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Higher cohomology for abelian groups of toral automorphisms II: the partially hyperbolic case, and corrigendum

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Abstract. In this paper we extend the results of an earlier paper, which deal with a description of the smooth untwisted cohomology for $\mathbb{Z}^k$-actions by hyperbolic automorphisms of a torus, to the partially hyperbolic case. Along the way we correct an error found in one of the steps in the proof for the hyperbolic case.

1. *Introduction; formulation of results*

In this paper we extend the results of [**3**], which deal with a description of the smooth untwisted cohomology for $\mathbb{Z}^k$-actions by hyperbolic automorphisms of a torus, to the partially hyperbolic case. Along the way we correct an error found in one of the steps in the proof. Most of the steps in the proof in [**3**] actually hold for the partially hyperbolic case. The principal difference lies in obtaining growth estimates from below for the dual orbits of the action. At the end of [**3**, §3] we indicate an approach to the partially hyperbolic case and acknowledge the fact that the estimates in that case cannot be uniform since the lattice points can be found arbitrary close to the eigenspaces corresponding to eigenvalues of absolute value 1. We outlined a scheme of handling the estimates in the partially hyperbolic case. Ironically, the uniform estimates from below claimed in [**3**, Theorem 3.1] are not correct for the general hyperbolic situation either, although they hold for some special situations, such as the totally non-symplectic (TNS) actions. Thus we need these more subtle estimates already in the hyperbolic case. We use definitions and notation from [**3**] without special notice. In several crucial instances we will provide references to specific places in that paper.

Let A be an invertible $N \times N$ matrix with integer entries. It generates an endomorphism of the N-dimensional torus $\mathbb{T}^N$ which we will denote by the same latter A. The dual endomorphism $\mathbb{Z}^N \to \mathbb{Z}^N$ is given by the transpose matrix ${}^{t}A$. Recall that the following conditions are equivalent:

(1) the endomorphism A is ergodic with respect to Lebesgue measure;
(2) the set of periodic points of A coincides with the set of points in $\mathbb{T}^N$ with rational coordinates;
(3) none of the eigenvalues of the matrix A are roots of unity;
(4) the matrix A has at least one eigenvalue of absolute value greater than one and has no eigenvectors with rational coordinates;
(5) all orbits of the dual map ${}^{t}A : \mathbb{Z}^N \to \mathbb{Z}^N$, other than the trivial zero orbit, are infinite.

Definition. An endomorphism A satisfying properties (1)–(5), as well as its matrix A, is called *partially hyperbolic*.

Notice that the transpose matrix of a partially hyperbolic matrix is partially hyperbolic.

Let α be an action of $\mathbb{Z}^k$ by partially hyperbolic automorphisms of $\mathbb{T}^N$, i.e. any element of the action, other than the identity, is partially hyperbolic. Let $\mathcal{P}(\alpha)$ be the set of all closed (finite) orbits of α. To each $\mathcal{C} \in \mathcal{P}$ one associates the α-invariant measure $\sigma_{\mathcal{C}}$ concentrated on that orbit: $\sigma_{\mathcal{C}} = (1/|\mathcal{C}|) \sum_{x \in \mathcal{C}} \delta_x$. We say that a k-cocycle over α vanishes on $\mathcal{C}$ if $[\varphi]_{\mathcal{C}} := \int_{\mathbb{T}^N} \varphi \, d\sigma_{\mathcal{C}} = 0$.

The new estimates enable us to establish the following results in the partially hyperbolic case.

THEOREM 1.1. *Let α be an action of $\mathbb{Z}^k$ by partially hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ ($\ell \geq 1$) that vanishes on all periodic orbits of $\mathbb{Z}^k$, i.e. $[\varphi]_{\mathcal{C}} = 0$ for each $\mathcal{C} \in \mathcal{P}(\alpha)$. Then for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^k$,*

$$\varphi(x, t) = \mathcal{D}\Phi(x, t), \tag{1}$$

where Φ is a C^∞ $(k-1)$-cochain.

