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Finite spanning sets for cusp forms and a related geometric result

By *Svetlana Katok*[1]) at Santa Cruz

1. Introduction

The two main results of this paper are united by the same method of proof. Our first result is concerned with cusp forms on Fuchsian groups. Let Γ be a cocompact Fuchsian group acting on the upper half-plane $\mathscr{H}$, i.e. a discrete subgroup of $SL(2, \mathbb{R})$ with $\Gamma \backslash \mathscr{H}$ compact. We study the sequence of finite dimensional vector spaces $\{S_{2k}(\Gamma), k=2, 3, \ldots\}$ of holomorphic cusp forms of weight $2k$ on Γ. For each conjugacy class of hyperbolic (i.e. diagonalizable over $\mathbb{R}$) elements in Γ, denoted by $[\gamma]$, $\gamma \in \Gamma$, there exists a special cusp form $\Theta_{k,[\gamma]} \in S_{2k}(\Gamma)$. Since conjugacy classes of primitive hyperbolic elements in Γ are in one-to-one correspondence with oriented (primitive) closed geodesics in $\Gamma \backslash \mathscr{H}$, we call these cusp forms *relative Poincaré series associated to closed geodesics*. They have been described by Petersson and studied by complex analysts as well as by number theorists. For bibliographical remarks and properties of relative Poincaré series see [5]. In [5] we proved that the set of *all* relative Poincaré series span $S_{2k}(\Gamma)$ (Theorem 1). Goldman and Millson [3] specified a priori a *finite* set of relative Poincaré series which span $S_{2k}(\Gamma)$. In this paper we use completely different methods to obtain a similar result, but with a better estimate on the number of relative Poincaré series needed.

Theorem 1. *For any* $\alpha > 0$ *there exists a constant* $C_0(\alpha) > 0$ *independent on k such that the set* $\{\Theta_{k,[\gamma]}, \text{length } [\gamma] \leqq C_0(\alpha) k^{8+\alpha}\}$ *spans* $S_{2k}(\Gamma)$.

Remark. The Selberg trace formula [4] implies that the number of closed geodesics of length $\leqq T$ grows with T as $\exp(T - \ln T)$. Thus our estimate for the number of relative Poincaré series which span $S_{2k}(\Gamma)$ is $\exp(C k^{8+\alpha} - \ln C k^{8+\alpha})$, compared to $\exp((2^{2k-1}-1) - \ln(2^{2k-1}-1))$ in [3]. Obviously both estimates greatly exceed the dimension of $S_{2k}(\Gamma)$ which grows linearly with k. However, obtaining a considerably better estimate seems to be beyond the reach of existing methods.

Our second result is a "finite version" of the following theorem which is a trivial consequence of Theorem 3. 6 by Guillemin and Kazhdan [2]:

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Theorem 2. *If a smooth function f on a compact negatively curved surface has zero integrals over all closed geodesics, then* $f=0$.

Let M be a compact negatively curved surface, and SM be its unit tangent bundle. We consider the space $L^2(SM)$ equipped with the scalar product $\langle,\rangle$, and we denote the corresponding norm in $L^2(SM)$ by $\|\cdot\|$. In this paper C with various subscripts denotes positive constants which may depend on the surface M. The dependence on a parameter, if any, is specified. We now can state our result.

Theorem 3. *Given* $\varepsilon>0$, *for any* $\alpha>0$, *there exists a constant* $C_1(\alpha)$ *such that if a function* $f\in C^2(M)$ *with* $\|f\|_{C^2}=1$ *has zero integrals over all closed geodesics of length* $\leqq C_1(\alpha)\varepsilon^{-2-\alpha}$, *then* $\|f\|\leqq\varepsilon$.

2. Geometric preliminaries

Let M be a compact negatively curved surface, SM be its unit tangent bundle, and $\frac{\partial}{\partial\theta}$ be the infinitesimal generator of the action of SO(2) on SM. $L^2(SM)$ can be decomposed as a direct sum

$$L^2(SM)=\bigoplus_{-\infty}^{\infty} H_m, \tag{2.1}$$

where H_m is the eigenspace of the differential operator $\mathscr{D}_\theta=-\sqrt{-1}\frac{\partial}{\partial\theta}$ with the eigenvalue m:

