} d(\pi\psi^T x_j, \pi\psi^T z_j)\le C_{33}d^2,$$

$$\Sigma_4=\left|\sum_{j=0}^{k-1}\int_{T}^{T_j} (f(\psi^t z_j)-f(\psi^t x_{j+1}))\,dt\right| \le C_{34}d^2.$$

This concludes the proof of Lemma 4.1. □

Corollary. *$F(x)$ is of class $C^{1+\lambda}_{K_0(\lambda)}$ on the set $\mathcal{S}$ (cf., Remark in § 3), for any λ, $0<\lambda<\alpha/\delta$.*

Lemma 4.2. *For any λ, $0<\lambda<\alpha/\delta$ there exist constants $K_1(\lambda)$, $K_2(\lambda)$, $K_3(\lambda)$ such that the function $F(x)$ can be extended from $\mathcal{S}$ to X as a function of class $C^{1+\lambda}_{K_1(\lambda)}$ in such a way that $\mathcal{D}F(x)\in C^{1+\lambda}_{K_2(\lambda)}(X)$, and for $h(x)=\mathcal{D}F(x)-f(x)$ we have $\|h(x)\|_{C^1}\le K_3(\lambda)\varepsilon^{\lambda}$.*

Proof. First, we show how to extend $F(x)$ from $\mathcal{S}\cap S_\rho(x_0)$ to $S_\rho(x_0)$ as a function of class $C^{1+\lambda}_{K_1(\lambda)}$. $\mathcal{S}\cap S_\rho(x_0)$ is a discrete ε-regular set. It is sufficient to extend $F(x)$ to a 'generating' quadrangle of $\mathcal{S}\cap S_\rho(x_0)$ which is a ε^2-perturbation of a square. We denote its vertices by A, B, C and D, and the directions $\vec{AB}$ and $\vec{AD}$ by x and y respectively. We may assume that F is a real-valued function since the following argument is valid for Re F and Im F. We extend $F(x)$ to the interval $[A, B]$ knowing $F(A)$, $F(B)$ and $F'_x(A)=l_A(v_{AB})=a_A$, $F'_x(B)=-l_B(v_{BA})=a_B$. There exist $t_{AB}\in[A, B]$ and $k_{AB}\in\mathbb{R}$ such that

$$F'_x(w)=\begin{cases} k_{AB}d(w, A)+a_A, & w\in[A, t_{AB}]\\ -k_{AB}d(w, A)+k_{AB}d(A, B)+a_B, & w\in[t_{AB}, B],\end{cases} \tag{4.8}$$

and $\int_A^B F'_x(w)\,dw=F(B)-F(A)$. $F(x)$ is of class $C^{1+\lambda}_{K_0(\lambda)}$. It follows from (4.8) that k_{AB} satisfies the following quadratic equation

$$d(A, B)^2k_{AB}^2-2[2(F(B)-F(A))-d(A, B)(a_A+a_B)]k_{AB}-(a_A-a_B)^2=0. \tag{4.9}$$

A direct calculation shows that

$$|k_{AB}|\le 4K_0(\lambda)d(A, B)^{\lambda-1},$$

Then for w, $w'\in[A, B]$

$$|F'_x(w)-F'_x(w')|\le|k_{AB}|d(w, w')\le 4K_0(\lambda)d(w, w')^{\lambda}.$$

We define $F'_y(w)$ linearly for $w\in[A, B]$:

$$F'_y(w)=\frac{F'_y(A)d(w, B)+F'_y(B)d(A, w)}{d(A, B)}. \tag{4.10}$$

Then $|F'_y(w)-F'_y(w')|\le K_0(\lambda)d(w, w')^{\lambda}$. Thus $F(x)$ is extended to $[AB]$, and analogously to $[BC]$, $[CD]$ and $[DA]$, as a $C^{1+\lambda}_{4K_0(\lambda)}$-function. We parametrize each interval $[AB]$ and $[CD]$ by its normalized length σ. Then we connect points having the same parameter by an interval of a geodesic, obtaining a family of coordinate curves, and extend $F(x)$ to each interval by the formulas (4.8) and (4.10) as a $C^{1+\lambda}_{C_1(\lambda)}$-function for some $C_1(\lambda)>0$. Thus we obtain a function inside the quadrangle $ABCD$. In order to prove that thus defined function is of class $C^{1+\lambda}_{K_1(\lambda)}$ inside $ABCD$, we construct a family of curves connecting intervals $[BC]$ and $[DA]$ as follows. For $\sigma\in[0, 1]$, let $z_\sigma\in[AB]$ and $w_\sigma\in[CD]$ be the points parametrized by σ, and $t_\sigma\in[z_\sigma, w_\sigma]$, $t_\sigma=t_{z_\sigma,w_\sigma}$ as in (4.8). We parametrize each interval $[z_\sigma, t_\sigma]$ and $[t_\sigma, w_\sigma]$ by its normalized length such that $\tau(z_\sigma)=0$, $\tau(t_\sigma)=\frac{1}{2}$, $\tau(w_\sigma)=1$, and $\tau=$ const. gives us the second family of coordinate curves. Let $P=(\sigma_1, \tau_1)$ and $Q=(\sigma_2, \tau_2)$, and $R=(\sigma_2, \tau_1)$. There exist $C_2(\lambda)$, $C_3(\lambda)$, $K_1(\lambda)>0$ such that for $i=\sigma, \tau$

$$\begin{aligned}|F'_i(Q)-F'_i(P)|&\le|F'_i(Q)-F'_i(R)|+|F'_i(R)-F'_i(P)|\\ &\le C_2(\lambda)d(Q, R)^{\lambda}+C_3(\lambda)d(R, P)^{\lambda})\le K_1(\lambda)d(P, Q)^{\lambda}.\end{aligned}$$

We use (4.8), (4.9) and (4.10) to obtain the second inequality. The last inequality follows from the regularity of the quadrangle $ABCD$ and the fact that the function d^{λ} is concave down for any λ, $0<\lambda\le 1$.

Let us choose a finite cover of X by cylinders $C_\rho(x_i)$, $i=0,\dots,N$ introduced in § 3. We extend $F(x)$ by the formula

$$F(\psi^t x)=\int_0^t f(\psi^s x)\,ds+F(x), \quad -\rho<t<\rho$$

to a $C^{1+\lambda}_{K_1(\lambda)}$-function on each $C_\rho(x_i)$. Thus for $i=0,\dots,N$ we obtain a function $F_i(x)$ defined on $C_\rho(x_i)$ and such that $F_i(x)=F(x)$ for $x\in\bar\Lambda_i$. Let $\{\lambda_0(x),\dots,\lambda_N(x)\}$, $\sum_{i=0}^N\lambda_i(x)=1$ be a C^∞ partition of unity corresponding to the cover $\{C_\rho(x_i)\}$, and $\bar F(x)=\sum_{i=0}^N\lambda_i(x)F_i(x)$. For

$$x\in\bigcap_{k=1}^M C_\rho(x_{i_k}), \quad \mathcal{D}\bar F(x)=\sum_{k=1}^M \mathcal{D}\lambda_{i_k}(x)F_{i_k}(x)+\lambda_{i_k}(x)\mathcal{D}F_{i_k}(x).$$

By construction, on each $C_\rho(x_{i_k})$ we have $\mathscr{D}F_{i_k}(x)=f(x)$ and $\lambda_i(x)=0$ if $i\neq i_k$. Thus

$$\mathscr{D}\bar{F}(x)=\sum_{k=1}^{M}\mathscr{D}\lambda_{i_k}(x)F_{i_k}(x)+f(x),$$

and therefore $\mathscr{D}\bar{F}(x)$ is of class $C^{1+\lambda}_{K_2(\lambda)}$ for some $K_2(\lambda)>0$.