THEOREM 1.2. *Let α be an action of $\mathbb{Z}^k$ by partially hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ n-cocycle over α with values in $\mathbb{R}^\ell$ ($\ell \geq 1$) and $1 \leq n \leq k-1$. Then φ is C^∞-cohomologous to a constant cocycle ψ, i.e. for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^n$,*

$$\varphi(x, t) = \psi(t) + \mathcal{D}\Phi(x, t), \tag{2}$$

where Φ is a C^∞ $(n-1)$-cochain.

Let α be an action of $\mathbb{Z}^k$ by partially hyperbolic automorphisms of $\mathbb{T}^N$, and let β be the dual action on $\mathbb{Z}^N$ with generators $B_1, \ldots, B_k \in GL(N, \mathbb{Z})$, the group of $N \times N$ matrices with integer entries and determinant ± 1. Since $B_1, \ldots, B_k$ are commuting real matrices, the space $\mathbb{R}^N$ can be decomposed into a direct sum of β-invariant subspaces

$$\mathbb{R}^N = \mathbb{I}_1 \oplus \cdots \oplus \mathbb{I}_r, \tag{3}$$

such that the minimal polynomial of B_j on $\mathbb{I}_i$ is a power of an irreducible polynomial (linear or quadratic) over $\mathbb{R}$. According to this decomposition, matrices $B_1, \ldots, B_k$ can be simultaneously brought to the following form with square blocks along the diagonal:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Lambda_{r1} \end{pmatrix}, \ldots, \Lambda_k = \begin{pmatrix} \Lambda_{1k} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Lambda_{rk} \end{pmatrix}.$$

For $1 \le i \le r$ the blocks Λ_{ij} correspond to either real eigenvalues λ_{ij} of B_j or the pairs of complex conjugate eigenvalues $(\lambda_{ij}, \overline{\lambda_{ij}})$. For more details on this decomposition see [**3**, §3].

For each $t = (t_1, \ldots, t_k) \in \mathbb{Z}^k \setminus \{0\}$, $\beta^t = B_1^{t_1} \cdots B_k^{t_k}$ is a partially hyperbolic automorphism, hence $\mathcal{O}(m)$, the orbit of the point $m \in \mathbb{Z}^N \setminus \{0\}$, is of rank k, i.e. $\mathcal{O}(m) \approx \mathbb{Z}^k$. For each $t \in \mathbb{Z}^k$ we have a decomposition of $\mathbb{R}^N$ into a direct sum of expanding, neutral, and contracting subspaces, $\mathbb{R}^N = V_t^+ \oplus V_t^\circ \oplus V_t^-$, such that $\beta^t(V_t^i) = V_t^i$, $i \in \{+, \circ, -\}$. These subspaces are direct sums of $\mathbb{I}_i$s with positive, zero, and negative Lyapunov exponents

$$\chi_i(t) = \sum_{j=1}^{k} t_j \ln|\lambda_{ij}|, \quad i = 1, \ldots, r,$$

respectively. Both V_t^+ and V_t^- are non-trivial for all $t \in \mathbb{Z}^k \setminus \{0\}$. We use the following norms: for $t \in \mathbb{Z}^k$, $\|t\| = \sum_{j=1}^k |t_j|$, and we decompose every $x \in \mathbb{Z}^N$ (or $\mathbb{R}^N$), according to (3), $x = (x_1, \ldots, x_r)$, and let $\|x\| = \sum_{i=1}^r \|x_i\|$, where $\|x_i\|$ is a norm on $\mathbb{I}_i$ (see [**3**, §3] for details).

2. *Orbit growth for the dual action*

The following result replaces Theorem 3.1 from [**3**] whose proof contains an error in the part dealing with the estimate from below. The error was found due to a comment by E. Lindenstrauss who pointed out an inaccuracy in an argument in the original version of [**2**] which was based on the incorrect estimate from below.

Our result here holds in a more general situation but contains a weaker estimate from below.

THEOREM 2.1. *Let α be an action by commuting partially hyperbolic automorphisms of $\mathbb{T}^N$, and β be the dual action. Then there exist constants $a, b, C_1, C_2 > 0$ depending on the action only such that for any initial point $m \in \mathbb{Z}^N$*

$$C_1\|m\|^{-N} \exp(b\|t\|) \le \|\beta^t m\| \le C_2\|m\| \exp(a\|t\|).$$

Remark. The difference with the statement of Theorem 3.1 of [**3**] (other than the latter only refers to the hyperbolic case) is in the estimate from below which is not uniform and that the estimates hold for any initial point m.

