

## Spanning sets for automorphic forms and dynamics of the frame flow on complex hyperbolic spaces

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*Dedicated to the memory of Jürgen Moser*

*Abstract.* Let  $G$  be a semisimple Lie group with no compact factors,  $K$  a maximal compact subgroup of  $G$ , and  $\Gamma$  a lattice in  $G$ . We study automorphic forms for  $\Gamma$  if  $G$  is of real rank one with some additional assumptions, using a dynamical approach based on properties of the homogeneous flow on  $\Gamma \backslash G$  and a Livshitz type theorem we prove for such a flow. In the Hermitian case  $G = SU(n, 1)$  we construct relative Poincaré series associated to closed geodesics on  $\Gamma \backslash G / K$  for one-dimensional representations of  $K$ , and prove that they span the corresponding spaces of holomorphic cusp forms.

### 0. Introduction

Let  $G$  be a real connected semisimple Lie group with no compact factors,  $K$  a maximal compact subgroup of  $G$ , and  $\Gamma$  a discrete subgroup of  $G$ . A general definition of automorphic forms for  $\Gamma$  (cf. [11] and [3]) depends on a finite-dimensional representation of  $K$ . It includes classical holomorphic automorphic forms on Fuchsian groups, Maass wave forms, and automorphic forms on bounded symmetric domains (these terms are defined in §1.1). A convenient construction of automorphic forms by means of relative Poincaré series introduced in [20] and [21] allows us to ‘extend’ an automorphic form for a subgroup  $\Gamma_0 \subset \Gamma$  to the whole discrete group  $\Gamma$ . For  $\Gamma_0 = \{e\}$  we obtain Poincaré series, which can be constructed for any smooth absolutely integrable function on  $G$  [1, §5]. For cocompact  $\Gamma$  and in some other cases which are discussed in §1.2, (relative) Poincaré series provide cusp forms. It may happen that a Poincaré series is identically zero. However, it has been shown (see [22, Ch. 3; 2, Ch. 5]) that for holomorphic automorphic forms on a symmetric bounded domain there exist non-zero Poincaré series for suitable weights. It is

not easy, however, to construct relative Poincaré series for non-trivial  $\Gamma_0$ , and even less so to prove that a certain collection spans the corresponding space of cusp forms.

In the case when  $G$  is of real rank one and with finite center, and  $\Gamma$  is a lattice in  $G$ , i.e.  $\Gamma \backslash G$  has finite Haar measure, we propose a program of constructing spanning sets for cusp forms based on dynamical properties of the homogeneous flow on  $\Gamma \backslash G$  and a Livshitz-type theorem that we prove for such a flow (Theorem 8). We carry out this program in the case when, in addition,  $K$  has non-trivial one-dimensional representations. In this case the symmetric space  $G/K$  is Hermitian of real rank one, and by the classification of symmetric spaces, it is a complex hyperbolic space

$$\mathbb{H}_{\mathbb{C}}^n = SU(n, 1)/S(U(n) \times U(1))$$

for some  $n \geq 1$ . Let  $\Gamma$  be a lattice in  $SU(n, 1)$ , and  $\kappa \geq 1$  be a half-integer such that  $(n + 1)\kappa$  is an integer. For each loxodromic element  $\gamma_0 \in \Gamma$  we construct a relative Poincaré series  $\Theta_{\gamma_0, \kappa}$  which is a  $\mathbb{C}$ -valued holomorphic cusp form in  $S_{2(n+1)\kappa}(\Gamma)$  (for exact definitions see §§1.1, 1.2, and 6.3). The main result of this paper is the following theorem.

**THEOREM 1.** *The relative Poincaré series  $\{\Theta_{\gamma_0, \kappa}, \gamma_0 \in \Gamma \text{ loxodromic}\}$  span  $S_{2(n+1)\kappa}(\Gamma)$ .*

This is an extension of a result by Katok [13] for the classical case of Fuchsian groups ( $n = 1$ ). The method is based on duality between the introduced relative Poincaré series and closed geodesics on  $\Gamma \backslash G/K$  (Theorems 11 and 14), dynamical properties of the frame flow on  $\Gamma \backslash G$  which imply a Livshitz theorem-type result (Theorem 8), and the existence of a three-dimensional Lie subalgebra associated to the frame flow which enables the harmonic analysis on  $\Gamma \backslash G$  that completes the proof. The proof utilizes two essential features of the complex hyperbolic space: real rank one and the presence of the complex structure. A natural question of finding finite spanning sets for cusp forms has been addressed in [15] for the classical  $n = 1$  case. The main ingredient there was an approximate version of the Livshitz theorem [16] for geodesic flows, based on more subtle dynamical considerations. The same scheme will probably work for complex hyperbolic spaces of all dimensions and will be pursued elsewhere. The paper is organized as follows. In §1 we review the general definition of automorphic forms and the relative Poincaré series, and discuss their convergence and further implications under additional assumptions. §2 describes the homogeneous flow on  $\Gamma \backslash G$  and its relation to the geodesic flow on the unit tangent bundle  $S(\Gamma \backslash G/K)$ . In §3 we prove a special Livshitz theorem for the homogeneous flow on  $\Gamma \backslash G$  and describe a program of constructing spanning sets for cusp forms in the real rank one case. In §§4–8 we restrict ourselves to the case of the complex hyperbolic space, construct relative Poincaré series associated to closed geodesics in  $\Gamma \backslash G/K$ , and prove Theorem 1.

## 1. Automorphic forms on symmetric spaces of semisimple Lie groups

1.1. *Definitions.* Let  $G$  be a real connected semisimple Lie group with no compact factors [5, 1.13.12],  $\mathfrak{g}$  its Lie algebra, and  $K$  a maximal compact subgroup of  $G$ . We shall denote the homogeneous space by  $X = G/K$ . The group  $G$  acts on itself by left

multiplications, and this action projects to the action on  $X$ . Let  $0$  be the fixed point of  $K$  in  $X$ , then the natural projection

$$\pi : G \rightarrow X$$

is given by

$$\pi : g \mapsto g(0). \quad (1)$$

The Cartan decomposition corresponding to  $K$  is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad (2)$$

where  $\mathfrak{k}$  is the Lie algebra of  $K$ ,  $\text{ad}(K)\mathfrak{p} \subset \mathfrak{p}$ , and  $\mathfrak{p}$  is the orthogonal complement to  $\mathfrak{k}$  with respect to the Killing form on  $\mathfrak{g}$ . The differential  $(d\pi)_e$  at the identity of  $G$  identifies  $\mathfrak{p}$  with  $T_0(X)$ . The kernel of  $(d\pi)_e$  is  $\mathfrak{k}$  [10, p. 208].

To each  $Y \in \mathfrak{g}$  is associated a left-invariant differential operator also denoted by  $Y$ ,

$$Yf(g) = \frac{d}{dt} f(g \cdot \exp tY)|_{t=0}; \quad (3)$$

this linear map from  $\mathfrak{g}$  into the algebra  $D(G)$  of left-invariant differential operators on  $G$  extends to an isomorphism of  $U(\mathfrak{g})$ , the universal enveloping algebra of the Lie algebra  $\mathfrak{g}$  (with complex coefficients) onto  $D(G)$ . In particular, one may consider differential operators  $Y$  for  $Y \in \mathfrak{g}^c$ , the complexification of  $\mathfrak{g}$ . On  $G^0$  the center  $Z(\mathfrak{g})$  of  $U(\mathfrak{g})$  corresponds to the left and right invariant operators and is isomorphic to a polynomial ring in  $\ell$  letters where  $\ell$  is the real rank of  $G$ . The general definition of *automorphic forms* in the sense of Harish-Chandra [11] and Borel [3] assumes only that  $\Gamma$  is a discrete subgroup of  $G$ . In this paper we study *cuspidal forms*, which are automorphic forms with some additional hypotheses imposed, in the case when  $\Gamma$  is not cocompact, concerning the behavior close to certain boundary points. Naturally, if  $\Gamma$  is cocompact, every automorphic form is a cusp form.

Let  $\Gamma$  be a lattice in  $G$ , and  $V$  be a finite-dimensional complex vector space. In what follows  $GL(V)$  will act on  $V$  on the right, and let  $\rho : K \rightarrow GL(V)$  be a fixed (anti-)representation of  $K$  in  $GL(V)$ . Let  $(\cdot, \cdot)$  be a Hermitian  $\rho(K)$ -invariant inner product on  $V$  and  $|\cdot|$  the corresponding norm. We define the norm  $\|\cdot\|_p$  ( $1 \leq p < \infty$ ) on measurable  $V$ -valued,  $\Gamma$ -invariant on the left functions on  $G$  as usual by

$$\|F\|_p = \left( \int_{\Gamma \backslash G} |F(g)|^p dg \right)^{1/p} < \infty,$$

where  $dg$  is the Haar measure on the group  $G$ , and the norm  $\|F\|_\infty$  as the essential supremum of  $|F|$  on  $G$ , and let  $L^p(\Gamma \backslash G) \otimes V$  ( $1 \leq p \leq \infty$ ) be the space of such  $F$  for which  $\|F\|_p < \infty$ .

*Definition 2.* A vector-valued function  $F : G \rightarrow V$  is called  *$Z(\mathfrak{g})$ -finite* if  $Z(\mathfrak{g}) \cdot F$  is annihilated by an ideal  $I$  of  $Z(\mathfrak{g})$  of finite codimension.

*Remark.* If  $I$  has codimension one, this means that  $F$  is an eigenfunction of every operator in  $Z(\mathfrak{g})$ .

*Definition 3.* A smooth vector-valued function  $F : G \rightarrow V$  is called a *cuspidal form* for  $\Gamma$  if:

- (3.1)  $F$  is  $\Gamma$ -invariant on the left and  $K$ -equivariant on the right, i.e.  $F(\gamma g k) = F(g)\rho(k)$  for any  $\gamma \in \Gamma, g \in G,$  and  $k \in K$ ;
- (3.2)  $F$  is  $Z(\mathfrak{g})$ -finite;
- (3.3)  $F \in L^\infty(\Gamma \backslash G) \otimes V$ .

*Remarks.* 1. Conditions (3.1) and (3.2) imply that  $F$  is a real-analytic function on  $\Gamma \backslash G$  [3], hence a function  $F$  which satisfies (3.1) and (3.2) satisfies (3.3) if and only if  $|F|$  is bounded on  $\Gamma \backslash G$ .

