ELLIPTIC OPERATORS AND SOLUTIONS OF COHOMOLOGICAL EQUATIONS FOR GEODESIC FLOWS WITH HYPERBOLIC BEHAVIOR

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In this paper we study the space of smooth functions on the unit tangent bundle SM to a compact negatively curved surface M that are eigenfunctions of the infinitesimal generator of the action of SO(2) on SM, and that have zero integrals over all periodic orbits of the geodesic flow on SM. It is proved that the space of such functions is finite dimensional. In the case of constant negative curvature a complete description of this space is obtained.

1. Notations and Statement of the Main Results

Let M be a compact negatively curved surface, SM its unit tangent bundle, $\{\psi^t\}$ the geodesic flow on SM, $C^\infty(SM)$ the space of infinitely differentiable functions on SM, and $\mathscr{D} = \dfrac{d}{dt}$ the operator of the differentiation in the direction of the geodesic flow $\{\psi^t\}$. We are going to study the space of functions in $C^\infty(SM)$ that have zero integrals over all periodic orbits of the geodesic flow $\{\psi^t\}$ (also called closed geodesics.) The Livshitz Theorem asserts that the following two conditions are equivalent:

(i) $f \in C^\infty(SM)$ has zero integrals over all closed geodesics,

(ii) the *Livshitz cohomological equation*

$$f = \mathscr{D}F \tag{1.1}$$

has a unique (up to an additive constant) solution $F \in C^\infty(SM)$.

The functions satisfying (ii) are usually called *coboundaries*. We shall denote the space of coboundaries by $\mathscr{B}^\infty(SM)$. The difficult part of the proof ((i) $\Rightarrow$ (ii)) consists of two parts: the existence of a solution of (1.1) which is given by a global construction, and the regularity of the constructed solution. The first part and the proof that the solution is C^1 is due to A. Livshitz [6], and the proof that the solution is C^∞ is due to Guillemin and Kazhdan [2]. For related more recent regularity results for Anosov flows see also [3] and [7]. In general, even for a very regular function f it is impossible

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to give a reasonable close expression for the solution F of (1.1). The goal of this paper is to show that in certain cases such an expression can be obtained through the interplay between the hyperbolic structure of the geodesic flow and the ellipticity of certain differential operators involved. I would like to thank Livio Flaminio for fruitful discussions which helped to clarify the above-mentioned connection.

Let $\frac{\partial}{\partial\theta}$ be the infinitesimal generator of the action of SO(2) on SM. We consider the $L^2(SM)$ with a scalar product $\langle\ ,\ \rangle$, and the orthogonal direct sum decomposition:

$$L^2(SM) = \bigoplus_{-\infty}^{\infty} H_m, \tag{1.2}$$

where H_m is the eigenspace of the differential operator $\mathscr{D}_\theta = -\sqrt{-1}\frac{\partial}{\partial\theta}$ with the eigenvalue m:

$$H_m = \{F \in L^2(SM) | \mathscr{D}_\theta F = mF\}.$$

The differential operator $\mathscr{D}$ is densely defined on $L^2(SM)$. It is skew-self-adjoint: $\mathscr{D}^* = -\mathscr{D}$ since the geodesic flow $\{\psi^t\}$ preserves the volume form on SM. The following properties can be found in [1]:

$\mathscr{D} = \mathscr{D}^+ + \mathscr{D}^-$ where $\mathscr{D}^+ : H_m \to H_{m+1}$ and $\mathscr{D}^- : H_m \to H_{m-1}$ are extensions of first order elliptic differential operators defined on the dense subspace $C_m^\infty = H_m \cap C^\infty(SM)$ of H_m, (1.3)

$$(\mathscr{D}^+)^* = -\mathscr{D}^-, \qquad (\mathscr{D}^-)^* = -\mathscr{D}^+, \tag{1.4}$$

$$[\mathscr{D}^+, \mathscr{D}^-] = \frac{-K}{2}\mathscr{D}_\theta, \qquad \text{where } K \text{ is the scalar curvature function on } M. \tag{1.5}$$