Proof. We first establish the estimates in the semisimple case, i.e. when the matrices $B_1, \ldots, B_k$ are simultaneously diagonalizable over $\mathbb{C}$. The estimate from above is a general fact; in particular, it follows from [**3**, Lemma 3.3(2)] for any choice of the initial point m.

Now we proceed to a proof of the crucial estimate from below. Let $V \subset \mathbb{R}^N$ be a β-invariant subspace, and $\Lambda = V \cap \mathbb{Z}^N$. Then Λ is either trivial or infinite. In the latter case $\Lambda \approx \mathbb{Z}^d$ for some $1 \le d \le N$, and $\beta|_\Lambda$ is dual to the restriction of α to an invariant d-dimensional subtorus. Hence it is also partially hyperbolic. This is because for each $t \in \mathbb{Z}^k$ each eigenvalue of $\beta^t|_\Lambda$ is also an eigenvalue of $\beta^t|_V$ and β^t, and if $\beta^t|_\Lambda$ has an eigenvalue which is a root of unity, then so does β^t. Moreover, $\mathbb{R}^d$ spanned by Λ is decomposed into a direct sum of β-invariant subspaces

$$\mathbb{R}^d = \bigoplus_{i\in I} \mathbb{I}'_i,$$

where $I \subset \{1, 2, \ldots, r\}$ and $\mathbb{I}'_i \subset \mathbb{I}_i$, so that the minimal polynomial of B_j on $\mathbb{I}'_i$ divides the minimal polynomial of B_j on $\mathbb{I}_i$. Thus we have $|I|$ Lyapunov exponents for $\beta|_\Lambda$:

$$\chi_i(t) = \sum_{j=1}^{k} t_j \ln|\lambda_{ij}|, \quad i \in I.$$

Each non-trivial β-invariant lattice Λ gives rise to a subset $I \subset \{1, \ldots, r\}$, and hence there are only finitely many types of such lattices. (Notice that there may be infinitely many lattices of the same type.)

We denote the collection of all subsets $I \subset \{1, 2, \ldots, r\}$ obtained by non-trivial β-invariant lattices by $\mathcal{I}_0$; $\mathcal{I}_0 \neq \emptyset$ since it includes $\{1, \ldots, r\}$.

For each Λ we make the following construction. Since $\beta|_\Lambda$ is a partially hyperbolic action, not all χ_i, $i \in I$, are identically zero, and hence, as follows from [**3**, Lemma 3.2], for any $t \in \mathbb{R}^k$ there exists an $i \in I$ such that $\chi_i(t) > 0$. The function $M(t) = \max_{i\in I} \chi_i(t)$ is continuous and achieves its minimum on the unit sphere $S^{k-1} \subset \mathbb{R}^k$ which must be positive by the above argument. Let $b_I = \min_{S^{k-1}} M(t)$. Then for any $t/\|t\| \in S^{k-1}$, $\max_{i\in I} \chi_i(t/\|t\|) \geq b_I$, and hence there exists a $i \in I$ for which $\chi_i(t) \geq b_I\|t\|$. Let $b = \min_{I\in\mathcal{I}_0} b_I$; $b > 0$.

Now let $m \in \mathbb{Z}^N$ be any non-zero initial point. It belongs to a β-invariant lattice Λ of minimal dimension d, $\Lambda \approx \mathbb{Z}^d$, therefore β is irreducible over $\mathbb{Q}$ on $\mathbb{R}^d$ spanned by Λ. Hence $\beta|_{\mathbb{R}^d}$ is separable (has no repeated eigenvalues) since otherwise the minimal polynomial of $\beta|_\Lambda$ would not be relatively prime with its derivative, i.e. the minimal polynomial would factor over $\mathbb{Q}$, and since it is monic, by Gauss' lemma, it would factor over $\mathbb{Z}$, which contradicts the fact that irreducibility of the action implies that the action contains a matrix with irreducible characteristic polynomial [**1**].