Now we estimate the C^1-norm of $h(x)=\mathscr{D}\bar{F}(x)-f(x)$. Let $x\in\bigcap_{k=1}^{M}C_\rho(x_{i_k})$. If $M=1$, $\bar{F}(x)=F_{i_1}(x)$, hence $\mathscr{D}\bar{F}(x)=f(x)$ and $h(x)=0$ in some neighborhood of the point x. Therefore, in the open set $X\setminus\bigcup_{j\neq i_1}C_\rho(x_j)$ $\|h(x)\|_{C^1}=0$. Suppose $M>1$. We notice that $\sum_{k=1}^{M}\mathscr{D}\lambda_{i_k}(x)=0$, and therefore $h(x)=\sum_{k=2}^{M}\mathscr{D}\lambda_{i_k}(x)(F_{i_k}(x)-F_{i_1}(x))$. Since the functions λ_{i_k} are of class C^∞, we have in $\bigcap_{k=1}^{M}C_\rho(x_{i_k})$

$$\|h(x)\|_{C^1}\leq C_4\sum_{k=2}^{M}\|F_{i_k}(x)-F_{i_1}(x)\|_{C^1}.$$

There exists a constant $C_5>0$ and two points $y\in\bar{\Lambda}_{i_k}$, $z\in\bar{\Lambda}_{i_1}$ such that $d(x,y)\leq\varepsilon$, $d(x,z)\leq\varepsilon$, $d(y,z)\leq C_5\varepsilon$. In the following estimate we use that the functions $F_{i_k}(x)$, $F_{i_1}(x)$ and $F(x)$ are of class C^1, and that $F_{i_k}(y)=F(y)$, $F_{i_1}(z)=F(z)$.

$$\begin{aligned}|F_{i_k}(x)-F_{i_1}(x)|&\leq|F_{i_k}(x)-F_{i_k}(y)|+|F_{i_1}(z)-F_{i_1}(x)|+|F_{i_k}(y)-F_{i_1}(z)|\\&=|F_{i_k}(x)-F_{i_k}(y)|+|F_{i_1}(z)-F_{i_1}(x)|+|F(y)-F(z)|\leq C_6\varepsilon.\end{aligned}$$

For $j=0$, s, u the functions $\mathscr{D}_jF_{i_k}(x)$ and $\mathscr{D}_jF_{i_1}(x)$ satisfy a Hölder condition of order λ and a constant $K_1(\lambda)$ for any λ, $0<\lambda<1$. By construction (Lemma 4.1) we have $\mathscr{D}_jF_{i_k}(y)=k^j(y)$, $\mathscr{D}_jF_{i_1}(z)=k^j(z)$ (see notations of § 2), and using Theorem 2.1 we obtain the following estimate:

$$\begin{aligned}|\mathscr{D}_jF_{i_k}(x)-\mathscr{D}_jF_{i_1}(x)|&\leq|\mathscr{D}_jF_{i_k}(x)-\mathscr{D}_jF_{i_k}(y)|\\&\quad+|\mathscr{D}_jF_{i_1}(z)-\mathscr{D}_jF_{i_1}(x)|+|\mathscr{D}_jF_{i_k}(y)-\mathscr{D}_jF_{i_1}(z)|\\&\leq C_7(\lambda)\varepsilon^\lambda.\end{aligned}$$

Thus, for some constant $K_3(\lambda)$ we have $\|h(x)\|_{C^1}\leq K_3(\lambda)\varepsilon^\lambda$, and the lemma follows. □

Now we can finish the proof of Theorem 1.1. For any λ, $0<\lambda<1$ there exists a constant $C_8(\lambda)>0$ such that for any $\varepsilon>0$ $\ln\varepsilon^{-1}\leq C_8(\lambda)\varepsilon^{\lambda-1}$. Given $T>0$, let

$$\varepsilon=C^{1/(3-\lambda)}C_8(\lambda)^{1/(3-\lambda)}T^{-1/(3-\lambda)},$$

where C is from Theorem 3.1. We apply Theorem 3.1 to construct an ε-dense piece of orbit $\mathcal{O}$ of length $C\ln\varepsilon^{-1}/\varepsilon^2\leq T$. Defining the function $F(x)$ on $\mathcal{O}$ by formula (4.1) and applying Lemmas 4.1 and 4.2 we obtain a function $h(x)$ with the following estimate on its C^1-norm:

$$\|h\|_{C^1}\leq K_2(\lambda)\varepsilon^\lambda\leq K_2(\lambda)C^{\lambda/(3-\lambda)}C_8(\lambda)^{\lambda/(3-\lambda)}T^{-\lambda/(3-\lambda)}=C(\lambda)T^{-\lambda/(3-\lambda)}.\quad\square$$

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