2. If  $\Gamma$  is cocompact, the condition (3.3) is automatic.

3. If  $\rho$  is trivial,  $F$  is  $K$ -invariant on the right and is often referred to as a cuspidal form of *weight zero*. In the case when  $\Gamma = SL(2, \mathbb{R})$  such cuspidal forms are called *Maass wave forms*. They are eigenfunctions of the non-Euclidean Laplacian.

Automorphic forms may be defined using an *automorphy factor*, i.e. a smooth map  $\mu : G \times X \rightarrow GL(V)$  that satisfies the 1-cocycle property, i.e.

$$\mu(g_1 g_2, x) = \mu(g_1, g_2(x))\mu(g_2, x) \tag{4}$$

for all  $g_1, g_2 \in G$  and  $x \in X$ .

If a smooth function  $f : X \rightarrow V$  satisfies the *automorphy equation*

$$(f|\gamma)(x) := f(\gamma(x))\mu(\gamma, x) = f(x), \tag{5}$$

for any  $\gamma \in \Gamma$  and  $x \in X$ , then its ‘lift’ to  $G$ ,  $\tilde{f} : G \rightarrow V$  defined by

$$\tilde{f}(g) = f(g(0))\mu(g, 0), \tag{6}$$

satisfies (3.1) with  $\rho(k) = \mu(k, 0)$ .

*Definition 4.* Let  $\mu$  be an automorphy factor. A smooth function  $f : X \rightarrow V$  is called a *cuspidal form of type  $\mu$*  if its lift to the group  $G$  defined by (6) is a cuspidal form of the Definition 3 with  $\rho(k) = \mu(k, 0)$ .

We denote the space of cuspidal forms of type  $\mu$  (as well as the space of their lifts to  $G$ ) by  $S_\mu(\Gamma)$ .

The space  $L^2(\Gamma \backslash G) \otimes V$  is a Hilbert space with the inner product

$$(F_1, F_2) = \int_{\Gamma \backslash G} (F_1(g), F_2(g)) dg \tag{7}$$

corresponding to the norm  $\|\cdot\|_2$  introduced above (inside the integral is the inner product on  $V$ ). Using the Cartan decomposition of the group  $G$  corresponding to (2),

$$g = g_x k, \tag{8}$$

where  $k \in K$  and  $g_x$  depends only on  $x = g(0) \in X$ , and the cocycle property of  $\mu$  (4), we obtain, for the lifts of automorphic forms  $\tilde{f}_1(g) = f_1(x)\mu(g, 0)$  and  $\tilde{f}_2(g) = f_2(x)\mu(g, 0)$  with  $x = g(0)$ , an analogue of the Petersson inner product

$$(f_1, f_2) := (\tilde{f}_1, \tilde{f}_2) = \int_{\Gamma \backslash G/K} (f_1(x)\mu(g_x, 0), f_2(x)\mu(g_x, 0)) dV, \tag{9}$$

where  $dV$  is a  $G$ -invariant volume form on the symmetric space  $X$ .

Since  $\Gamma$  is a lattice, we have

$$L^\infty(\Gamma \backslash G) \otimes V \subset L^2(\Gamma \backslash G) \otimes V \subset L^1(\Gamma \backslash G) \otimes V.$$

Therefore, the integral (9) converges for cusp forms; in fact, it converges if either  $f_1$  or  $f_2$  is a cusp form. Thus  $S_\mu(\Gamma) \subset L^1(\Gamma \backslash G) \otimes V$ . Notice that in general a function satisfying (3.1) and (3.2) and belonging to  $L^1(\Gamma \backslash G) \otimes V$  is not necessarily a cusp form (e.g. for  $G = SL(2, \mathbb{R})$ ,  $\Gamma = SL(2, \mathbb{Z})$  and Maass wave forms, a residual Eisenstein series has behavior like  $y^a$ , for  $a < \frac{1}{2}$ , in the cusp; it satisfies (3.1) and (3.2), is in  $L^1(\Gamma \backslash G)$  but is not bounded), although for holomorphic forms on a symmetric bounded domain it is (see Remark 1 below).

Let  $X$  be a symmetric bounded domain. If  $f$  is a holomorphic function on  $X$ , condition (3.2) is satisfied [3]. Let  $\mathcal{H}_\mu^p(\Gamma \backslash G)$  be the subspace of  $L^p(\Gamma \backslash G) \otimes V$  consisting of holomorphic functions on  $X$  which satisfy (5). Then  $S_\mu(\Gamma) = \mathcal{H}_\mu^\infty(\Gamma \backslash G)$ .

*Remarks.* 1. For holomorphic automorphic forms on a symmetric bounded domain  $X$  all spaces  $\mathcal{H}_\mu^p(\Gamma \backslash G)$  ( $1 \leq p \leq \infty$ ) coincide for suitable  $\mu$  by Satake's theorem [24; 2, Ch. 11, §5].

2. Although, in general,  $\dim S_\mu = \infty$ , for holomorphic automorphic forms on a symmetric bounded domain  $X$ ,  $\dim S_\mu < \infty$  [22, Ch. 4; 12; 2, Ch. 11, §5].

3. The classical case of holomorphic automorphic forms on Fuchsian groups corresponds to  $G = SL(2, \mathbb{R}) \approx SU(1, 1)$ .

1.2. *The relative Poincaré series.* Let  $\Gamma_0$  be a subgroup of  $\Gamma$ . The following construction allows us to 'extend' an automorphic form for  $\Gamma_0$  to an automorphic form for  $\Gamma$ .

**THEOREM 5.** *Let  $\Gamma_0$  be a subgroup of  $\Gamma$ ,  $f : X \rightarrow V$  a function satisfying the automorphy equation (5) for  $\Gamma_0$ , and  $\tilde{f}$  its lift to the group  $G$  by the formula (6) such that:*

- (1)  $\tilde{f}$  is  $Z(\mathfrak{g})$ -finite;
- (2)  $\tilde{f} \in L^1(\Gamma_0 \backslash G) \otimes V$ .

*Then the series  $\Theta_{\Gamma_0} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} f|_\gamma$ , called the relative Poincaré series for  $\Gamma_0$ , converges absolutely and uniformly on compact sets, and represents a function satisfying (3.1) and (3.2), and belonging to  $L^1(\Gamma \backslash G) \otimes V$ .*

The proof follows the lines of the argument of Harish-Chandra for the Poincaré series (see [2, 3]). It is proved that the series converges absolutely and uniformly on compact sets and satisfies (3.1) and (3.2). For cocompact  $\Gamma$  this proves that  $\Theta_{\Gamma_0}$  is a cusp form. If  $\Gamma$  is a lattice, it follows that  $\tilde{\Theta}_{\Gamma_0} \in L^1(\Gamma \backslash G) \otimes V$  which, without additional assumptions, does not imply that  $\Theta_{\Gamma_0}$  is a cusp form. However, for holomorphic automorphic forms on a symmetric bounded domain, according to Satake's theorem, the relative Poincaré series  $\Theta_{\Gamma_0}$  are cusp forms.

If  $X = G/K$  is a symmetric bounded domain, then any absolutely integrable function on  $X$ , for example any polynomial on  $X$ , produces a holomorphic cusp form for the trivial  $\Gamma_0 = \{e\}$  [2, Ch. 11, §1].

If  $\Gamma_0$  is a parabolic subgroup of  $\Gamma$ , the above construction gives so-called Poincaré–Eisenstein series.

For general  $\Gamma_0$ , it is not easy to find a function  $f$  which satisfies conditions (1) and (2) of Theorem 5. We shall give a construction for cyclic loxodromic subgroups of isometries of complex hyperbolic spaces in §6.3.

## 2. The homogeneous flow

2.1. *Definitions.* From now on we suppose, in addition to the standing assumptions of §1.1, that  $G$  is of real rank one and has finite center (and therefore non-compact simple). Then any non-zero  $\mathfrak{D} \in \mathfrak{p}$  defines a maximal Abelian subspace of  $\mathfrak{p}$ ,  $\mathfrak{a} = \mathbb{R}\mathfrak{D}$ , consisting of real semisimple elements (cf. [10, pp. 401, 431]). We fix a non-zero  $\mathfrak{D} \in \mathfrak{p}$  and let

$$A = \{\exp t\mathfrak{D} = a_t \mid t \in \mathbb{R}\}$$

be the corresponding maximal split Abelian subgroup of  $G$ . Acting by right multiplications on  $G$ , it defines a standard flow  $\tilde{\alpha}_t$  on  $G$ ,

$$\tilde{\alpha}_t(g) = ga_t. \quad (10)$$

An orbit of  $g \in G$ ,  $\{ga_t \mid t \in \mathbb{R}\}$ , projects by (1) to a geodesic  $\{ga_t(0) = ga_tg^{-1}(x_g) \mid t \in \mathbb{R}\}$  on  $X$  passing through  $x_g = g(0)$  [10, p. 208]. The subgroup  $A$  itself projects on  $X$  to the ‘standard geodesic’

$$\mathfrak{J} = \{a_t(0) \mid t \in \mathbb{R}\}$$

passing through 0. There is a subgroup  $W \subset K$  which, acting on  $X$  on the left by isometries, fixes  $\mathfrak{J}$  pointwise. Then  $AW = WA = Z(A)$ , the centralizer of  $A$ .

*Definition 6.* An element  $g \in G$  is called *hyperbolic* (or *regular*) if it is conjugate to an element in  $A$ .

*Definition 7.* An element  $g \in G$  is called *loxodromic* if it is conjugate to an element in  $Z(A)$ .

Any loxodromic (and, therefore, hyperbolic) element in  $G$  fixes a geodesic in  $X$ , called its *axis*, and has two fixed points in the boundary  $\partial X$ . It is easy to see that geodesics in  $X$  are exactly the axes of loxodromic elements in  $G$ .