Now, for $I \subset \mathcal{I}_0$ corresponding to the lattice Λ we choose an index i as above, so that $\chi_i(t) \geq b_I\|t\| \geq b\|t\| > 0$, and take the corresponding eigenspace $\mathbb{I}'_i$. Then $\mathbb{R}^d = \mathbb{I}'_i \oplus \bigoplus_{j\in I-\{i\}} \mathbb{I}'_j$, where $\beta^t|_{\mathbb{I}'_i}$ and $\beta^t|\bigoplus_{j\in I-\{i\}} \mathbb{I}'_j$ have no common eigenvalues, and also $\bigoplus_{j\in I-\{i\}} \mathbb{I}'_j \cap \mathbb{Z}^d = \{0\}$.

Let m_i be a projection of m to $\mathbb{I}'_i$. Then, by Katznelson's lemma [**4**, Lemma 3], there exists a constant γ_I such that

$$\|m_i\| \geq d(m, V) \geq \gamma_I \|m\|^{-N},$$

where d is the Euclidean distance, and the constant γ_I depends only on the splitting (3) for the action β. Thus, we have

$$\begin{aligned}\|\beta^t m\| &= \sum_{j=1}^{r} \exp \chi_j(t)\|m_j\| \geq \exp \chi_i(t)\|m_i\| \\ &\geq \exp(b\|t\|)\|m_i\| \geq \exp(b\|t\|)\gamma_I\|m\|^{-N}. \end{aligned} \tag{4}$$

So, our estimate holds with $C_1 = \gamma_I$ for any initial point m.

If the action is not semisimple, only the polynomial growth in $\|t\|$ may occur in addition due to the presence of unipotent factors. Thus, the same estimates will hold with slightly smaller b and slightly larger a. This completes the proof of the theorem. □

3. *Estimates for the solution of the coboundary equation*

PROPOSITION 3.1. *Let α be an action of $\mathbb{Z}^k$ by partially hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ ($\ell \geq 1$) such that for any non-trivial dual orbit $\mathcal{O}$, $\sum_{m\in\mathcal{O}} \hat{\varphi}(m) = 0$. Then φ is C^∞-cohomologous to a constant cocycle ψ, i.e. for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^k$,*

$$\varphi(x, t) = \psi(t) + \mathcal{D}\Phi(x, t), \tag{5}$$

where Φ is a C^∞ $(k-1)$-cochain.

Proof. We proceed by first constructing a dual cochain $\hat{\Phi}$ on each non-trivial dual orbit as in [**3**, Proposition 2.2]. We follow the scheme of the proof of [**3**, Proposition 4.1] with modifications due to non-uniformity of the estimates for the growth of dual orbits.

Since the cocycle φ is C^∞ we have the following estimate on the decay of the dual cocycle $\hat{\varphi}$: for any positive integer B there exists $C = C(B)$ such that

$$|\hat{\varphi}(m)| \leq C\|m\|^{-B}. \tag{6}$$

We want to obtain a similar estimate on the decay of each component of the dual cochain $\hat{\Phi}_j$ ($1 \leq j \leq k$). Each $0 \neq m \in \mathbb{Z}^N$ belongs to some dual orbit $\mathcal{O}(m^*)$, where now we choose the initial point m^* to be one of 'the lowest': $\|m^*\| = \min_{s\in\mathbb{Z}^k} \|\beta^s(m^*)\|$. Then $\|m^*\| \geq 1$ and $m = \beta^t m^*$ for some $t \in \mathbb{Z}^k$.

Let $t = (t_1, \ldots, t_k)$. Formula (2.5) of [**3**] shows that $\hat{\Phi}_j(\beta^t m^*) = 0$ if at least one of the coordinates $t_1, \ldots, t_{j-1}$ is not equal to zero, hence it is sufficient to consider only the case when $t_1 = \cdots = t_{j-1} = 0$. Fix $s = (0, \ldots, 0, t_j, \ldots, t_k)$ and consider the following half-lattice

$$\mathbb{H}^j = \{r \in \mathbb{Z}^k \mid r = (r_1, \ldots, r_{j-1}, r_j, 0, \ldots, 0),\ r_j \geq 0 \text{ if } t_j \geq 0,\ r_j < 0 \text{ if } t_j < 0\}.$$