Let $S_m = \operatorname{Ker} \mathscr{D}^-|_{H_m}$, and $V_m = \operatorname{Ker} \mathscr{D}^+|_{H_m}$. The following properties are implied by the fact that $\mathscr{D}^+$ and $\mathscr{D}^-$ are elliptic operators, and can be found in [8]:

$$S_m \subseteq C_m^\infty, \qquad V_m \subseteq C_m^\infty, \qquad [8, \text{Theorem 4.8(a)}]; \tag{1.6}$$

$$\dim S_m < \infty, \qquad \dim V_m < \infty, \qquad [8, \text{Theorem 4.8(b)}]. \tag{1.7}$$

For $\tau \in S_m{}^\perp \cap C_m^\infty$ there exists a unique $\xi \in C_{m-1}^\infty \cap V_{m-1}{}^\perp$ such that $\mathscr{D}^+\xi = \tau$, and for $\tau \in V_m{}^\perp \cap C_m^\infty$ there exists a unique $\xi \in C_{m+1}^\infty \cap S_{m+1}{}^\perp$ such that $\mathscr{D}^-\xi = \tau$ ($\perp$ denotes the orthogonal complement with respect to the scalar product on $L^2(SM)$) [8, Theorem 4.11]. (1.8)

Guillemin and Kazhdan obtained the following *finiteness* result [1, Theorem 3.6]: *Let f be a smooth function on SM of the form $f = \sum_{|i| \le n} f_i$, $f_i \in H_i$ having zero integrals over all closed geodesics. Then there exists a smooth solution of* (1.1) *of the form*

$F = \sum_{|i| \le n-1} F_i$, $F_i \in H_i$. In particular, if f is a function on the surface M itself, the above result enables us to conclude that the only solutions of (1.1) are constants, and the following *vanishing* theorem holds [5, Theorem 2]: *If a smooth function f on a compact negatively curved surface has zero integrals over all closed geodesics, then $f = 0$.* In other words, $H_0 \cap \mathscr{B}^\infty(SM) = \{0\}$. In this paper we generalize this result in the following way: we prove that the space of coboundaries $\mathscr{B}_k^\infty = H_k \cap \mathscr{B}^\infty(SM)$ for $k > 0$ is always *finite-dimensional.*

Theorem 1.1. *Let M be a compact negatively curved surface, $k \ge 2$ an integer, and $f \in C_k^\infty$ be a function having zero integrals over all closed geodesics on SM. Then f belongs to a finite-dimensional subspace of C_k^∞ of dimension $\sum_{i=0}^{[k/2]-1} \dim(S_{k-1-2i})$.*

Remark. For the case $k = 1$ see Theorem 6.1.

In the case of *constant* negative curvature the fact that the operators $\mathscr{D}^+$, $\mathscr{D}^-$ and $\mathscr{D}_\theta$ generate a 3-dimensional Lie algebra enables us to prove the following *algebraicity* result that gives a complete description of coboundaries.

Theorem 1.2. *Let M be a compact surface of negative curvature $K = -1$, $k \ge 2$ an integer. A function $f \in C_k^\infty$ has zero integrals over all closed geodesics on SM if and only if it can be represented in the form $f = \sum_{i=0}^{[k/2]-1} (\mathscr{D}^+)^{2i+1} h_{k-1-2i}$, where $h_{k-1-2i} \in S_{k-1-2i}$. In the latter case a solution F of (1.1) can be written explicitly: $F = \sum_{m=0}^{[k/2]-1} F_{k-1-2m}$, where $F_{k-1-2m} = \sum_{j=m}^{[k/2]-1} (\mathscr{D}^+)^{2(j-m)} g_{k-1-2j}$, and each $g_{k-1-2j} \in S_{k-1-2j}$ is a multiple of h_{k-1-2j}.*