Geodesics naturally lift to  $G/W$  which can be identified with the unit tangent bundle  $S(G/K)$ . Recall that as subgroups of  $G$  acting on the space  $X = G/K$  on the left by isometries,  $K$  fixes the point  $0 \in X$ , and  $W \subset K$  fixes  $\mathfrak{J}$  pointwise, and hence the unit tangent vector to  $\mathfrak{J}$  at 0 is denoted by  $\iota$ . Thus we have two natural mappings:

$$\pi : G \rightarrow G/K$$

given by (1) and

$$\sigma : G \rightarrow G/W$$

given by

$$\sigma(g) = g_*(\iota),$$

where  $g_*$  is the differential of  $g$ . Defining the mapping

$$\tau : G/W \rightarrow G/K$$

such that  $\tau \circ \sigma = \pi$ , we see that  $G/W$  is a sphere bundle over  $G/K$  with the fiber at each point identified with  $K/W$ , the unit tangent space at this point. For any geodesic  $C$  in  $X$  passing through a point  $x_0$  there exists a  $g \in G$  mapping  $\mathcal{J}$  into  $C$  in such a way that  $g(0) = x_0$ . This follows from the fact that the group  $G$  acts transitively on the unit tangent bundle  $S(X) = G/W$ . The transformation  $g$  is not unique but is determined up to the right multiplication by the subgroup  $W$  which fixes every point of  $\mathcal{J}$ . Thus we obtain a family of lifts of  $C$ ,  $\{gwa_t \mid w \in W, t \in \mathbb{R}\}$  to the group  $G$  parametrized by the group  $W$ , which are orbits of the flow  $\tilde{\alpha}_t$ . Since the subgroups  $A$  and  $W$  commute, the flow  $\tilde{\alpha}_t$  projects to the *geodesic flow* on the factor  $G/W$ , denoted by  $\alpha_t$ :

$$\alpha_t(\sigma(g)) = \sigma(\tilde{\alpha}_t(g)).$$

We see that all lifts  $\{gwa_t \mid w \in W, t \in \mathbb{R}\}$  project to the orbit  $\alpha_t(gW)$  of the geodesic flow on  $G/W$ .

Let  $\Gamma$  be a lattice in  $G$  and  $\Gamma \backslash X = M$ . The action of  $A$  descends to the factor  $\Gamma \backslash G$  and defines a *homogeneous flow*  $\tilde{\varphi}_t$  on  $\Gamma \backslash G$  denoted by  $\tilde{\varphi}_t$ ,

$$\tilde{\varphi}_t(\Gamma g) = \Gamma ga_t.$$

The mappings  $\pi$ ,  $\sigma$ , and  $\tau$  also descend to the corresponding left factors by  $\Gamma$ , and the homogeneous flow  $\tilde{\varphi}_t$  projects to the *geodesic flow* on  $S(M) = \Gamma \backslash G/W$ , denoted by  $\varphi_t$ , by the formula

$$\varphi_t(\sigma(g)) = \sigma(\tilde{\varphi}_t(g)).$$

For the base point  $x = g(0) = \pi(g) \in X$  we shall use the same notation:

$$\varphi_t(x) = \pi(\tilde{\varphi}_t(g)).$$

Respectively, we have two differential operators:

$$\mathfrak{D}f(g) = \frac{d}{dt}f(\tilde{\varphi}_t(g))|_{t=0},$$

defined on the set of functions on  $\Gamma \backslash G$  differentiable along the orbits of  $\tilde{\varphi}_t$ , and

$$\mathcal{D}f(v) = \frac{d}{dt}f(\varphi_t(v))|_{t=0},$$

defined on the set of functions on  $\Gamma \backslash G/W$  differentiable along the orbits of  $\varphi_t$ .

The following characterization of closed geodesics in  $M = \Gamma \backslash X$  is immediate. Closed geodesics in  $M$  are the axes of loxodromic elements in  $\Gamma$ . They are in one-to-finite correspondence with conjugacy classes of primitive loxodromic elements in  $\Gamma$ .

It is easy to see that closed geodesics in  $M$  lift to closed orbits of the geodesic flow  $\varphi_t$  on  $\Gamma \backslash G/W$  while their lifts to  $\Gamma \backslash G$  are not necessarily closed.

We shall denote a closed geodesic in  $M$  corresponding to the axis of a loxodromic element  $\gamma_0 \in \Gamma$  in  $\Gamma \backslash G/K$  as well as its lift to  $\Gamma \backslash G/W$  by  $[\gamma_0]$ , and a family of lifts to  $\Gamma \backslash G$  by  $\{[\gamma_0]_w \mid w \in W\}$ .

2.2. *Dynamics of homogeneous and geodesic flows.* It is a standard fact that there exists a left  $G$ -invariant Riemannian metric on  $X = G/K$  relative to which  $X$  is a symmetric space [18, Ch. 11, Theorem 8.6]. Since the real rank of  $G$  is equal to one, this metric is of negative sectional curvature, which is the main reason for the following properties of the homogeneous flow  $\tilde{\varphi}_t$  and the geodesic flow  $\varphi_t$  which we review below. A standard procedure [18, Ch. 11, §6] provides  $G$ -invariant Riemannian metrics on  $G$  and  $G/W$  (this can be done for any compact subgroup of  $G$  in place of  $W$ ) in such a way that the distance (which we will denote in both spaces by  $d$ ) does not increase after projection: for any  $g_1, g_2 \in G$ ,

$$d(\sigma(g_1), \sigma(g_2)) \leq d(g_1, g_2). \tag{11}$$

The Riemannian volume on  $G$  coincides with the Haar measure.

We have seen that the homogeneous flow  $\tilde{\varphi}_t$  on  $\Gamma \backslash G$ , considered as a fibered bundle over  $\Gamma \backslash G/W$ , projects to the geodesic flow  $\varphi_t$  on the base  $\Gamma \backslash G/W$ . In addition, if  $g_1$  and  $g_2$  belong to the same fiber, i.e.  $g_2 = g_1 w$  for some  $w \in W$ , we have

$$d(\tilde{\varphi}_t(g_1), \tilde{\varphi}_t(g_2)) = d(g_1, g_2)$$

by left-invariance of the metric and since  $w$  and  $a_t$  commute.

The geodesic flow  $\varphi_t$  on  $\Gamma \backslash G/W$  is *Anosov (hyperbolic)* [17, Theorem 17.6.2] with the corresponding  $D\varphi_t$ -invariant splitting of the tangent bundle of  $\Gamma \backslash G/W$ :

$$T(\Gamma \backslash G/W) = E^0 \oplus E^s \oplus E^u,$$

and foliations  $W^0$  (the orbit foliation),  $W^s$ , and  $W^u$ , respectively.

It follows that the homogeneous flow  $\tilde{\varphi}_t$  is an isometric extension of  $\varphi_t$  and, as such, is *partially hyperbolic*, i.e. there is a  $C^\infty$   $D\tilde{\varphi}_t$ -invariant splitting of the tangent bundle of  $\Gamma \backslash G$ :

$$T(\Gamma \backslash G) = \tilde{E}^0 \oplus \tilde{E}^s \oplus \tilde{E}^u$$

with the following properties. The integral manifolds of the distribution  $\tilde{E}^0$  form the *neutral* foliation denoted by  $\tilde{W}^0$ ; its leaf through a point  $g$  being  $gW \times \tilde{O}(g)$ , where  $\tilde{O}(g)$  is the orbit of the flow. The flow restricted to the leaves of  $\tilde{W}^0$  is isometric. The integral manifolds of the distributions  $\tilde{E}^s$  ( $\tilde{E}^u$ ) form the *stable (unstable)* foliation denoted by  $\tilde{W}^s$  ( $\tilde{W}^u$ ).

Let us denote the distance along the leaves of the foliations  $\tilde{W}^s$  and  $\tilde{W}^u$  by  $d^s$  and  $d^u$ , respectively. Then there exist  $C, \lambda > 0$  such that for any  $g_1, g_2$  lying on the same leaf of  $\tilde{W}^j$  ( $j = u, s$ ),

$$d^j(\tilde{\varphi}_t(g_1), \tilde{\varphi}_t(g_2)) \leq C e^{-\lambda|t|} d^j(g_1, g_2) \tag{12}$$

for  $j = s, t \geq 0$  and for  $j = u, t \leq 0$ .

The foliations  $\tilde{W}^0, \tilde{W}^s$ , and  $\tilde{W}^u$  are transversal and project to the corresponding foliations  $W^0, W^s$ , and  $W^u$  of  $\varphi_t$  with the same estimates (12). The homogeneous flow  $\tilde{\varphi}_t$  preserves the Haar measure on  $\Gamma \backslash G$  while  $\varphi_t$  preserves the Riemannian volume on  $\Gamma \backslash G/W$ .



3. *Dynamical approach to construction of cusp forms in the real rank one case*

3.1. *Cohomological equation for the homogeneous flow.* The hyperbolic properties of the geodesic flow on  $\Gamma \backslash G / W$  imply the Anosov closing lemma and its strengthening (see [17, Theorem 6.4.15 and Proposition 6.4.16]). As a consequence, the Livshitz theorem holds. Since the space  $S(M)$  may not be compact, it is natural to formulate it for the class  $\mathcal{BL}$  of bounded Lipschitz functions.

THE LIVSHITZ THEOREM. *Let  $f \in \mathcal{BL}(S(M))$  be such that it has zero integrals over all closed orbits of the geodesic flow on  $S(M)$ . Then there exists a Lipschitz function  $F$  on  $S(M)$  which is differentiable in the direction of the flow  $\varphi_t$ , and such that*

$$\mathcal{D}F = f.$$

This theorem has been proved in [19] for Anosov flows on compact manifolds. It has been proved later [4] that if the function  $f$  is  $C^\infty(S(M))$ , the solution is also  $C^\infty(S(M))$ . The original proof works with minor alterations for manifolds with cusps (the proof for Fuchsian groups with cusps is given in [14, Appendix]).

By the Moore’s ergodicity theorem [26, Theorem 2.2.6] the homogeneous flow on  $\Gamma \backslash G$  is ergodic and hence topologically transitive, but it is not Anosov, the closing lemma does not hold, and hence a straightforward generalization of the Livshitz theorem does not hold either. However, the same conclusion holds under a stronger hypothesis, and the proof is similar to the proof of the Livshitz theorem for the geodesic flow.

THEOREM 8. (Special Livshitz theorem) *Let  $f \in \mathcal{BL}(\Gamma \backslash G)$  be such that for every closed geodesic  $[\gamma_0]$  in  $S(M)$  and every  $w \in W$  the integral in  $\Gamma \backslash G$*

$$\int_{[\gamma_0]_w} f(\tilde{\varphi}_s(g)) ds = 0.$$

*Then there exists a Lipschitz function  $F$  on  $\Gamma \backslash G$  which is constant on  $W$ -cosets and differentiable in the direction of the flow  $\tilde{\varphi}_t$ , such that*

$$\mathfrak{D}F = f. \tag{13}$$

*Remark.* We shall give a proof for real-valued functions. Then the theorem will obviously hold for functions valued in any  $\mathbb{R}^m$ .