We write $s = \underline{s} + \bar{s}$, where $\underline{s} = (0, \ldots, 0, \underline{s}_j, \ldots, \underline{s}_k)$ with

$$\underline{s}_i = \begin{cases} \dfrac{t_i}{2} & \text{if } t_i \text{ is even,} \\[2mm] \dfrac{t_i - 1}{2} & \text{if } t_i > 0 \text{ is odd,} \\[2mm] \dfrac{t_i + 1}{2} & \text{if } t_i < 0 \text{ is odd,} \end{cases}$$

and $\bar{s} = (0, \ldots, 0, \bar{s}_j, \ldots, \bar{s}_k)$ with

$$\bar{s}_i = \begin{cases} \dfrac{t_i}{2} & \text{if } t_i \text{ is even,} \\[2mm] \dfrac{t_i + 1}{2} & \text{if } t_i \text{ is odd,} \\[2mm] \dfrac{t_i - 1}{2} & \text{if } t_i \text{ is odd.} \end{cases}$$

Then $\|\underline{s}\| \le \|\bar{s}\|$, and, since the norm we use is additive, $\|s + r\| = \|\underline{s}\| + \|\bar{s}\| + \|r\|$. By formula (2.5) of [**3**]

$$|\hat{\Phi}_j(\beta^s m^*)| \le \sum_{r \in \mathbb{H}^j} |\hat{\varphi}(\beta^{r+s} m^*)|. \tag{7}$$

The following method for estimation of the right-hand side of (7) is a generalization and modification of an argument used by Veech [**5**]. Let us fix $t_0 = \lceil ((N+1)/b) \ln \|m^*\| \rceil$ so that the inequality

$$\|m^*\|^{-N} \ge \|m^*\| \exp(-bt_0) \tag{8}$$

holds. Here $\lceil x \rceil$ is the smallest integer greater than x.

We split the right-hand side of (7) into two sums, $S_1(\beta^s m^*)$ and $S_2(\beta^s m^*)$, where $S_1(\beta^s m^*)$ is a finite sum over r such that $\|\underline{s} + r\| < 4t_0$, where we are going to use the simple estimate

$$\|\beta^{r+s} m^*\| \ge \|m^*\|, \tag{9}$$

and $S_2(\beta^s m^*)$ is the infinite sum over r with $\|\underline{s}+r\| \ge 4t_0$, where the exponential estimates of Theorem 2.1 prevail and become uniform.

Estimate of $S_2(\beta^s m^*)$. Since $\|\underline{s} + r\| \ge 4t_0$, then $\|\underline{s}\| \ge 2t_0$, or $\|r\| \ge 2t_0$.

Case 1: $\|r\| \ge 2t_0$. Using (8) we obtain

$$\begin{aligned} \|\beta^{r+s} m^*\| &\ge C_1 \|m^*\|^{-N} \exp(b(\|r+s\|)) \ge C_1 \|m^*\| \exp(b(\|r+s\| - t_0)) \\ &= C_1 \|m^*\| \exp(b(\|s\|)) \exp(b(\|r\| - t_0)). \end{aligned} \tag{10}$$

The estimate from above of Theorem 2.1 gives

$$\|\beta^s m^*\| \le C_2 \|m^*\| \exp(a\|s\|).$$

Then for some constant $C_3 > 0$, since $\|m^*\| \ge 1$ and $b \le a$, we obtain

$$\|\beta^s m^*\|^{b/a} \le C_3 \|m^*\|^{b/a} \exp(b\|s\|) \le C_3 \|m^*\| \exp(b\|s\|). \tag{11}$$

Now we use (10) to obtain

$$\|\beta^{r+s} m^*\| \ge C_4 \|\beta^s m^*\|^{b/a} \exp(b(\|r\| - t_0))$$

for yet another constant $C_4 > 0$. Using (6) we obtain

$$\begin{aligned} |\hat{\varphi}(\beta^{r+s} m^*)| &\le C \|\beta^{r+s} m^*\|^{-B} \\ &\le C C_4^{-B} \|\beta^s m^*\|^{-B(b/a)} \exp(-Bb(\|r\| - t_0)). \end{aligned}$$