This theorem puts very strong restrictions on the functions that can be coboundaries, in particular, it includes our earlier *vanishing* result, Theorem 2(i) of [4]: *If a function $f \in S_k$ has zero integrals over all closed geodesics, then $f = 0$.* This result was used in [4] to show that for any Fuchsian group of the first kind Γ and $k \ge 2$, relative Poincaré series associated to closed geodesics span the space of cusp forms of weight $2k$ on Γ. Incidentally, the proof of this theorem given in [4] for the constant negative curvature works for the variable curvature as well, and hence we have a class of functions (S_k) that cannot be coboundaries. Certain other classes of funtions that cannot be or always are coboundaries in the variable curvature case are identified and dicussed in Sec. 6.

2. The Variable Curvature Case

The following lemma makes (1.7) more precise.

Lemma 2.1

(i) *If $m > 0$, $V_m = \{0\}$, i.e., the operator $\mathscr{D}^+|_{H_m}$ is injective.*
(ii) *If $m < 0$, $S_m = \{0\}$, i.e., the operator $\mathscr{D}^-|_{H_m}$ is injective.*
(iii) *$S_0 = V_0$ consists of constants, i.e., $\dim S_0 = \dim V_0 = 1$.*

Proof. By (1.6), it suffices to consider $f \in C_m^\infty$, and the lemma follows immediately from Lemma 3.4 by Guillemin and Kazhdan [1]. For the completeness sake we repeat their argument. By (1.5)

$$\mathscr{D}^+\mathscr{D}^- f = \mathscr{D}^-\mathscr{D}^+ f - \frac{K}{2}\mathscr{D}_\theta f = \mathscr{D}^-\mathscr{D}^+ f - \frac{K}{2} mf,$$

and hence

$$\langle \mathscr{D}^+\mathscr{D}^- f, f\rangle = \langle \mathscr{D}^-\mathscr{D}^+ f, f\rangle + m\left\langle -\frac{K}{2} f, f\right\rangle,$$

and

$$\|\mathscr{D}^- f\|^2 + a_0 m\|f\|^2 \le \|\mathscr{D}^+ f\|^2 \le \|\mathscr{D}^- f\|^2 + a_1 m\|f\|^2, \tag{2.1}$$

where $a_0 = \min\left(-\frac{K}{2}\right)$ and $a_1 = \max\left(-\frac{K}{2}\right)$, $a_1 \ge a_0 > 0$. If $m > 0$, $\|\mathscr{D}^+ f\| = 0$ implies $\|f\| = 0$, and (i) follows. If $m < 0$, $\|\mathscr{D}^- f\| = 0$ implies $\|f\| = 0$, and (ii) follows. Let $m = 0$, and $\mathscr{D}^+ f = 0$. Then (2.1) implies $\mathscr{D}^- f = 0$, hence $\mathscr{D} f = 0$, and by the ergodicity of the geodesic flow $f = \text{const}$. The case $\mathscr{D}^- f = 0$ follows similarly. □

The next observation plays a very important role in what follows.

Lemma 2.2. *For any* $f \in C^1(SM) \cap H_k$, $\mathscr{D}^+ f \perp S_{k+1}$.

Proof. For any $h_{k+1} \in S_{k+1}$ we have by (1.4) $\langle \mathscr{D}^+ f, h_{k+1}\rangle = -\langle f, \mathscr{D}^- h_{k+1}\rangle = 0$. □

Lemma 2.3. *Let $k > 0$ be an integer. Then any $f_k \in C_k^\infty$ can be written uniquely in the form $f_k = \sum_{i=0}^{k-1} (\mathscr{D}^+)^i h_{k-i} + (\mathscr{D}^+)^k h_0$, where $h_{k-i} \in S_{k-i}$, $h_0 \in C_0^\infty \cap V_0{}^\perp$, and any $f_{-k} \in C_{-k}^\infty$ can be written uniquely in the form $f_k = \sum_{i=0}^{k-1} (\mathscr{D}^-)^i h_{k+i} + (\mathscr{D}^-)^k h_0$, where $h_{-k+i} \in V_{-k+i}$, $h_0 \in C_0^\infty \cap S_0{}^\perp$.*