*Proof.* The homogeneous flow on  $\Gamma \backslash G$  is topologically transitive, i.e. there exists a  $g \in G$  whose orbit  $\tilde{\mathcal{O}}(g) = \{\tilde{\varphi}_t(g) \mid t \in \mathbb{R}\}$  is dense in  $\Gamma \backslash G$ . We define a function  $F$  on this orbit by the formula

$$F(\tilde{\varphi}_t(g)) = \int_0^t f(\tilde{\varphi}_s(g)) ds.$$

We need to prove that  $F$  satisfies a Lipschitz condition on  $\tilde{\mathcal{O}}(g)$  and hence can be extended to  $\Gamma \backslash G$  as a Lipschitz function. Given  $\epsilon > 0$ , let  $t_1 < t_2$  be such that

$$d(\tilde{\varphi}_{t_1}(g), \tilde{\varphi}_{t_2}(g)) < \epsilon. \tag{14}$$

We will show that

$$\int_{t_1}^{t_2} f(\tilde{\varphi}_s(g)) ds = O(\epsilon). \tag{15}$$

As is customary, ‘ $= O(x)$ ’ means that the expression is ‘ $\leq Cx$ ’ for some constant  $C$ . We will use this notation to avoid an accumulation of constants.

Since the homogeneous flow is not Anosov, we cannot hope to be able to approximate any  $\epsilon$ -close piece of the dense orbit by a closed one as for the geodesic flow. Nevertheless, the close relation with the geodesic flow allows us to prove the following.

LEMMA 9. (Approximation lemma) *Using the notation above, for a piece of a dense orbit of  $\tilde{\varphi}_t$  (14), there is a lift of a closed orbit of  $\varphi_t$  to  $\Gamma \backslash G$ ,  $\{\tilde{\varphi}_t(\tilde{p}) \mid 0 \leq t \leq T\}$  with  $|T - (t_2 - t_1)| = O(\epsilon)$ , such that for  $0 \leq t \leq t_2 - t_1$*

$$d(\tilde{\varphi}_{t_1+t}(g), \tilde{\varphi}_t(\tilde{p})) = O(\epsilon e^{-\lambda(\min(t, t_2-t_1-t))}). \tag{16}$$

*Proof.* By (11) the projection of  $\tilde{O}(g)$  onto  $\Gamma \backslash G/W$  is a dense orbit  $\mathcal{O}(v)$  of the geodesic flow  $\varphi_t$  with  $v = \sigma(g)$ , and it follows that

$$d(\varphi_{t_1}(v), \varphi_{t_2}(v)) < \epsilon.$$

Let  $\mathcal{O}(v)|_{t_1}^{t_2} = \{\varphi_t(v) \mid t_1 \leq t \leq t_2\}$  and  $\tilde{O}(g)|_{t_1}^{t_2} = \{\tilde{\varphi}_t(g) \mid t_1 \leq t \leq t_2\}$  be the corresponding pieces of orbits. Let us denote  $\varphi_{t_1}v = v_1$ ,  $\tilde{\varphi}_{t_1}g = g_1$ ,  $\varphi_{t_2}v = v_2$  and  $\tilde{\varphi}_{t_2}g = g_2$ . Since the geodesic flow  $\varphi_t$  is Anosov, by the Anosov closing lemma there exists a  $p \in \Gamma \backslash G/W$  such that  $d(p, v_1) = O(\epsilon)$  and its orbit  $\mathcal{O}(p) = \{\varphi_t(p) \mid t \in \mathbb{R}\}$  is closed, i.e.  $\varphi_T(p) = p$  with the period  $T$  satisfying  $|T - (t_2 - t_1)| = O(\epsilon)$ . The point  $p$  can be chosen such that the leaf of the unstable foliation containing  $v_1$ ,  $W^u(v_1)$ , and the leaf of the stable foliation containing  $p$ ,  $W^s(p)$ , intersect, and since they are transversal, they intersect at a point  $W^s(p) \cap W^u(v_1) = q$ , and  $d^u(v_1, q) = O(\epsilon)$ .

Lift  $W^u(v_1)$  to  $\tilde{W}^u(g_1)$ . Since  $q \in W^u(v_1)$  its lift  $\tilde{W}^u(g_1)$  intersects  $qW$ , so let  $\tilde{W}^u(g_1) \cap qW = \tilde{q}$ , and similarly,  $\tilde{W}^s(\tilde{q})$  intersects  $pW$ ,  $\tilde{W}^s(\tilde{q}) \cap pW = \tilde{p}$ , and  $\tilde{\varphi}_T(\tilde{p}) \in pW$  (see Figure 1 which represents the picture in the direction transversal to the orbits).

Since the distance between  $\varphi_t(p)$  and  $\varphi_t(q)$  decreases exponentially for  $t > 0$ , and  $d^s(p, q) = O(\epsilon)$ , we have

$$d^s(\varphi_t(p), \varphi_t(q)) = O(\epsilon e^{-\lambda t}),$$

and since all leaves are transversal we have the same estimate on  $\tilde{W}^s(\tilde{q})$ ;

$$d^s(\tilde{\varphi}_t(\tilde{p}), \tilde{\varphi}_t(\tilde{q})) = O(\epsilon e^{-\lambda t}).$$

Similarly, since the distance between  $\varphi_{t_2-t_1-t}(v_1)$  and  $\varphi_{t_2-t_1-t}(q)$  decreases exponentially for  $t > 0$ , and  $d^u(\varphi_{t_2-t_1}(v_1), \varphi_{t_2-t_1}(q)) = O(\epsilon)$ , we have

$$d^u(\varphi_t(v_1), \varphi_t(q)) = O(\epsilon e^{-\lambda(t_2-t_1-t)}),$$

and the same estimate on  $\tilde{W}^s(\tilde{q})$ :

$$d^u(\tilde{\varphi}_t(g_1), \tilde{\varphi}_t(\tilde{q})) = O(\epsilon e^{-\lambda(t_2-t_1-t)}).$$

Therefore, for  $0 \leq t \leq t_2 - t_1$  we have simultaneous estimates

$$d(\varphi_{t_1+t}(v), \varphi_t(p)) = O(\epsilon e^{-\lambda(\min(t, t_2-t_1-t))}),$$

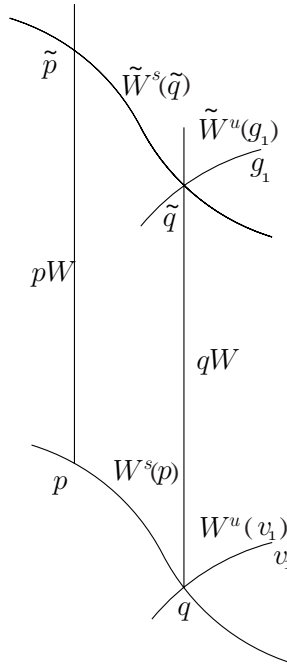


FIGURE 1.

and

$$d(\tilde{\varphi}_{t_1+t}(g), \tilde{\varphi}_t(\tilde{p})) = O(\epsilon e^{-\lambda(\min(t, t_2-t_1-t))}).$$

□

It follows from (16) that

$$\left| \int_{t_1}^{t_2} f(\tilde{\varphi}_s(g)) ds - \int_0^T f(\tilde{\varphi}_s(\tilde{p})) ds \right| = O(\epsilon),$$

and since by the hypothesis

$$\int_0^T f(\tilde{\varphi}_s(\tilde{p})) ds = 0,$$

we obtain the required estimate (15). This proves the claim. Thus  $F$  can be extended from the dense orbit to a Lipschitz function in  $\Gamma \backslash G$ . Since  $\mathcal{D}F = f$  on the dense orbit, it follows that  $F$  is differentiable in the direction of the homogeneous flow and  $\mathcal{D}F = f$  in  $\Gamma \backslash G$ .

A similar argument shows that the function  $F$  is constant on  $W$ -cosets. Let  $\sigma(g_1) = \sigma(g_2) = v$ . There exist  $g'_1$  and  $g'_2$  on the dense orbit of  $\tilde{\varphi}_t$   $\epsilon$ -close to  $g_1$  and  $g_2$ , respectively. Then the projection to  $\Gamma \backslash G / W$  is an  $\epsilon$ -close orbit of  $\varphi_t$ . Find a closed orbit of  $\varphi_t$   $\epsilon$ -close to the projection, and lift it back to  $\Gamma \backslash G$  in the same manner as we described before. It follows from the exponential estimates that

$$|F(g'_2) - F(g'_1)| = O(\epsilon),$$

and as  $\epsilon \rightarrow 0$ , we obtain  $F(g_1) = F(g_2)$ .

□

3.2. *Spanning of cusp forms via relative Poincaré series in the real rank one case.* Our program of constructing spanning sets for  $S_\mu(\Gamma)$  consists of three steps.

(1) *Construction of relative Poincaré series associated to closed geodesics in  $M = \Gamma \backslash G/K$ .* Fix a finite-dimensional complex vector space  $V$  and an automorphy factor  $\mu : G \times X \rightarrow GL(V)$ . For each loxodromic element  $\gamma_0 \in \Gamma$  find a function  $q_{\gamma_0} : X \rightarrow V$  which satisfies conditions of Theorem 5 for the subgroup  $\Gamma_0 = \langle \gamma_0 \rangle$ . Having such a function, we can apply Theorem 5 to obtain a relative Poincaré series  $\Theta_{\gamma_0}$ , and prove that they are cusp forms.

(2) *The period formula.* Let  $f \in S_\mu(\Gamma)$ ,  $\gamma_0 \in \Gamma$  loxodromic, and  $\tilde{f}$  be the lift of  $f$  to  $G$ . Then for any lift  $[\gamma_0]_w$ ,  $w \in W$  of closed geodesic  $[\gamma_0]$  to  $G$ ,

$$(f, \Theta_{\gamma_0}) = C \int_{[\gamma_0]_w} \tilde{f} dt$$

with an explicit constant  $C$  depending only on  $[\gamma_0]_w$ .