Since $\|r\| > t_0$, the series

$$\sum_{r \in \mathbb{H}^j} \exp(-Bb(\|r\| - t_0))$$

converges and is estimated by some constant $C_5 > 0$. Therefore, for some constants $C_6, C_7 > 0$ we obtain

$$S_2(\beta^s m^*) \le C_6 \|\beta^s m^*\|^{-B(b/a)} \sum_{r \in \mathbb{H}^j} \exp(-Bb(\|r\| - t_0)) \le C_7 \|\beta^s m^*\|^{-B(b/a)}, \tag{12}$$

i.e. for $m = \beta^s m^*$ we obtain a super-polynomial estimate for $S_2(m)$:

$$S_2(m) \le C_7 \|m\|^{-B(b/a)}. \tag{13}$$

Case 2: $\|\underline{s}\| \ge 2t_0$. Since $\|\underline{s} + r\| \ge t_0$, we have

$$\begin{aligned}\|\beta^{r+s} m^*\| &\ge C_1 \|m^*\| \exp(b(\|r+s\| - t_0)) \\ &= C_1 \|m^*\| \exp(b(\|\bar{s}\|)) \exp(b(\|\underline{s} + r\| - t_0)).\end{aligned} \tag{14}$$

Using the estimate (11) with s replaced by $\bar{s}$,

$$\|\beta^{\bar{s}} m^*\|^{b/a} \le C_3 \|m^*\| \exp(b\|\bar{s}\|),$$

and (14) we obtain

$$\|\beta^{r+s} m^*\| \ge C_8 \|\beta^{\bar{s}} m^*\|^{b/a} \exp(b(\|\underline{s} + r\| - t_0))$$

for yet another constant $C_8 > 0$. By (6) we have

$$\begin{aligned}|\hat{\varphi}(\beta^{r+s} m^*)| &\le C \|\beta^{r+s} m^*\|^{-B} \\ &\le C C_4^{-B} \|\beta^{\bar{s}} m^*\|^{-B(b/a)} \exp(-Bb(\|\underline{s} + r\| - t_0)).\end{aligned}$$

Since $\|\underline{s} + r\| \ge t_0$, as before, $\sum_{r \in \mathbb{H}^j} \exp(-Bb(\|\underline{s} + r\| - t_0)) < C_9$ for some constant $C_9 > 0$, and for some constants $C_{10}, C_{11} > 0$ we obtain

$$S_2(\beta^s m^*) \le C_{10} \|\beta^{\bar{s}} m^*\|^{-B(b/a)} \sum_{r \in \mathbb{H}^j} \exp(-Bb(\|\underline{s} + r\| - t_0)) \le C_{11} \|\beta^{\bar{s}} m^*\|^{-B(b/a)}. \tag{15}$$

Now, since $\|\underline{s}\| \ge 2t_0$, $\|\bar{s}\| \ge 2t_0$, hence $\|\bar{s}/2\| \ge t_0$ and

$$\exp(b(\|\bar{s}\| - t_0)) \ge \exp(b\|\bar{s}/2\|).$$

Therefore, by the lower estimate of Theorem 2.1, using (8) and the fact that $\|m^*\| \ge 1$, we have

$$\begin{aligned}\|\beta^{\bar{s}} m^*\| &\ge C_1 \|m^*\| \exp(b(\|\bar{s}\| - t_0)) \\ &\ge C_1 \|m^*\| \exp(b\|\bar{s}/2\|) \ge C_1 \exp(b\|\bar{s}/2\|).\end{aligned} \tag{16}$$

On the other hand, by the upper estimate of Theorem 2.1, and using (16) we obtain for some constant $C_{12} > 0$

$$\begin{aligned}\|\beta^s m^*\| = \|\beta^{\underline{s}}(\beta^{\bar{s}} m^*)\| &\le C_2 \|\beta^{\bar{s}} m^*\| \exp(a\|\underline{s}\|) \le C_2 \|\beta^{\bar{s}} m^*\| \exp(a\|\bar{s}\|) \\ &= C_2 \|\beta^{\bar{s}} m^*\| \exp(b\|\bar{s}/2\|)^{2a/b} \le C_{12} \|\beta^{\bar{s}} m^*\|^{1+(2a/b)}.\end{aligned}$$