Proof. First we notice that any $f_k \in C_k^\infty$ can be written uniquely in the form $f_k = h_k + \tau$, where $h_k \in S_k$, and $\tau \in S_k{}^\perp$. By (1.6) $\tau \in S_k{}^\perp \cap C_k^\infty$, and hence by (1.8) f_k can be written uniquely in the form $f_k = h_k + \mathscr{D}^+\xi$, where $h \in S_k$, $\xi \in C_{k-1}^\infty$. Now we give a proof by induction on k. For $k = 1$ the statement is true: $f_1 = h_1 + \mathscr{D}^+ h_0$ with $h_1 \in S_1$, $h_0 \in C_0^\infty \cap V_0{}^\perp$. Suppose it is true for $k = n$. Then $f_{n+1} = h_{n+1} + \mathscr{D}^+ f_n$, where $h_{n+1} \in S_{n+1}$, $f_n \in C_n^\infty$, and hence by the induction hypothesis and by the linearity of $\mathscr{D}^+$ we have

$$f_{n+1} = h_{n+1} + \mathscr{D}^+\left(\sum_{i=0}^{n-1} (\mathscr{D}^+)^i h_{n-i} + (\mathscr{D}^+)^n h_0\right) = h_{n+1} + \sum_{i=1}^{n} (\mathscr{D}^+)^i h_{n-i} + (\mathscr{D}^+)^{n+1} h_0.$$

To prove the uniqueness we suppose that $0 = h_n + \sum_{i=1}^{n-1} (\mathscr{D}^+)^i h_{n-i} + (\mathscr{D}^+)^n h_0$ with $h_0 \in C_0^\infty \cap V_0^\perp$, $h_i \in S_i$, $0 < i \le n$, i.e., $h_n = \mathscr{D}^+\zeta$, where $\zeta = -\sum_{i=1}^{n} (\mathscr{D}^+)^i h_{n-i}$. By Lemma 2.2 $h_n \perp S_n$, which implies $h_n = 0$, and hence, by the injectivity of $\mathscr{D}^+$ (Lemma 2.1(i)), $\zeta = 0$, and the assertion follows from the uniqueness for $k = n$. The statement for f_{-k} is proved similarly. □

3. Proof of Theorem 1.1

By the Livshitz Theorem the cohomological equation (1.1) has a smooth solution F. We decompose the functions F and f according to (1.2): $F = \sum_{-\infty}^{\infty} F_m$, $f = f_k$. Then the equation (1.1) is equivalent to the following system of equations:

$$\begin{aligned} &\mathscr{D}^- F_{k+1} + \mathscr{D}^+ F_{k-1} = f_k \\ &\mathscr{D}^- F_{i+1} + \mathscr{D}^+ F_{i-1} = 0 \qquad \text{for } i \ne k. \end{aligned} \tag{3.1}$$

By the finiteness result of Guillemin and Kazhdan [1, Theorem 3.6] $F_i = 0$ for $i \ge k$ and $i \le -k$, and Lemma 2.1 implies that $F_i = 0$ for $i < 0$, $\mathscr{D}^+ F_0 = \mathscr{D}^- F_0 = 0$, and $F_i = 0$ for $i \equiv k \pmod 2$, i.e., $F_0 = \text{const.}$, and (3.1) is equivalent to the following system:

$$\begin{aligned} &\mathscr{D}^+ F_{k-1} = f = f_k \\ &\mathscr{D}^- F_{k-1} + \mathscr{D}^+ F_{k-3} = 0 \\ &\qquad\cdots\cdots\cdots\cdots \\ &\mathscr{D}^- F_{\alpha(k)} = 0, \end{aligned} \tag{3.2}$$