(3) *Cohomological equation for cusp forms.* In order to prove that the relative Poincaré series  $\{\Theta_{\gamma_0}, \gamma_0 \in \Gamma \text{ loxodromic}\}$  span  $S_\mu(\Gamma)$ , we assume that there is a cusp form  $f \in S_\mu(\Gamma)$  such that  $(f, \Theta_{\gamma_0}) = 0$  for all relative Poincaré series  $\Theta_{\gamma_0}$ . In order to apply the special Livshitz theorem (Theorem 8) we first need to show that  $\tilde{f} \in \mathcal{BL}(\Gamma \backslash G)$  and then use (2). Thus we obtain a solution  $F$  of the cohomological equation  $\mathfrak{D}F = \tilde{f}$ , which is Lipschitz and differentiable in the direction of the flow  $\tilde{\varphi}_t$ , hence  $\tilde{f}$  is a *coboundary*. Proving that cusp forms cannot be coboundaries will imply the result.

We are able to carry out this program in the case when  $K$  has non-trivial one-dimensional representations. In this case the symmetric space is Hermitian of real rank one, and by the classification of symmetric spaces [10, p. 518], it is a complex hyperbolic space

$$\mathbb{H}_{\mathbb{C}}^n = SU(n, 1)/S(U(n) \times U(1))$$

for some  $n \geq 1$ . The rest of the paper is devoted to this case.

#### 4. The complex hyperbolic space

4.1. *The unit ball model.* The group  $G = SU(n, 1)$  is the group of  $(n + 1) \times (n + 1)$  complex unimodular matrices preserving the Hermitian form

$$\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n - z_{n+1} \bar{w}_{n+1}$$

on  $\mathbb{C}^{n,1}$ . In other words,

$$\begin{aligned} G &= \{A \in SL(n + 1, \mathbb{C}) \mid \langle A \cdot z, A \cdot w \rangle = \langle z, w \rangle \forall z, w \in \mathbb{C}^{n,1}\} \\ &= \{A \in SL(n + 1, \mathbb{C}) \mid A^T \cdot S \cdot \bar{A} = S\}, \end{aligned}$$

where  $1_n$  denotes the  $n \times n$  identity matrix,  $S = \begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix}$ , and  $\cdot$  is used for matrix multiplication.

The maximal compact subgroup of  $G$

$$K = S(U(n) \times U(1)) = \left\{ \begin{pmatrix} u_n & 0 \\ 0 & a \end{pmatrix} \mid u_n \in U(n), a = (\det u_n)^{-1} \right\}.$$

The symmetric space  $G/K$  is called the *complex hyperbolic space* and is denoted by  $\mathbb{H}_{\mathbb{C}}^n$ . It can be identified with the projectivized space of *negative* vectors  $z \in \mathbb{C}^{n,1}$ , i.e. such that  $\langle z, z \rangle < 0$ :

$$\mathbb{H}_{\mathbb{C}}^n \cong \mathbb{P}(\{z \in \mathbb{C}^{n,1} \mid \langle z, z \rangle < 0\}),$$

or, equivalently, with the set of *negative* lines in  $\mathbb{C}^{n,1}$ , or in homogeneous coordinates, with the unit ball in  $\mathbb{C}^n$ :

$$B^n = \{z \in \mathbb{C}^n \mid z_1 \bar{z}_1 + \dots + z_n \bar{z}_n < 1\}.$$

The last identification is obtained by the biholomorphic embedding

$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{P}(\mathbb{C}^{n,1}) \\ z &\mapsto \begin{pmatrix} z \\ 1 \end{pmatrix} \end{aligned} \tag{17}$$

of  $\mathbb{C}^n$  onto the affine chart of  $\mathbb{P}(\mathbb{C}^{n,1})$  defined by  $z_{n+1} \neq 0$ .  $B^n$  is a bounded domain in  $\mathbb{C}^n$  and hence a complex manifold. An invariant Riemannian metric on  $B^n$  of the form  $ds^2 = g_{ij} dz_i d\bar{z}_j$  obtained by a standard construction [18, Ch. 9, §6] is of negative sectional curvature and is called the *Bergman metric*. We normalize it according to [8, III.1.3] so that the sectional curvature is pinched between  $-1$  and  $-\frac{1}{4}$ . Using [10, Ch. 8, Proposition 2.5], one easily obtains the following relation between the volume form  $dV$  on  $B^n$  corresponding to the Riemannian metric and the Euclidean volume form  $dV_E$  on  $\mathbb{C}^n$ :

$$dV = \frac{4^n dV_E}{(-\langle z, z \rangle)^{n+1}}. \tag{18}$$

Using the above terminology, we call a vector  $z \in \mathbb{C}^{n,1}$  *positive* if  $\langle z, z \rangle > 0$  and *null* if  $\langle z, z \rangle = 0$ . We shall always assume that negative and null vectors are represented in the form (17), and keep the notation  $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n - 1$  for them.

4.2. *The tangent bundle.* For each point  $z = (z_1, \dots, z_n) \in B^n$  we write  $z_j = x_j + iy_j$ . The real tangent space at  $z$ ,  $T_z(B^n) = T_z$ , refers to the tangent space of the underlying  $2n$ -dimensional real  $C^\infty$  manifold and has a basis

$$\frac{\partial}{\partial x_1}(z), \frac{\partial}{\partial y_1}(z), \dots, \frac{\partial}{\partial x_n}(z), \frac{\partial}{\partial y_n}(z).$$

The vector fields

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

form a basis of the complexification  $T_z^c$  of  $T_z$ . The real tangent space  $T_z$  can be identified with the  $n$ -dimensional complex subspace of  $T_z^c$  of the vectors in the form

$$\eta = \sum_{j=1}^n \eta_j \frac{\partial}{\partial z_j} + \bar{\eta}_j \frac{\partial}{\partial \bar{z}_j},$$

where  $\eta_j \in \mathbb{C}$ . We shall always refer to the tangent vectors as being in the form  $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$  ( $\eta_j \in \mathbb{C}$ ).

4.3. *The action of  $SU(n, 1)$ .* The group  $G = SU(n, 1)$  acts on  $B^n$  by biholomorphic transformations (automorphisms): for

$$g = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix}, \tag{19}$$

$$g(z) = \left( \frac{a_{11}z_1 + \cdots + a_{1n}z_n + b_1}{c_1z_1 + \cdots + c_nz_n + d}, \dots, \frac{a_{n1}z_1 + \cdots + a_{nn}z_n + b_n}{c_1z_1 + \cdots + c_nz_n + d} \right) \tag{20}$$

(sometimes we will also use  $gz$  for this action), which are isometries of  $B^n$  with respect to the Bergman metric. This action corresponds to the left multiplication on  $G$ . The differential of  $g : B^n \rightarrow B^n$  at  $z$ ,

$$g_* : T_z \rightarrow T_{g(z)},$$

is given by the Jacobian matrix which can be written as

$$J(g, z) = (c \cdot z + d)^{-2} (A(c \cdot z + d) - (A \cdot z + b) \cdot c), \tag{21}$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix}, \quad c = (c_1 \quad \cdots \quad c_n), \quad z = \begin{pmatrix} z_1 \\ \cdots \\ z_n \end{pmatrix}.$$

A direct calculation shows that

$$\det J(g, z) = (c \cdot z + d)^{-(n+1)} = (c_1z_1 + \cdots + c_nz_n + d)^{-(n+1)}. \tag{22}$$

The group of biholomorphic automorphisms of  $B^n$  is actually

$$PU(n, 1) = G/Z,$$

where

$$Z = \{1_{n+1}, \epsilon 1_{n+1}, \dots, \epsilon^n 1_{n+1}\}$$

is the center of  $G$  ( $\epsilon$  is the primitive  $(n + 1)$ th root of 1).

4.4. *The Cartan decomposition.* The Lie algebra of the group  $G$  is  $\mathfrak{g} = \mathfrak{su}(n, 1)$ . The Cartan decomposition (2) is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

$\mathfrak{k}$  is the Lie algebra of  $K$  consisting of  $(n + 1) \times (n + 1)$  complex matrices of the form

$$\begin{pmatrix} U & 0 \\ 0 & i\lambda \end{pmatrix},$$

where  $U$  is an  $n \times n$  skew Hermitian matrix,  $\lambda \in \mathbb{R}$ , and  $\text{tr } U + i\lambda = 0$ , and  $\mathfrak{p}$  consists of matrices

$$\begin{pmatrix} 0_n & \eta \\ {}^t\bar{\eta} & 0 \end{pmatrix}, \tag{23}$$

where  $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathbb{C}^n$ .

The differential at the identity  $(d\pi)_e$  of the natural mapping

$$\pi : G \rightarrow B^n, \quad \pi : g \mapsto g(0), \tag{24}$$

maps  $\begin{pmatrix} 0_n & \eta \\ {}^t\bar{\eta} & 0 \end{pmatrix} \in \mathfrak{p}$  to the tangent vector  $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in T_{g(0)}$  thus identifying  $\mathfrak{p} = T_0(G/K)$  with  $T_{g(0)}(B^n)$ .