Therefore, for yet another constant $C_{13} > 0$,

$$\|\beta^{\bar{s}} m^*\|^{-B(b/a)} \le C_{13} \|\beta^s m^*\|^{-BL},$$

where $L = b^2/a(b + 2a)$. Plugging this into (15) and letting $\beta^s m^* = m$, we obtain a super-polynomial estimate for $S_2(m)$:

$$S_2(m) \le C\|m\|^{-BL}. \tag{17}$$

Estimate of $S_1(\beta^s m^*)$. We first use the upper estimate of Theorem 2.1 to obtain

$$\|\beta^s m^*\| \le C_2 \|m^*\| \exp(a\|s\|).$$

Since $\|s\| \le 8t_0 + k$ (k is the rank of the action), $\exp(a\|s\|) \le C_{14}\|m^*\|^{8a(N+1)/b}$, for some constant $C_{14} > 0$, and hence

$$\|\beta^s m^*\| \le C_{14}\|m^*\|^{1+(8a(N+1)/b)},$$

which implies that for $C_{15} = C_{14}^{-1}$

$$\|m^*\| \ge C_{15}\|\beta^s m^*\|^K, \tag{18}$$

where $K = b/(b + 8a(N+1))$, a constant depending only on the action β, not on the particular orbit, $0 < K < 1$. The simple estimate (9) combined with (18) gives us

$$\|\beta^{r+s} m^*\| \ge \|m^*\| \ge C_{15}\|\beta^s m^*\|^K.$$

Therefore

$$\begin{aligned}|\hat{\varphi}(\beta^{r+s} m^*)| &\le C\|\beta^{r+s} m^*\|^{-B} \\ &\le CC_{15}^{-B}\|\beta^s m^*\|^{-BK}.\end{aligned}$$

Now, since $\|r + \underline{s}\| \le 4t_0$, the number of terms in the finite sum $S_1(m)$, where $m = \beta^s m^*$, is $\le (4t_0)^k$, and we obtain for some constant $C_{16} > 0$,

$$S_1(m) \le C_{16}\|m\|^{-BK}(\ln\|m^*\|)^k \le C_{16}\|m\|^{-BK}(\ln\|m\|)^k. \tag{19}$$

But since for every $\epsilon > 0$ there exists $C_{17} > 0$ such that $(\ln\|m\|)^k \le C_{17}\|m\|^{\epsilon}$ (for $\|m\| \ge 2$), taking $\epsilon < BK/2$ we obtain for some constant C_{18},

$$S_1(m) \le C_{18}\|m\|^{-BK+\epsilon} = C_{18}\|m\|^{-BK/2}. \tag{20}$$

Completion of the proof. Combining (20) with (13) or (17) we obtain a super-polynomial estimate for $\hat{\Phi}$ for some $C_{19} > 0$ and $M > 0$,

$$|\hat{\Phi}_j(m)| \le C_{19}\|m\|^{-BM}.$$

Thus we have obtained global estimates on the decay of $\hat{\Phi}_j$. Letting $\hat{\Phi}_j(0) = 0$ and using (2.1) and (2.2) of [**3**], we therefore obtain a C^∞ $(k-1)$-cochain $\Phi = (\Phi_1, \dots, \Phi_k)$ such that

$$\mathcal{D}\Phi = \varphi - \hat{\varphi}(0),$$

i.e. this is a solution of our equation (5). □

The proofs of Theorems 1.1 and 1.2 now follow exactly as in [**3**].

For the proof of Theorem 1.1 we first apply [**3**, Corollary 1.4] to conclude that if a C^∞ k-cocycle over α, φ, vanishes on all periodic orbits of α, then for any dual orbit $\mathcal{O}$, including 0, $\sum_{m\in\mathcal{O}} \hat{\varphi}(m) = 0$. Now, by Proposition 3.1, $\mathcal{D}\Phi = \varphi - \hat{\varphi}(0)$, and since $\hat{\varphi}(0) = 0$ we obtain a solution of (1).

For Theorem 1.2, the assertion (2) for 1-cocycles is proved using [**3**, Proposition 2.3] and estimates of Proposition 3.1. The assertion (2) for n-cocycles, $1 \le n \le k-1$, follows by induction on k. Our hypothesis holds for the highest cocycles for which their dual cocycles vanish over each dual orbit (Proposition 3.1) and for 1-cocycles. These cases are considered as the basis step in our induction argument which goes exactly as in [**3**, p. 591].

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