where $\alpha(k) = 1$ or 2 depending on whether k is even or odd. It is easy to see that $\alpha(k) = k + 1 - 2\left[\frac{k}{2}\right]$, and the number of equations in (3.2) is equal to $\left[\frac{k}{2}\right] + 1$. It is convenient to number them from the bottom: $(3.2)_0, \ldots, (3.2)_{[k/2]}$. We prove by induction on n that for $n = 0, \ldots, \left[\frac{k}{2}\right] - 1$, $F_{\alpha(k)+2n}$ belongs to a subspace of $C^\infty_{\alpha(k)+2n}$ of dimension $\sum_{i=0}^{n} \dim(S_{\alpha(k)+2i})$. For $n = 0$ from the equation $(3.2)_0$ we obtain that $F_{\alpha(k)} \in S_{\alpha(k)}$, i.e., the assertion holds. Suppose it holds for $n = j$, and consider the equation $(3.2)_{j+1}$: $\mathscr{D}^- F_{\alpha(k)+2j+2} = -\mathscr{D}^+ F_{\alpha(k)+2j}$. According to (1.8) for each $F_{\alpha(k)+2j}$ satisfying to the equation $(3.2)_j$, the equation $(3.2)_{j+1}$ has a unique solution orthogonal to $S_{\alpha(k)+2j+2}$. It follows from the injectivity of $\mathscr{D}^+$ (Lemma 2.1(i)) that the space of all solutions of the equation $(3.2)_{j+1}$ has dimension $\dim(S_{\alpha(k)+2(j+1)}) + \sum_{i=0}^{j} \dim(S_{\alpha(k)+2i})$, and the assertion holds for $n = j + 1$. Applying this result to $n = \left[\frac{k}{2}\right] - 1$ we conclude that F_{k-1} belongs to a subspace of C^∞_{k-1} of dimension $\sum_{i=0}^{[k/2]-1} \dim(S_{k-1-2i})$, and by the injectivity of $\mathscr{D}^+$, f belongs to a subspace of C^∞_k of the same dimension. □

4. The Constant Curvature Case

In this section we assume that M is a compact surface of constant negative curvature $K = -1$.

Lemma 4.1. *Let n and k be two positive integers. Then*
(i) *for any $h_0 \in C^\infty_0$, $\mathscr{D}^-(\mathscr{D}^+)^n h_0 = (\mathscr{D}^+)^n \mathscr{D}^- h_0 - \frac{n(n-1)}{4}(\mathscr{D}^+)^{n-1} h_0$;*
(ii) *for any $h_k \in S_k$, $\mathscr{D}^-(\mathscr{D}^+)^n h_k = -\frac{n(2k+n-1)}{4}(\mathscr{D}^+)^{n-1} h_k$.*

Proof. Since $K = -1$, the commutation relation (1.5) can be rewritten as $\mathscr{D}^+\mathscr{D}^- - \mathscr{D}^-\mathscr{D}^+ = \frac{1}{2}\mathscr{D}_\theta$. Applying it n times we move $\mathscr{D}^-$ to the right to get (i). To obtain the formula in the part (ii) we also need to use the fact that $\mathscr{D}^- h_k = 0$. □

Lemma 4.2. *Let n and k be two positive integers. Then*
(i) *for $f_k \in C^\infty_k$, $(\mathscr{D}^-)^n(\mathscr{D}^+)^n f_k \in S_k$ if and only if $f_k \in S_k$;*
(ii) *for $h_0 \in C^\infty_0$, $(\mathscr{D}^-)^n(\mathscr{D}^+)^n h_0 = 0$ if and only if $h_0 = \text{const}$.*