$$H_m=\{F\in L^2(SM)\mid \mathscr{D}_\theta F=mF\}. \tag{2.2}$$

Let $\{\psi^t\}$ be the geodesic flow on SM, and $\mathscr{D}$ be the operator of differentiation along the orbits of $\{\psi^t\}$. It is defined on a dense set of functions differentiable along the orbits of $\{\psi^t\}$, and is skew-self-adjoint: $\mathscr{D}^*=-\mathscr{D}$, since the geodesic flow $\{\psi^t\}$ preserves the volume form on SM. We will need the following properties of $\mathscr{D}$ (cf. e.g. [2]):

$$\mathscr{D}=\mathscr{D}^+ + \mathscr{D}^-, \text{ where } \mathscr{D}^+ : H_m\to H_{m+1},\ \mathscr{D}^- : H_m\to H_{m-1}, \tag{2.3}$$

$$(\mathscr{D}^+)^*=-\mathscr{D}^-,\ (\mathscr{D}^-)^*=-\mathscr{D}^+, \tag{2.4}$$

$$[\mathscr{D}^+,\mathscr{D}^-]=\frac{K}{2}\mathscr{D}_\theta, \text{ where } K \text{ is the scalar curvature function on } M. \tag{2.5}$$

Let $M=\Gamma\backslash\mathscr{H}$ be as in § 1. It is of constant negative curvature $K=-1$, and all of the above applies. Following [5] we parametrize the unit tangent bundle to the upper half-plane, $S\mathscr{H}$, by local coordinates (z,ζ), where $z\in\mathscr{H}$, $\zeta\in\mathbb{C}$, $|\zeta|=\operatorname{Im} z$. For any $f(z)\in S_{2k}(\Gamma)$ the function $f(z)\zeta^k$ is well-defined on SM. The scalar product on $L^2(SM)$ is given by the formula

$$\langle F, G\rangle=\int_{SM} F\cdot\bar{G}\,dV\,d\theta,$$

where $dV = \frac{dx\,dy}{y^2}$, $\theta = \frac{1}{2\pi}\operatorname{Arg}\zeta$, and $dV d\theta$ is the $SL(2, \mathbb{R})$-invariant volume on SM. Obviously, $f(z)\zeta^k \in L^2(SM)$. The subspaces H_m in this case are defined as follows: $H_m = \{G(z, \zeta) \in L^2(SM) \mid G(z, \zeta) = g(z)\zeta^k\}$; notice that $g(z)$ is not supposed to be holomorphic. Let $\iota_k\colon S_{2k}(\Gamma) \to L^2(SM)$ be given by the formula $\iota_k f(z) = f(z)\zeta^k$. Then $\iota_k S_{2k}(\Gamma) = \{f(z)\zeta_k \in H_k, f(z) \text{ holomorphic}\}$; it is a finite dimensional subspace of H_k. The explicit formula for $\mathcal{D}^-$ which can be found in [5], and holomorphy of cusp forms imply the following property:

$$\mathcal{D}^-(\iota_k S_{2k}(\Gamma)) = 0. \tag{2.6}$$

By [5], Proposition 3, Theorem 1 follows immediately from the following statement.

Theorem 4. *For any $\alpha > 0$ there exists a constant $C_0(\alpha) > 0$ independent on k such that if $f(z) \in S_{2k}(\Gamma)$ and $\int_{[\gamma]} f(z)\zeta^k dt = 0$ for all closed geodesics $[\gamma]$ of length $\leqq C_0(\alpha)k^{8+\alpha}$, then $f(z) = 0$.*

The first ingredient that goes into the proofs of Theorems 3 and 4 is the following result in dynamical systems.

Theorem 5 (Finite Livshitz Theorem). *Let M be a compact negatively curved surface, $X = SM$, $\{\psi^t\}$ be the geodesic flow on X, and $T > 0$. Then there is $\lambda_0 \leqq 1$, and for any λ, $0 < \lambda < \lambda_0$ a constant $C(\lambda)$ such that if $f \in C^2(X)$, $\|f\|_{C^2} = 1$, and $\int_{[o]} f\,dt = 0$ for all periodic orbits $[o]$ of $\{\psi^t\}$ of length $\leqq T$, then there exist $F, h \in C^{1+\lambda}(X)$ such that $f = \mathcal{D}F + h$ with $\|h\|_{C^1} \leqq C(\lambda)T^{-\frac{\lambda}{3-\lambda}}$.*

Remark 1. The constant λ_0 is computable in terms of characteristics of the geodesic flow $\{\psi^t\}$ [6], and $\lambda_0 = 1$ in the case of constant negative curvature -1 (Theorem 4).