Using the Cartan decomposition of the group  $G$  (8), we can write any element

$$g = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix} \in G$$

as a product  $g = g_z \cdot k$ . The matrix  $k \in K$  is in the form  $k = \begin{pmatrix} u_n & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$ , where  $u_n \in U(n)$  and  $\det u_n = e^{i\psi}$ ; the matrix  $g_z$  is hyperbolic where the entries depend only on  $z = g(0)$ , and it represents the projection on the symmetric space  $G/K$ , and by [8, III.2.7],

$$d = \frac{e^{-i\psi}}{\sqrt{-\langle z, z \rangle}}. \tag{25}$$

4.5. *The complex structure.* Recall [18, Ch. 9] that the canonical complex structure on  $B^n$  is a tensor field  $J : z \rightarrow J_z$ , where for each  $z \in B^n$ ,  $J_z$  is an automorphism of the tangent space  $T_z$  such that  $J_z^2 = -I$  ( $I$  denotes the identity map of  $T_z$ ). In local coordinates it is given by

$$J_z \left( \frac{\partial}{\partial x_j}(z) \right) = \frac{\partial}{\partial y_j}(z), \quad J_z \left( \frac{\partial}{\partial y_j}(z) \right) = -\frac{\partial}{\partial x_j}(z).$$

Restricting  $J$  to  $T_0(B^n) = \mathfrak{p}$  we obtain an automorphism  $J_0 : \mathfrak{p} \rightarrow \mathfrak{p}$  such that  $J_0^2 X = -X$  for all  $X \in \mathfrak{p}$ . Since  $G$  is semisimple, there exists an element  $Z_0 \in Z(\mathfrak{k})$  such that  $J_0 = \text{ad}_{\mathfrak{p}}(Z_0)$  and  $\mathfrak{k} = \{X \in \mathfrak{g} \mid [Z_0, X] = 0\}$  [18, Ch. 11, Theorem 9.6]. We have

$$Z_0 = \begin{pmatrix} \frac{i}{n+1} 1_n & 0 \\ 0 & -\frac{ni}{n+1} \end{pmatrix},$$

and

$$J_0 \left( \begin{pmatrix} 0_n & \eta \\ {}^t\bar{\eta} & 0 \end{pmatrix} \right) = \begin{pmatrix} 0_n & i\eta \\ -i{}^t\bar{\eta} & 0 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} 0_n & \eta \\ {}^t\bar{\eta} & 0 \end{pmatrix} \in \mathfrak{p}.$$

For each  $z \in B^n$  the automorphism  $J_z$  can be extended uniquely to a complex linear mapping of the complexification  $T_z^c$  of  $T_z$  onto itself, also denoted by  $J_z$ , and satisfying  $(J_z)^2 = -I$ . The eigenvalues of  $J_z$  are therefore  $i$  and  $-i$ . For  $z = 0$  we have  $T_0^c(B^n) = \mathfrak{p}^c$  and the corresponding eigenspaces are

$$\mathfrak{p}^+ = \{Z \in \mathfrak{p}^c \mid J(Z) = iZ\} \quad \text{and} \quad \mathfrak{p}^- = \{Z \in \mathfrak{p}^c \mid J(Z) = -iZ\}.$$

Then

$$\mathfrak{p}^+ = \{X - iJ(X) \mid X \in \mathfrak{p}\}, \quad \mathfrak{p}^- = \{X + iJ(X) \mid X \in \mathfrak{p}\}, \quad (26)$$

and  $\mathfrak{p}^c$  can be written as a direct sum of complex vector spaces

$$\mathfrak{p}^c = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Thus by (26) and (23)

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0_n & \eta \\ 0 & 0 \end{pmatrix} \mid \eta \in \mathbb{C}^n \right\}$$

and

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0_n & 0 \\ {}^t\eta & 0 \end{pmatrix} \mid \eta \in \mathbb{C}^n \right\}.$$

Any  $g \in G$  can also be decomposed according to Harish-Chandra [3, p. 688, 10, Ch. VIII, §7] as

$$g = g_+ \cdot g_0 \cdot g_-,$$

where  $g^+ = \exp p^+$ ,  $p^+ \in \mathfrak{p}^+$ ,  $g^- = \exp p^-$ ,  $p^- \in \mathfrak{p}^-$ , and  $g_0 \in K^c$ , the complexification of  $K$ . For

$$g = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix}$$

we obtain

$$g^+ = \begin{pmatrix} 1_n & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g^- = \begin{pmatrix} 1_n & 0 \\ {}^t w & 1 \end{pmatrix}, \quad (27)$$

where

$$z = \begin{pmatrix} \frac{b_1}{d} \\ \cdots \\ \frac{b_n}{d} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} \frac{c_1}{d} \\ \cdots \\ \frac{c_n}{d} \end{pmatrix}.$$

Since  $z = g(0)$  we see that the Harish-Chandra map

$$\chi : G \rightarrow \mathfrak{p}^+,$$

given by

$$g \mapsto \log g^+,$$

is exactly the map  $\pi$  of (24) and the bounded domain  $\chi(G) \subset \mathfrak{p}^+$  is the unit ball  $B^n \subset \mathbb{C}^n$ .

*Remark.* Notice that in (27)  $w \neq \bar{z}$ .

The Bergman metric on  $B^n$  introduced in §4.1 is Hermitian, i.e. it is invariant by the complex structure  $J$ , and thus defines a Hermitian inner product on each tangent space  $T_z(B^n)$ .



5. The frame flow

5.1. Geometric interpretation. Let  $\Gamma$  be a lattice in  $G$  and

$$\mathfrak{D} = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathfrak{p}.$$

The corresponding maximal split Abelian subgroup of  $G$

$$A = \left\{ a_t = \begin{pmatrix} 1_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\} \tag{28}$$

projects to the ‘standard geodesic’

$$\mathfrak{J} = \left\{ a_t(0) = \begin{pmatrix} 0 \\ \dots \\ 0 \\ \tanh t \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The subgroup of  $K$  which commutes with  $A$  is

$$W = \left\{ \begin{pmatrix} u_{n-1} & 0 & 0 \\ 0 & e^{-i\psi} & 0 \\ 0 & 0 & e^{-i\psi} \end{pmatrix} \mid \psi \in \mathbb{R}/2\pi\mathbb{Z}, \det u_{n-1} = e^{2i\psi} \right\}.$$

As has been explained in §2.1, acting by right multiplications on  $G$ ,  $A$  defines a homogeneous flow  $\tilde{\varphi}_t$  on  $\Gamma \backslash G$ , which is called the *frame flow* in this case. The terminology comes from the fact that the group  $PU(n, 1)$  may be identified with a principal  $U(n)$ -bundle over  $B^n$ . A fiber at the point  $z \in B^n$  geometrically represents the space of  $n$ -frames: unitary frames of tangent vectors at  $z$  with respect to the Hermitian inner product. Geometrically, the flow  $\tilde{\varphi}_t$  acts as follows: along the geodesic it leaves invariant, it acts by hyperbolic isometries whose differentials move the last vector along the geodesic at constant speed (this represents the projection of  $\tilde{\varphi}_t$  to the geodesic flow  $\varphi_t$  on  $\Gamma \backslash G/W$ ), and the remaining vectors of the frame by parallel translation [18, Ch. 9, Theorem 3.2]. Notice that the distance with respect to the Bergman metric between two points differs from the parameter  $t$  of the geodesic flow on the corresponding geodesic by a factor of two:  $d(z, \varphi_t(z)) = 2t$ .

5.2. Loxodromic elements in  $SU(n, 1)$ . As has been explained in the general setup in §2.1, a loxodromic element  $\gamma_0$  has axis in  $B^n$ , and two fixed points in the boundary  $\partial B^n$ , one *attracting* and one *repelling*. The fixed points correspond to the null eigenvectors of  $\gamma_0$  with eigenvalues  $\lambda$  and  $\bar{\lambda}^{-1}$ , and the remaining  $n - 1$  eigenvectors are positive with eigenvalues of absolute value 1.

As we have pointed out earlier, the automorphisms of  $B^n$  are in fact elements of  $PU(n, 1)$ . Therefore, each loxodromic automorphism may also be represented by an element of  $U(n, 1)$  with a real eigenvalue not equal to 1. Sometimes it is possible to

find such an element in  $SU(n, 1)$ . Eigenvectors (in  $\mathbb{C}^{n+1}$ ) and hence fixed points (on the boundary  $\partial B^n$ ) do not depend on the choice of a representative in  $U(n, 1)$ . The following proposition gives an explicit formula for the transformation  $T$  conjugating a loxodromic element in  $U(n, 1)$  to an ‘almost hyperbolic’ element with a nice fundamental domain. This formula will be used in §7.

PROPOSITION 10. *Let  $\gamma_0$  be a loxodromic element in  $U(n, 1)$  with eigenvectors  $V_1, \dots, V_{n-1}, X, Y$  and corresponding eigenvalues  $\tau_1, \dots, \tau_{n-1}, \lambda, \lambda^{-1}$ ,  $|\tau_j| = 1$ ,  $j = 1, \dots, n-1$ ,  $\lambda \in \mathbb{R}$ ,  $|\lambda| > 1$ . Let  $V_j$ ,  $j = 1, \dots, n-1$ , be normalized so that  $\langle V_j, V_j \rangle = 1$  and  $V_1$  be chosen so that the matrix*

$$T := \begin{pmatrix} V_1 & \cdots & V_{n-1} & \frac{X}{\langle X, Y \rangle} + \frac{Y}{2} & \frac{X}{\langle X, Y \rangle} - \frac{Y}{2} \end{pmatrix}$$

has real positive determinant. Let  $F_0$  be the Dirichlet fundamental domain for  $\langle \gamma_0 \rangle$  centered at  $T(0)$ . Then

(1)  $T \in SU(n, 1)$ ,

(2)

$$\gamma := T^{-1} \cdot \gamma_0 \cdot T = \begin{pmatrix} \tau_1 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \tau_{n-1} & 0 & 0 \\ 0 & \cdots & 0 & \frac{\lambda^2+1}{2\lambda} & \frac{\lambda^2-1}{2\lambda} \\ 0 & \cdots & 0 & \frac{\lambda^2-1}{2\lambda} & \frac{\lambda^2+1}{2\lambda} \end{pmatrix}$$

has null eigenvectors

$$X_0 = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

(3) the fundamental domain  $T^{-1}(F_0)$  of  $\langle \gamma \rangle$  is bounded by two hypersurfaces:

$$H_1 = \left\{ w \left| \left| \frac{\lambda^2 - 1}{2\lambda} w_n - \frac{\lambda^2 + 1}{2\lambda} \right| = 1 \right. \right\}$$

and

$$H_2 = \left\{ w \left| \left| \frac{\lambda^2 - 1}{2\lambda} w_n + \frac{\lambda^2 + 1}{2\lambda} \right| = 1 \right. \right\}.$$

### 6. Automorphic forms on the complex hyperbolic space

6.1. *Automorphy factor.* For any  $g \in G$ , along with the Jacobian matrix  $J(g, z)$  and its determinant, the function

$$j(g, z) = (\det J(g, z))^{1/(n+1)} = (c_1 z_1 + \cdots + c_n z_n + d)^{-1}, \tag{29}$$

is a 1-cocycle. Taking  $\mu(g, z) = j(g, z)^{(n+1)\kappa}$  for a half-integer  $\kappa \geq 1$  such that  $(n+1)\kappa$  is an integer as an automorphy factor, we see that for  $k \in K$ ,  $\rho(k) = \mu(k, 0) = e^{i(n+1)\kappa\psi}$  is

a one-dimensional representation of  $K$ . The operator  $f|_\gamma$  defined in (5) is well-defined on  $PU(n, 1)$ . Thus we obtain a space of holomorphic cusp forms of weight  $(n + 1)\kappa$  denoted classically by  $S_{(n+1)\kappa}(\Gamma)$  (see, e.g., [23]).