Proof. Suppose $f_k \in S_k$, then n applications of Lemma 4.1(ii) imply that $(\mathscr{D}^-)^n(\mathscr{D}^+)^n f_k \in S_k$. Conversely, if $(\mathscr{D}^-)^n(\mathscr{D}^+)^n f_k \in S_k$, we decompose f_k as follows: $f_k = h_k + g_k$, where $h_k \in S_k$, and $g_k \perp S_k$. Then as we have just seen, $(\mathscr{D}^-)^n(\mathscr{D}^+)^n h_k \in S_k$, and hence $(\mathscr{D}^-)^n(\mathscr{D}^+)^n g_k \in S_k$. Applying (1.4) n times we obtain that $0 = \langle(\mathscr{D}^-)^n(\mathscr{D}^+)^n g_k, g_k\rangle = \|(\mathscr{D}^+)^n g_k\|^2$, which implies $(\mathscr{D}^+)^n g_k = 0$, and by Lemma 2.1(i) $g_k = 0$, hence $f_k \in S_k$, and (i) follows. Obviously, if $h_0 = \text{const.}$ then $(\mathscr{D}^-)^n(\mathscr{D}^+)^n h_0 = 0$. If $(\mathscr{D}^-)^n(\mathscr{D}^+)^n h_0 = 0$ then $(\mathscr{D}^-)^{n-1}(\mathscr{D}^+)^n h_0 \in S_1$, and hence by part (i) $\mathscr{D}^+ h_0 \in S_1$. However, by Lemma 2.2 $\mathscr{D}^+ h_0 \perp S_1$, so $\mathscr{D}^+ h_0 = 0$, and $h_0 = \text{const.}$ by Lemma 2.1(iii). □

5. Proof of Theorem 1.2

As in Sec. 3, the equation (1.1) is equivalent to the system (3.2), the nth equation being $(3.2)_n$: $\mathscr{D}^- F_{\alpha(k)+2n} = -\mathscr{D}^+ F_{\alpha(k)+2n-2}$. We shall prove by induction on n that $F_{\alpha(k)+2n} = \sum_{i=0}^{n} (\mathscr{D}^+)^{2i} g_{\alpha(k)+2n-2i}$ with some $g_{\alpha(k)+2n-2i} \in S_{\alpha(k)+2n-2i}$ for $n = 0, \ldots, \left[\frac{k}{2}\right] - 1$. Then for each $m = \left[\frac{k}{2}\right] - 1, \ldots, 0$ we shall obtain a required formula for F_{k-1-2m}. For $n = 0$ from the equation $(3.2)_0$ we have $F_{\alpha(k)} \in S_{\alpha(k)}$, so the statement is true. Suppose the statement is true for $n = j$:

$$F_{\alpha(k)+2j} = \sum_{i=0}^{j} (\mathscr{D}^+)^{2i} g_{\alpha(k)+2j-2i}. \tag{5.1}$$

Let us decompose $F_{\alpha(k)+2j+2}$ according to Lemma 2.3:

$$F_{\alpha(k)+2j+2} = \sum_{i=0}^{\alpha(k)+2j+1} (\mathscr{D}^+)^i f_{\alpha(k)+2j+2-i} + (\mathscr{D}^+)^{\alpha(k)+2j+2} f_0, \tag{5.2}$$

where $f_0 \in C_0^\infty$, $f_{\alpha(k)+2j+2-i} \in S_{\alpha(k)+2j+2-i}$ for $i < \alpha(k) + 2j + 2$. First we want to show that the last term of (5.2) is equal to 0. Using the equation $(3.2)_{j+1}$: $\mathscr{D}^- F_{\alpha(k)+2j+2} = -\mathscr{D}^+ F_{\alpha(k)+2j}$ and the induction hypothesis for $n = j$ we conclude that

$$\begin{aligned}(\mathscr{D}^-)^{\alpha(k)+2j+2} F_{\alpha(k)+2j+2} &= -(\mathscr{D}^-)^{\alpha(k)+2j+1} \mathscr{D}^+ F_{\alpha(k)+2j} \\ &= -\sum_{i=0}^{j} (\mathscr{D}^-)^{\alpha(k)+2j+1} (\mathscr{D}^+)^{2i+1} g_{\alpha(k)+2j-2i}.\end{aligned}$$