Remark 2. This theorem was proved in [6] for contact Anosov flows, which include geodesic flows on compact negatively curved surfaces as a particular case.

3. Regularity of cusp forms

In this section we shall prove the following general result about cusp forms.

Let $\alpha = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{|\alpha|})$ be a formal vector of length $|\alpha| > 0$ with $\mathcal{D}_i = \mathcal{D}^+, \mathcal{D}^-$ or $\mathcal{D}_\theta$. A partial derivative of order α we define as the differential operator $\mathcal{D}^\alpha = \mathcal{D}_1\mathcal{D}_2 \ldots \mathcal{D}_{|\alpha|}$. For $|\alpha| = 0$ we let $\mathcal{D}^0 f = f$. Let $H^n(SM)$ be a subspace of $L^2(SM)$ consisting of functions whose partial derivatives of order α belong to $L^2(SM)$ for all α with $|\alpha| \leqq n$. The space $H^n(SM)$ is a Sobolev-like space for non-commuting differential operators $\mathcal{D}^+$, $\mathcal{D}^-$ and $\mathcal{D}_\theta$. It can be equipped with the norm

$$\|f\|_{H^n} = \sum_{0 \leqq |\alpha| \leqq n} \|\mathcal{D}^\alpha f\|.$$

Similarly, we define the space $C^m(SM)$ of functions having continuous partial derivatives of order α for all α with $|\alpha| \leqq m$ with the norm

$$\|f\|_{C^m} = \sum_{0 \leqq |\alpha| \leqq m} \sup |\mathscr{D}^\alpha f|.$$

Theorem 6 (Regularity Theorem). *Let m be a positive integer. Then there exists a constant $C>0$ such that for any $F \in \iota_k S_{2k}(\Gamma)$, $\|F\|_{C^m} \leqq C k^{m+2} \|F\|$.*

Proof. We shall prove by induction that for $n \geqq 1$ there exists a constant $C_1(n) > 0$ such that for any $F \in \iota_k S_{2k}(\Gamma)$, $\|F\|_{H^n} \leqq C_1(n) k^n \|F\|$. For $n=1$ we have

$$\|F\|_{H^1} = \|F\| + \|\mathscr{D}_\theta F\| + \|\mathscr{D}^+ F\| + \|\mathscr{D}^- F\|.$$

According to (2. 4)—(2. 6) we have $\|\mathscr{D}_\theta F\| = k\|F\|$, $\|\mathscr{D}^- F\| = 0$,

$$\|\mathscr{D}^+ F\|^2 = \langle \mathscr{D}^+ F, \mathscr{D}^+ F\rangle = -\langle F, \mathscr{D}^- \mathscr{D}^+ F\rangle = \frac{1}{2}\langle F, \mathscr{D}_\theta F\rangle - \langle F, \mathscr{D}^+ \mathscr{D}^- F\rangle = \frac{k}{2}\|F\|^2.$$

Hence $\|\mathscr{D}^+ F\| = \sqrt{\frac{k}{2}}\|F\|$, and the claim follows. Suppose that for $n-1$ our hypothesis is true. First we compute $\|\mathscr{D}^{+n} F\| = \|\mathscr{D}^+ \mathscr{D}^+ \cdots \mathscr{D}^+ F\|$ (all $\mathscr{D}^+$s) using (2. 4)—(2. 6).

$$\begin{aligned}
\|\mathscr{D}^{+n} F\|^2 &= \langle \mathscr{D}^+ \mathscr{D}^+ \cdots \mathscr{D}^+ F, \mathscr{D}^+ \mathscr{D}^+ \cdots \mathscr{D}^+ F\rangle = -\langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \mathscr{D}^- \mathscr{D}^+ \cdots \mathscr{D}^+ F\rangle \\
&= \langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \frac{1}{2}\mathscr{D}_\theta \mathscr{D}^+ \cdots \mathscr{D}^+ F\rangle - \langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \mathscr{D}^+ \mathscr{D}^- \cdots \mathscr{D}^+ F\rangle \\
&= \langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \frac{1}{2}\mathscr{D}_\theta \mathscr{D}^+ \cdots \mathscr{D}^+ F\rangle + \langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \frac{1}{2}\mathscr{D}^+ \mathscr{D}_\theta \mathscr{D}^+ \cdots \mathscr{D}^+ F\rangle + \cdots \\
&\quad + \langle \mathscr{D}^+ \cdots \mathscr{D}^+ F, \frac{1}{2}\mathscr{D}^+ \cdots \mathscr{D}^+ \mathscr{D}_\theta F\rangle = \frac{1}{4} n(2k+n-1)\,\|\mathscr{D}^{+(n-1)} F\|^2 \\
&\leqq C_2(n) k \cdot k^{2(n-1)} \|F\|^2 = C_2(n) k^{2n-1} \|F\|^2,
\end{aligned}$$