Using the formula (25) we see that

$$\mu(g, 0) = d^{-(n+1)\kappa} = \left( e^{i\psi} \sqrt{-\langle z, z \rangle} \right)^{(n+1)\kappa},$$

and therefore the inner product is given by the formula

$$(f_1, f_2) = \int_{\Gamma \backslash \mathbb{H}_{\mathbb{C}}^n} f_1 \overline{f_2} (-\langle z, z \rangle)^{(n+1)\kappa} dV. \tag{30}$$

6.2. *Lift to the group.* It is convenient to choose the following local (partial) coordinates  $(z, \eta, \zeta)$  on  $SU(n, 1)$ :

$$z(g) = g(0) \in B^n, \quad \eta(g) = J(g, 0) \cdot \iota \in S_z(B^n), \tag{31}$$

where  $\iota$  is the unit tangent vector at zero to the ‘standard geodesic’  $\mathfrak{J}$ , introduced in §2.1, and

$$\zeta(g) = j(g, 0) = d^{-1} = \sqrt{-\langle z, z \rangle} e^{i\psi} \neq 0. \tag{32}$$

It is easy to check that the left multiplication by  $g' \in G$  corresponds to the action on  $z \in B^n$  by a biholomorphic transformation  $g'(z)$  (20), on  $\eta \in S_z(B^n)$  by the Jacobian matrix  $J(g', z)$  (21), and on  $\zeta$  by  $j(g', z)$  (29). The lift of the automorphic form  $f \in S_{(n+1)\kappa}(\Gamma)$  to  $G$  (6) has a nice expression in these coordinates:

$$\tilde{f}(g) = f(z)\zeta^{(n+1)\kappa}.$$

6.3. *Construction of relative Poincaré series associated to closed geodesics.* The following construction associates a cusp form to each loxodromic element

$$\gamma_0 = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix} \in \Gamma.$$

As a loxodromic automorphism,  $\gamma_0$  has two fixed points on the boundary  $\partial B^n$ ,  $X$  and  $Y$ :  $\langle X, X \rangle = 0$  and  $\langle Y, Y \rangle = 0$ ,  $X$  repelling and  $Y$  attracting. Then  $Q_{\gamma_0}(z) = \langle z, X \rangle \langle z, Y \rangle \neq 0$  on the closure of  $B^n$  except for the points  $X$  and  $Y$ , and transforms under  $\gamma_0$  as follows:

$$Q_{\gamma_0}(\gamma_0 z) = j(\gamma_0, z)^2 Q_{\gamma_0}(z) = (c_1 z_1 + \cdots + c_n z_n + d)^{-2} Q_{\gamma_0}(z).$$

Let  $\mathcal{K}_n = \{\kappa \geq 1 \mid \kappa \in \frac{1}{2}\mathbb{Z}, (n + 1)\kappa \in \mathbb{Z}\}$ . For the rest of the paper we shall assume that  $\kappa \in \mathcal{K}_n$ . The function  $q(z) = 1/Q_{\gamma_0}^{(n+1)\kappa}(z)$  is an automorphic form of type  $\mu(g, z) = j(g, z)^{2(n+1)\kappa}$  for the subgroup  $\Gamma_0 = \langle \gamma_0 \rangle$ , and it satisfies condition (1) of Theorem 5 since it is holomorphic. Condition (2) is also satisfied. To see that we write, using (18),

$$\int_{\Gamma_0 \backslash B^n} |q(z)| (-\langle z, z \rangle)^{(n+1)\kappa} dV = 4^n \int_{F_0} \frac{(-\langle z, z \rangle)^{(n+1)(\kappa-1)}}{|\langle z, X \rangle \langle z, Y \rangle|^{(n+1)\kappa}} dV_E, \tag{33}$$

where  $F_0$  is a Dirichlet fundamental domain for  $\Gamma_0$ . Since the denominator of the expression in the second integral is equal to zero only at  $z = X$  and  $z = Y$ , the expression represents a continuous function on  $F_0$ . Since  $F_0$  is a bounded domain, the integral is finite. Thus for any  $\kappa \in \mathcal{K}_n$  we produce a relative Poincaré series

$$\Theta_{\gamma_0, \kappa}(z) = \sum_{\gamma \in \Gamma_0 \setminus \Gamma} (q|\gamma)(z) \tag{34}$$

of weight  $2(n + 1)\kappa$  which belongs to  $L^1(\Gamma \setminus G)$ , and, by Satake’s theorem, is a cusp form. If  $\gamma_1$  is conjugate to  $\gamma_0$  in  $\Gamma$ , then it is easy to see that  $\Theta_{\gamma_1, \kappa}(z) = \Theta_{\gamma_0, \kappa}(z)$ . If two primitive loxodromic elements  $\gamma_0, \gamma_1 \in \Gamma$  have the same axis, then  $\gamma_1 = \gamma_0 w$ , where  $w$  belongs to a compact subgroup of  $K$  conjugate to  $W$ . It follows from discreteness of  $\Gamma$  that if  $n \geq 2$ , finitely many elements in  $\Gamma$  may have the same axis. It is a consequence of the period formula (Corollary 13) that their relative Poincaré series coincide.

7. The period formula

THEOREM 11. For any  $f \in S_{2(n+1)\kappa}(\Gamma)$  and any loxodromic element  $\gamma_0 \in \Gamma$ ,

$$(f, \Theta_{\gamma_0, \kappa}) = |\alpha|^{-2(n+1)\kappa} C \int_{z_0}^{\gamma_0 z_0} f(z) Q_{\gamma_0, \kappa}(z)^{(n+1)\kappa} dt.$$

Here  $z_0$  is any point on the axis of  $\gamma_0$ , the integration is over the axis of  $\gamma_0$  in  $B^n$ ,  $t$  is the parameter of the geodesic flow,  $\alpha = -\langle X, Y \rangle / 2$ , and

$$C = \frac{(2(n + 1)\kappa - n - 1)!}{(((n + 1)\kappa - 1)!)^2} \pi^n 2^{2(n+1)(1-\kappa)-1}$$

is a constant.

Proof. Let

$$I := (f, \Theta_{\gamma_0, \kappa}).$$

Since the series (34) converges absolutely, we can interchange summation and integration and, using the standard ‘Rankin–Selberg method’ (see [25, p. 246]), obtain

$$\begin{aligned} I &= \int_{F=\Gamma \setminus B^n} f(z) \sum_{\gamma \in \Gamma_0 \setminus \Gamma} \overline{(q|\gamma)(z)} (-\langle z, z \rangle)^{2(n+1)\kappa} dV \\ &= \sum_{\gamma \in \Gamma_0 \setminus \Gamma} \int_F f(z) \overline{(q|\gamma)(z)} (-\langle z, z \rangle)^{2(n+1)\kappa} dV \\ &= \sum_{\gamma \in \Gamma_0 \setminus \Gamma} \int_{\gamma F} f(z) \overline{q(z)} (-\langle z, z \rangle)^{2(n+1)\kappa} dV \\ &= \int_{F_0} f(z) \frac{(-\langle z, z \rangle)^{2(n+1)\kappa}}{(\langle X, z \rangle \langle Y, z \rangle)^{(n+1)\kappa}} dV, \end{aligned}$$

where  $F_0$  is the fundamental domain for  $\langle \gamma_0 \rangle$  of Proposition 10.

We now make the change of variables  $w = T^{-1}z$ , where  $T$  is as in Proposition 10, mapping the ‘standard geodesic’  $\mathfrak{J}$  into the axis of  $\gamma_0$ . The following lemma is easily checked.

LEMMA 12.  $\langle z, X \rangle \langle z, Y \rangle = \bar{\alpha} j(T, w)^2 (1 - w_n^2)$ , where  $\alpha = -\langle X, Y \rangle / 2$ .

According to (18) we have

$$dV = (2i)^n \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(-\langle z, z \rangle)^{(n+1)}}.$$

Let us denote  $dV_i = dw_i \wedge d\bar{w}_i \wedge \cdots \wedge dw_n \wedge d\bar{w}_n$ . Then

$$\begin{aligned} I &= (2i)^n \int_{F_0} f(z) \frac{(-\langle z, z \rangle)^{2(n+1)\kappa}}{(\langle X, z \rangle \langle Y, z \rangle)^{(n+1)\kappa}} \cdot \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(-\langle z, z \rangle)^{(n+1)}} \\ &= (2i)^n \int_{T^{-1}F_0} f(Tw) \frac{(-\langle Tw, Tw \rangle)^{2(n+1)\kappa}}{(\langle X, Tw \rangle \langle Y, Tw \rangle)^{(n+1)\kappa}} \cdot \frac{dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n}{(-\langle w, w \rangle)^{(n+1)}} \\ &= (2i)^n \alpha^{-(n+1)\kappa} \int_{T^{-1}F_0} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa} |j(T, w)|^{4(n+1)\kappa}}{(1 - \bar{w}_n^2)^{(n+1)\kappa} j(T, w)^{2(n+1)\kappa}} \cdot \frac{dV_1}{(-\langle w, w \rangle)^{(n+1)}} \\ &= (2i)^n \alpha^{-(n+1)\kappa} \int_{T^{-1}F_0} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - (n+1)} j(T, w)^{2(n+1)\kappa}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} dV_1. \end{aligned}$$

Further calculations can be divided onto three parts:

Step 1. Show that

$$\begin{aligned} I &= -\frac{2\pi i}{2(n+1)\kappa - (n+1) + 1} (2i)^n \alpha^{-(n+1)\kappa} \\ &\quad \times \int_{T^{-1}F_0 \cap \{w|w_1=0\}} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - (n+1) + 1}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_2. \end{aligned}$$

Step 2. Show that

$$\begin{aligned} I &= 2^n i \alpha^{-(n+1)\kappa} \frac{(2(n+1)\kappa - n - 1)!}{(2(n+1)\kappa - 2)!} (2\pi)^{n-1} \\ &\quad \times \int_{T^{-1}F_0 \cap \{w|w_1=\dots=w_{n-1}=0\}} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - 2}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_n. \end{aligned}$$