Since the maximal power of $\mathscr{D}^+$ in the last sum is $2j + 1$ and $\alpha(k) = 1$ or 2, we conclude using Lemma 4.2(i) that this sum is equal to 0. By the same argument $(\mathscr{D}^-)^{\alpha(k)+2j+2} \sum_{i=0}^{\alpha(k)+2j+1} (\mathscr{D}^+)^i f_{\alpha(k)+2j+2-i} = 0$, and hence $(\mathscr{D}^-)^{\alpha(k)+2j+2} (\mathscr{D}^+)^{\alpha(k)+2j+2} f_0 = 0$. By Lemma 4.2(ii) we have $(\mathscr{D}^+)^{\alpha(k)+2j+2} f_0 = 0$. We can rewrite (5.2) now:

$$F_{\alpha(k)+2j+2} = \sum_{i=0}^{\alpha(k)+2j+1} (\mathscr{D}^+)^i f_{\alpha(k)+2j+2-i}. \tag{5.3}$$

The following expression is obtained by an application of Lemma 4.1(ii).

$$\mathscr{D}^- F_{\alpha(k)+2j+2} = \sum_{i=1}^{\alpha(k)+2j+1} -\frac{i(2\alpha(k) + 4j + 3 - i)}{4} (\mathscr{D}^+)^{i-1} f_{\alpha(k)+2j+2-i}. \tag{5.4}$$

Using the expression (5.1) we obtain

$$\mathscr{D}^+ F_{\alpha(k)+2j} = \sum_{i=0}^{j} (\mathscr{D}^+)^{2i+1} g_{\alpha(k)+2j-2i}. \tag{5.5}$$

Plugging the expressions (5.4) and (5.5) into the equation $(3.2)_{j+1}$ and applying the uniqueness of Lemma 2.3 we conclude that $f_{\alpha(k)+2j+2-i} = 0$ for odd i, and hence the statement is true for $n = j + 1$. Thus we obtain the required expression for F_{k-1-2m} for $m = 0, \ldots, \left[\frac{k}{2}\right] - 1$. For $m = 0$ we obtain $F_{k-1} = \sum_{j=0}^{[k/2]-1} (\mathscr{D}^+)^{2j} g_{k-1-2j}$ with some $g_{k-1-2j} \in S_{k-1-2j}$, and hence

$$f = \sum_{i=0}^{[k/2]-1} (\mathscr{D}^+)^{2i+1} h_{k-1-2i} \tag{5.6}$$

with some $h_{k-1-2i} \in S_{k-1-2i}$. It is easy to see now that for any f of the form (5.6) one can solve the system (3.2) descending from $(3.2)_{[k/2]}$ to $(3.2)_0$ and using Lemma 4.1(ii). The solution F will be of the required form, and in the expression for F_{k-1-2m} each g_{k-1-2j} is a multiple of h_{k-1-2j}. $\square$

6. Related Results for the Variable Curvature

Now we return to the variable curvature case and discuss some additional results. Besides the two vanishing results mentioned in Sec. 1 dealing with coboundaries from H_0 and S_k for $k \geq 2$ we have one more.

Theorem 6.1. *Let M be a compact negatively curved surface. No function $0 \neq f \in C_1^\infty$ can have zero integrals over all closed geodesics.*

Proof. It follows immediately from Lemma 2.1(iii).

The next theorem identifies another class of functions in C_k^∞ that cannot be coboundaries.