and therefore $\|\mathscr{D}^{+n}\| \leqq C_1(n) k^{n-\frac{1}{2}} \|F\|$. Now let $|\alpha| = n$ and $\mathscr{D}^\alpha = \mathscr{D}_1 \mathscr{D}_2 \cdots \mathscr{D}_n$. If for some $i = 1, \ldots, n$ $\mathscr{D}_i = \mathscr{D}_\theta$, we use (2. 2) and (2. 4) to conclude that

$$\|\mathscr{D}^\alpha F\| \leqq \frac{1}{2}(k+n-1)\, C_1(n-1) k^{n-1} \|F\| \leqq C_1(n) k^n \|F\|.$$

Now suppose that for some $i = 1, \ldots, n$ $\mathscr{D}_i = \mathscr{D}^-$. If $i = n$, by (2. 6) $\mathscr{D}^\alpha F = 0$. If $i < n$, we may assume that $\mathscr{D}_j = \mathscr{D}^+$ for $j > i$. Then

$$\|\mathscr{D}^\alpha F\| \leqq \frac{1}{2}(n-1)(k+n-2)\, C_1(n-1) k^{n-1} \|F\| \leqq C_1(n) k^n \|F\|.$$

Let us cover SM by a finite number of coordinate charts. Inside each chart the operators $\mathscr{D}^+$, $\mathscr{D}^-$ and $\mathscr{D}_0$ are expressed as linear combinations of partial derivatives relative to this chart with smooth coefficients. Using the chain rule one sees that our C^m and H^n norms are locally equivalent to the standard C^m and Sobolev norms defined relative to each coordinate system. Therefore we can use the Sobolev embedding theorem [7] to conclude that for $n > \frac{3}{2} + m$ $H^n \subset C^m$, in particular, $H^{m+2} \subset C^m$, and therefore for some constant $C_3 > 0$, $\|F\|_{C^m} \leqq C_3 \|F\|_{H^{m+2}}$, and the theorem follows. □

4. The proof of Theorem 4

Suppose $f(z) \neq 0$. We may assume then that $\|f(z)\zeta^k\|_{C^2} = 1$. Applying Theorem 5 to the function $f(z)\zeta^k \in \iota_k S_{2k}(\Gamma)$, we obtain for every λ, $0 < \lambda \leqq 1$, two functions $F, h \in C^{1+\lambda}(SM)$ which satisfy the equation

$$\mathscr{D} F = f(z)\zeta^k + h. \tag{4.1}$$

Lemma 1. *Let $f(z)\zeta^k \in \iota_k S_{2k}(\Gamma)$ and (4.1) holds. Then $\|f(z)\zeta^k\| \leqq 6(\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}$.*

Proof. We decompose the functions F and h according to (2.1): $F(z,\zeta) = \sum_{-\infty}^{\infty} F_m$, $h(z,\zeta) = \sum_{-\infty}^{\infty} h_m$, $F_m, h_m \in H_m$. Then the equation (4.1) is equivalent to the following system

$$\begin{aligned} \mathscr{D}^- F_{k+1} + \mathscr{D}^+ F_{k-1} &= f(z)\zeta^k + h_k, \\ \mathscr{D}^- F_{j+1} + \mathscr{D}^+ F_{j-1} &= h_j \quad \text{for all} \quad j \neq k. \end{aligned} \tag{4.2}$$

We have

$$\|\mathscr{D}^- F_{j+1}\|^2 = \|\mathscr{D}^+ F_{j-1} - h_j\|^2.$$

By [2], Lemma 3.4, we have

$$\|\mathscr{D}^+ F_{j+1}\|^2 = \frac{j+1}{2}\|F_{j+1}\|^2 + \|\mathscr{D}^- F_{j+1}\|^2 = \|\mathscr{D}^+ F_{j-1} - h_j\|^2 + \frac{j+1}{2}\|F_{j+1}\|^2.$$

Let $j \geqq k+1$. Then $j+1 > 0$, and therefore

$$\|\mathscr{D}^+ F_{j+1}\|^2 \geqq \|\mathscr{D}^+ F_{j-1} - h_j\|^2,$$

$$\|\mathscr{D}^+ F_{j+1}\| \geqq \|\mathscr{D}^+ F_{j-1} - h_j\| \geqq \|\mathscr{D}^+ F_{j-1}\| - \|h_j\|,$$

and

$$\|\mathscr{D}^+ F_{j+2n-1}\| \geqq \|\mathscr{D}^+ F_{j-1}\| - \sum_{i=0}^{n-1} \|h_{j+2i}\|.$$

Since $\|\mathscr{D}^+ F_{j+2n-1}\| \to 0$ as $n \to \infty$, we have for $j-1=k+1>0$

$$\|\mathscr{D}^- F_{k+1}\| \leqq \|\mathscr{D}^+ F_{k+1}\| \leqq \sum_{m \neq 0} \|h_m\|. \tag{4. 3}$$

We shall prove that

$$\sum_{m \neq 0} \|h_m\| \leqq \frac{\pi}{\sqrt{3}} (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}. \tag{4. 4}$$

We recall that

$$\|h\|_{H^1} = \|h\| + \|\mathscr{D}_\theta h\| + \|\mathscr{D}^+ h\| + \|\mathscr{D}^- h\|.$$

Obviously, $\|h\|_{H^1} \leqq (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}$. By the Cauchy-Schwartz inequality

$$\sum_{m \neq 0} \|h_m\| \leqq \left(\sum_{m \neq 0} m^2 \|h_m\|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{m \neq 0} \frac{1}{m^2}\right)^{\frac{1}{2}} \leqq \frac{\pi}{\sqrt{3}} \left(\sum_{-\infty}^{\infty} m^2 \|h_m\|^2\right)^{\frac{1}{2}},$$

and since

$$\|h\|_{H^1} \geqq \|\mathscr{D}_\theta h\| = \left\|\sum_{-\infty}^{\infty} m h_m\right\| = \left(\sum_{-\infty}^{\infty} m^2 \|h_m\|^2\right)^{\frac{1}{2}},$$

(4. 4) follows. The next inequality follows now from (4. 3) and (4. 4).

$$\|\mathscr{D}^- F_{k+1}\| \leqq \|\mathscr{D}^+ F_{k+1}\| \leqq \frac{\pi}{\sqrt{3}} (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}. \tag{4. 5}$$

Let us denote $\mathscr{D}^- F_{k+1} + h_k = g_k$ and rewrite the first equation of (4. 2) as follows:

$$f(z)\zeta^k = \mathscr{D}^+ F_{k-1} + g_k. \tag{4. 6}$$

We also have

$$\|h_k\| \leqq \|h\| \leqq \|h\|_{H^1} \leqq (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}. \tag{4. 7}$$

Then using (4. 5) and (4. 7) we obtain

$$\|g_k\| \leqq \|\mathscr{D}^- F_{k+1}\| + \|h_k\| \leqq \left(\frac{\pi}{\sqrt{2}} + 1\right) (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}.$$

Taking $\mathscr{D}^-$ from both sides of (4. 6) and using that by (2. 6) $\mathscr{D}^- f(z)\zeta^k = 0$ we obtain

$$0 = \mathscr{D}^- \mathscr{D}^+ F_{k-1} + \mathscr{D}^- g_k. \tag{4. 8}$$

The next equality is obtained by taking the scalar product of both sides of (4. 8) with F_{k-1} and using (2. 4).

$$0 = \langle F_{k-1}, \mathscr{D}^- \mathscr{D}^+ F_{k-1}\rangle + \langle F_{k-1}, \mathscr{D}^- g_k\rangle = -\langle \mathscr{D}^+ F_{k-1}, \mathscr{D}^+ F_{k-1}\rangle - \langle \mathscr{D}^+ F_{k-1}, g_k\rangle.$$

Therefore

$$\|\mathscr{D}^+ F_{k-1}\|^2 = |\langle \mathscr{D}^+ F_{k-1}, g_k\rangle| \leqq \|\mathscr{D}^+ F_{k-1}\| \cdot \|g_k\|.$$