Theorem 6.2. *Let M be a compact negatively curved surface. If $k > 2$, no $f \in H_k$ of the form $f = h_k + \mathscr{D}^+ h_{k-1} + (\mathscr{D}^+)^2 h_{k-2}$ with $h_i \in S_i$, $h_{k-2} \neq 0$ can have zero integrals over all closed geodesics.*

Proof. Suppose f has zero integrals over all closed geodesics. Then by Livshitz Theorem the cohomological equation (1.1) has a smooth solution $F = \sum_{-\infty}^{\infty} F_m$, and it is equivalent to the following system of equations:

$$\begin{aligned} &\mathscr{D}^+ F_{k-1} = h_k + \mathscr{D}^+ h_{k-1} + (\mathscr{D}^+)^2 h_{k-2} \\ &\mathscr{D}^- F_{k-1} + \mathscr{D}^+ F_{k-3} = 0 \\ &\cdots\cdots\cdots\cdots \\ &\mathscr{D}^- F_{\alpha(k)} = 0. \end{aligned} \tag{6.1}$$

First we observe that by Lemma 2.2 $h_k = 0$, and by Lemma 2.1(i) $F_{k-1} = h_{k-1} + \mathscr{D}^+ h_{k-2}$. From the second equation of (6.1) we obtain $\mathscr{D}^+ F_{k-3} = -\mathscr{D}^- F_{k-1} = -\mathscr{D}^- \mathscr{D}^+ h_{k-2} = -\frac{K}{2}(k-2)h_{k-2} - \mathscr{D}^+ \mathscr{D}^- h_{k-2} = -\frac{K}{2}(k-2)h_{k-2}$. If $F_{k-3} \neq 0$ by Lemma 2.2 $\mathscr{D}^+ F_{k-3} \perp S_{k-2}$, in particular, $\langle \mathscr{D}^+ F_{k-3}, h_{k-2} \rangle = 0$. Using (1.4) we obtain $0 = \langle \mathscr{D}^+ F_{k-3}, h_{k-2} \rangle = (k-2)\left\langle -\frac{K}{2} h_{k-2}, h_{k-2} \right\rangle$. But

$$a_0 \|h_{k-2}\|^2 \leq \left\langle -\frac{K}{2} h_{k-2}, h_{k-2} \right\rangle \leq a_1 \|h_{k-2}\|^2,$$

where $a_0 = \min\left(-\frac{K}{2}\right)$ and $a_1 = \max\left(-\frac{K}{2}\right)$, $a_1 \geq a_0 > 0$ which implies (since

$h_{k-2} \neq 0) \left\langle -\frac{K}{2} h_{k-2}, h_{k-2} \right\rangle > 0$, a contradiction. Therefore $F_{k-3} = 0$, hence $\mathscr{D}^+ F_{k-3} = -\frac{K}{2}(k-2)h_{k-2} = 0$ which contradicts $h_{k-2} \neq 0$. □

If $k = 2$ we obtain a stronger result.

Theorem 6.3. *If $f \in C_2^\infty$ has zero integrals over all closed geodesics then $f = \mathscr{D}^+ h_1$ with $h_1 \in S_1$.*

Proof. Let us decompose f according to Lemma 2.3: $f = h_2 + \mathscr{D}^+ h_1 + (\mathscr{D}^+)^2 h_0$ where $h_2 \in S_2$, $h_1 \in S_1$, and $h_0 \in C_0^\infty$. Then the cohomological equation has a solution $F = F_1 \in C_1^\infty$ such that

$$\mathscr{D}^+ F_1 = h_2 + \mathscr{D}^+ h_1 + (\mathscr{D}^+)^2 h_0$$

$$\mathscr{D}^- F_1 = 0.$$

From the first equation we conclude that $h_2 = 0$, and thus $F_1 = h_1 + \mathscr{D}^+ h_0$. From the second equation we see that $F_1 \in S_1$, and by uniqueness of Lemma 2.3 we obtain $\mathscr{D}^+ h_0 = 0$. Thus $f = \mathscr{D}^+ h_1$. □

The converse to Theorem 6.3 is also true, moreover, it is a particular case of a "trivial" class of coboundaries: *If $k \geq 2$, then any function $f \in H_k$ of the form $f = \mathscr{D}^+ h_{k-1}$ with $h_{k-1} \in S_{k-1}$ has zero integrals over all closed geodesics.*

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