If $\|\mathscr{D}^+ F_{k-1}\| \neq 0$ we divide both sides by $\|\mathscr{D}^+ F_{k-1}\|$ and obtain the estimate

$$\|\mathscr{D}^+ F_{k-1}\| \leqq \|g_k\| \leqq \left(\frac{\pi}{\sqrt{3}} + 1\right) (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}, \tag{4. 9}$$

which is also true for $\|\mathscr{D}^+ F_{k-1}\| = 0$. The desired inequality follows now from (4. 5), (4. 7) and (4. 9):

$$\|f(z)\zeta^k\| \leqq \mathscr{D}^+ F_{k-1}\| + \|\mathscr{D}^- F_{k+1}\| + \|h_k\| \leqq 6(\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}. \qquad \square$$

An application of Theorem 6 to the function $f(z)\zeta^k$ for $m=2$ gives us the following inequality

$$\|f(z)\zeta^k\| \geqq \frac{C}{k^4} \|f(z)\zeta^k\|_{C^2} = \frac{C}{k^4}. \tag{4. 10}$$

By the Finite Livshitz Theorem, for any $T>0$, the cohomological equation can be solved in such a way that for any λ, $0<\lambda<1$, $\|h\|_{C^1} \leqq C(\lambda) T^{-\frac{\lambda}{3-\lambda}}$, and hence for $C_0 = 6(\operatorname{vol} M)^{\frac{1}{2}}$

$$\|f(z)\zeta^k\| \leqq C_0 \cdot C(\lambda) T^{-\frac{\lambda}{3-\lambda}}. \tag{4. 11}$$

Choose $\lambda = \frac{12}{12+\alpha}$ and let $C'(\alpha) = \left(\frac{C_0 C(\lambda)}{C}\right)^{\frac{3-\lambda}{\lambda}}$, and $C_0(\alpha) > C'(\alpha)$. Then for

$$C_0(\alpha) k^{8+\alpha} \geqq T > C'(\alpha) k^{8+\alpha}$$

we have

$$C_0 \cdot C(\lambda) T^{-\frac{\lambda}{3-\lambda}} < \frac{C}{k^4}.$$

The last inequality leads to a contradiction between (4. 10) and (4. 11). $\square$

Remark. Let Δ be the Laplace (Casimir) operator for $S\mathscr{H}$ in $L^2(SM)$ [1]. Δ operates as a scalar on each subspace $\iota_k S_{2k}(\Gamma) \subset H_k$, i.e. for $f \in \iota_k S_{2k}(\Gamma)$

$$\Delta f = -\frac{k(k-2)}{4} f.$$

Then $\|f\|_{C^2} \geqq C' \|f\|_{H^2} \geqq C k^2 \|f\|$, i.e.

$$\|f\| \leqq \frac{1}{C k^2}. \tag{4. 12}$$

Therefore, an estimate

$$\|f\| \geqq \frac{C}{k^2}$$

instead of (4. 10) would have given us the estimate $T \leqq C_0(\alpha) k^{4+\alpha}$ instead of $C_0(\alpha) k^{8+\alpha}$, and a better estimate cannot be obtained by our methods.

5. The proof of Theorem 3.

We notice first that $L^2(M) = H_0$. Applying Theorem 5 to the function $f \in H_0$ we obtain the following system of equations:

$$(5.\,1)\qquad \begin{aligned} \mathscr{D}^- F_1 + \mathscr{D}^+ F_{-1} &= f + h_0, \\ \mathscr{D}^- F_{j+1} + \mathscr{D}^+ F_{j-1} &= h_j \quad \text{for all} \quad j \neq 0, \end{aligned}$$

where $F, h \in C^{1+\lambda}(SM)$, $F = \sum_{-\infty}^{\infty} F_m$, $h(z, \zeta) = \sum_{-\infty}^{\infty} h_m$, where $F_m, h_m \in H_m$.

Lemma 2. *Let $f \in H_0$ and (5. 1) holds. Then* $\|f\| \leqq 5 (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}$.

Proof. Using the same argument as in Lemma 1 we obtain that

$$(5.\,2)\qquad \|\mathscr{D}^- F_1\| \leqq \|\mathscr{D}^+ F_1\| \leqq \frac{\pi}{\sqrt{3}} (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}$$

and

$$(5.\,3)\qquad \|h_0\| \leqq (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}.$$

In order to estimate $\|\mathscr{D}^+ F_{-1}\|$ we need an argument different from holomorphy used in the proof of Lemma 1. Let $a_0 = \min\left(-\frac{K}{2}\right)$, $a_1 = \max\left(-\frac{K}{2}\right)$. By [2], Lemma 3. 4, for $j+1 < 0$ we have

$$\|\mathscr{D}^+ F_{j+1}\|^2 \leqq a_0 \frac{j+1}{2} \|F_{j+1}\|^2 + \|\mathscr{D}^- F_{j+1}\|^2 = \|\mathscr{D}^+ F_{j-1} - h_j\|^2 + a_0 \frac{j+1}{2} \|F_{j+1}\|^2.$$

Then

$$\|\mathscr{D}^+ F_{j+1}\|^2 \leqq \|\mathscr{D}^+ F_{j-1} - h_j\|^2 \leqq \|\mathscr{D}^+ F_{j-1}\| + \|h_j\|^2,$$

and

$$\|\mathscr{D}^+ F_{j+1}\| \leqq \|\mathscr{D}^+ F_{j-2n-1}\| + \sum_{i=0}^{n} \|h_{j-2i}\|.$$

Since $\|\mathscr{D}^+ F_{j-2n-1}\| \to 0$ as $n \to \infty$, we have for $j+1=-1<0$

$$(5.4) \qquad \|\mathscr{D}^+ F_{-1}\| \leqq \sum_{m \neq 0} \|h_m\| \leqq \frac{\pi}{\sqrt{3}} (\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}.$$

Thus

$$\|f\| \leqq \mathscr{D}^+ F_{-1}\| + \|\mathscr{D}^- F_{-1}\| + \|h_0\| \leqq 5(\operatorname{vol} M)^{\frac{1}{2}} \|h\|_{C^1}. \qquad \square$$

By the Finite Livshitz Theorem, for any $T>0$, the cohomological equation can be solved in such a way that for any λ, $0<\lambda<1$ $\|h\|_{C^1} \leqq C(\lambda)\, T^{-\frac{\lambda}{3-\lambda}}$, and hence

$$(5.5) \qquad \|f\| \leqq 5(\operatorname{vol} M)^{\frac{1}{2}} \cdot C(\lambda)\, T^{-\frac{\lambda}{3-\lambda}}.$$

Choose $\lambda = \dfrac{3}{3+\alpha}$ and let $C_1'(\alpha) = 5^{2+\alpha} (\operatorname{vol} M)^{\frac{2+\alpha}{2}} C(\lambda)^{2+\alpha}$, and $C_1(\alpha) > C_1'(\alpha)$. Then for $T = C_1'(\alpha)\, \varepsilon^{-2-\alpha}$ we have $\|f\| \leqq 5(\operatorname{vol} M)^{\frac{1}{2}} C(\lambda)\, T^{-\frac{1}{2+\alpha}} \leqq \varepsilon$. $\square$

Corollary. *Let $C_L^2(M) = \{f \in C^2(M) \mid L\|f\| > \|f\|_{C^2}\}$. For any $\alpha > 0$ there exists $T(\alpha, L)$ such that if $f \in C_L^2(M)$ and $\int_{[\gamma]} f\,dt = 0$ for all $[\gamma]$ of length $\leqq T(\alpha, L)$, then $f = 0$.*

Proof. Suppose $f \neq 0$. We may assume then that $\|f\|_{C^2} = 1$. Take $\varepsilon = \dfrac{1}{L}$, $T(\alpha, L) = C_1(\alpha) L^{2+\alpha}$. Applying Theorem 3 we obtain $\|f\| \leqq \varepsilon$, which contradicts the inequality $\|f\| > \varepsilon$.

6. A concluding remark

Theorem 2 and Theorem 1(i) of [5] could be included as particular cases in the following conjecture.

Conjecture. *Let $f \in H_k$, and $f \in C^1(SM)$. If the function f has zero integrals over all closed geodesics, then $\|f\| = 0$.*

The corresponding generalizations of our "finite" results, Theorems 3 and 4, would be a positive solution to the following

Problem. *Given $\varepsilon > 0$, find an effective estimate $T(\varepsilon)$ such that if a function $f \in H_k$, $f \in C^2(SM)$, and $\|f\|_{C^2} = 1$ has zero integrals over all closed geodesics of length $\leqq T(\varepsilon)$, then $\|f\| \leqq \varepsilon$.*

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Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

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