Step 3. Show that

$$I = |\alpha|^{-2(n+1)\kappa} C \int_{z_0}^{\gamma_0 z_0} f(z) (\langle z, X \rangle \langle z, Y \rangle)^{(n+1)\kappa} dt.$$

Step 1.  $\gamma = T^{-1}\gamma_0 T$  leaves invariant  $\{w \mid w_1 = 0\}$ . The fundamental domain  $T^{-1}F_0$  for  $\gamma$  over which we take the integral can be described as  $D_1 = T^{-1}F_0 \cap \{w \mid w_1 = 0\}$  with the disc  $\{w \mid w_2 = \text{constant}, \dots, w_n = \text{constant}\}$  of radius  $(1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)^{1/2}$  ‘attached’ at every point  $(0, w_2, \dots, w_n) \in D_1$  (this disc has only one common point with  $D_1$ ). Change coordinates on the disc  $\{w \mid w_2 = \text{constant}, \dots, w_n = \text{constant}\}$ :

$$(\text{Re } w_1, \text{Im } w_1) \rightarrow (R, \Theta),$$

$$w_1 = (1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)^{1/2} \text{Re}^{i\Theta}, \quad 0 \leq R \leq 1, \quad 0 \leq \Theta < 2\pi.$$

Then on the disc  $\{w \mid w_2 = \text{constant}, \dots, w_n = \text{constant}\}$ ,

$$\begin{aligned} dw_1 \wedge d\bar{w}_1 &= -2i d(\operatorname{Re} w_1) \wedge d(\operatorname{Im} w_1) \\ &= -2i \left| \frac{\partial(\operatorname{Re} w_1, \operatorname{Im} w_1)}{\partial(R, \Theta)} \right| dR \wedge d\Theta \\ &= -2i(1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)R dR \wedge d\Theta, \\ d\Theta &= \frac{dw_1}{iw_1} \Big|_{R=\text{constant}}. \end{aligned}$$

We have

$$\begin{aligned} -\langle w, w \rangle &= (1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)(1 - R^2), \\ -\langle w, w \rangle|_{D_1} &= 1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n, \end{aligned}$$

so

$$I = -2i(2i)^n \alpha^{-(n+1)\kappa} \int_0^1 R(1 - R^2)^{2(n+1)\kappa - (n+1)} dR \frac{1}{i} \oint_{R=\text{constant}} \frac{F(w_1)}{w_1} dw_1,$$

where

$$F(w_1) = \int_{D_1} f(Tw) \frac{(1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)^{2(n+1)\kappa - (n+1)+1}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_2$$

is a holomorphic function of  $w_1$ .

By the Cauchy integral formula

$$\oint_{R=\text{constant}} \frac{F(w_1)}{w_1} dw_1 = 2\pi i F(0),$$

hence this integral does not depend on the value of  $R$  and, taking into account the fact that

$$\int_0^1 R(1 - R^2)^{2(n+1)\kappa - (n+1)} dR = \frac{1}{2(2(n+1)\kappa - (n+1) + 1)},$$

we get

$$\begin{aligned} I &= -(2i)^{n+1} \alpha^{-(n+1)\kappa} \frac{1}{i} 2\pi i \frac{1}{2(2(n+1)\kappa - (n+1) + 1)} \\ &\quad \times \int_{D_1} f(Tw) \frac{(1 - w_2\bar{w}_2 - \dots - w_n\bar{w}_n)^{2(n+1)\kappa - (n+1)+1}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_2 \\ &= -\frac{2\pi i}{2(n+1)\kappa - (n+1) + 1} (2i)^n \alpha^{-(n+1)\kappa} \\ &\quad \times \int_{D_1} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - (n+1)+1}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_2. \end{aligned}$$

Step 2. Denote  $D_p = T^{-1}F_0 \cap \{w \mid w_1 = \dots = w_p = 0\}$ .  $\gamma$  leaves invariant  $\{w \mid w_1 = \dots = w_p = 0\}$  for any  $1 \leq p \leq n - 1$ . Repeat Step 1  $n - 2$  times more (i.e. totally we perform Step 1  $n - 1$  times). It is proved by induction that

$$\begin{aligned} I &= (2i)^n \alpha^{-(n+1)\kappa} (-2\pi i)^p \frac{1}{2(n+1)\kappa - (n+1) + 1} \dots \frac{1}{2(n+1)\kappa - (n+1) + p} \\ &\quad \times \int_{D_p} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - (n+1)+p}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dV_{p+1}, \end{aligned}$$

and the step  $p - 1 \rightarrow p$  essentially repeats the argument in Step 1.

For  $p = n - 1$  we obtain

$$\begin{aligned} I &= (2i)^n \alpha^{-(n+1)\kappa} (-2\pi i)^{n-1} \frac{1}{2(n+1)\kappa - (n+1) + 1} \cdots \frac{1}{2(n+1)\kappa - (n+1) + n - 1} \\ &\quad \times \int_{D_{n-1}} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - (n+1) + n - 1}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dw_n \wedge d\bar{w}_n \\ &= 2^n i \alpha^{-(n+1)\kappa} \frac{(2(n+1)\kappa - n - 1)!}{(2(n+1)\kappa - 2)!} (2\pi)^{n-1} \\ &\quad \times \int_{D_{n-1}} f(Tw) \frac{(-\langle w, w \rangle)^{2(n+1)\kappa - 2}}{(1 - \bar{w}_n^2)^{(n+1)\kappa}} j(T, w)^{2(n+1)\kappa} dw_n \wedge d\bar{w}_n. \end{aligned}$$

Step 3. The last integral is over  $D_{n-1} = T^{-1}F_0 \cap \{w \mid w_1 = 0 = \cdots = w_{n-1} = 0\}$ , the fundamental domain for a ‘standard hyperbolic element’

$$\gamma = \begin{pmatrix} \frac{\lambda^2 + 1}{2\lambda} & \frac{\lambda^2 - 1}{2\lambda} \\ \frac{\lambda^2 - 1}{2\lambda} & \frac{\lambda^2 + 1}{2\lambda} \end{pmatrix}$$

of  $SU(1, 1)$  acting on the unit disc in  $\mathbb{C}$ . The change of coordinates

$$(\operatorname{Re} w_n, \operatorname{Im} w_n) \rightarrow (r, \phi), \quad (r > 0, 0 < \phi < \pi),$$

where

$$w_n = \frac{u - i}{u + i}, \quad u = r e^{i\phi},$$

maps the unit disc to the upper half-plane with polar coordinates  $(r, \phi)$ , so that  $D_{n-1}$  is mapped onto the upper half-annulus

$$\{|\lambda|^{-1} < r < |\lambda|, 0 < \phi < \pi\}.$$

Using formulas

$$\begin{aligned} dw_n \wedge d\bar{w}_n &= -2i \frac{4r dr \wedge d\phi}{|re^{i\phi} + i|^4}, \\ -\langle w, w \rangle &= 1 - w_n \bar{w}_n = \frac{4r \sin \phi}{|re^{i\phi} + i|^2}, \\ 1 - \bar{w}_n^2 &= -\frac{4ire^{-i\phi}}{(re^{-i\phi} - i)^2}, \\ 1 - w_n^2 &= \frac{4ire^{i\phi}}{(re^{i\phi} + i)^2}, \end{aligned}$$

we obtain

$$\begin{aligned} I &= 2^{n-1} \alpha^{-(n+1)\kappa} \frac{(2(n+1)\kappa - n - 1)!}{(2(n+1)\kappa - 2)!} (2\pi)^{n-1} \int_0^\pi (\sin \phi)^{2(n+1)\kappa - 2} d\phi \\ &\quad \times \int_{\phi=\text{constant}, |\lambda|^{-1} \leq r \leq |\lambda|} f(Tw) (1 - w_n^2)^{(n+1)\kappa} j(T, w)^{2(n+1)\kappa} \frac{1}{r} dr. \end{aligned}$$

But

$$dw_n|_{\phi=\text{constant}} = \frac{\partial w_n}{\partial r} dr = \frac{2ie^{i\phi}}{(re^{i\phi} + i)^2} dr,$$

hence

$$\frac{dr}{r} \Big|_{\phi=\text{constant}} = \frac{(re^{i\phi} + i)^2}{2ie^{i\phi}r} dw_n|_{\phi=\text{constant}} = 2 \frac{1}{1 - w_n^2} dw_n|_{\phi=\text{constant}},$$

and we have

$$I = 2^{n-1} \alpha^{-(n+1)\kappa} \frac{(2(n+1)\kappa - n - 1)!}{(2(n+1)\kappa - 2)!} (2\pi)^{n-1} \int_0^\pi (\sin \phi)^{2(n+1)\kappa-2} d\phi \\ \times 2 \int_{\phi=\text{constant}} f(Tw)(1 - w_n^2)^{(n+1)\kappa} j(T, w)^{2(n+1)\kappa} \frac{dw_n}{1 - w_n^2}.$$

The last integral does not depend on  $\phi$  as the integral of a  $\gamma$ -invariant holomorphic function of  $w_n$ . For the same reason it does not depend on the choice of the point  $w_0$  and the path from  $w_0$  to  $\gamma w_0$  in  $D_{n-1}$ . Hence we may take  $w_0, \gamma w_0 \in \mathfrak{J}$ , and since on  $\mathfrak{J}$   $dt = dw_n/(1 - w_n^2)$ , the last integral can be rewritten as

$$\int_{w_0}^{\gamma w_0} f(Tw)(1 - w_n^2)^{(n+1)\kappa} j(T, w)^{2(n+1)\kappa} dt,$$

where the integration is over a segment of  $\mathfrak{J}$ .

Using Lemma 12 again, we go back to the integral over the axis of  $\gamma_0$ , and obtain

$$I = 2^n |\alpha|^{-2(n+1)\kappa} \frac{(2(n+1)\kappa - n - 1)!}{(2(n+1)\kappa - 2)!} (2\pi)^{n-1} \\ \times \int_0^\pi (\sin \phi)^{2(n+1)\kappa-2} d\phi \int_{z_0}^{\gamma_0 z_0} f(z) (\langle z, X \rangle \langle z, Y \rangle)^{(n+1)\kappa} dt.$$

Finally, we obtain

$$(f, \Theta_{\gamma_0, \kappa}) = |\alpha|^{-2(n+1)\kappa} C \int_{z_0}^{\gamma_0 z_0} f(z) (\langle z, X \rangle \langle z, Y \rangle)^{(n+1)\kappa} dt \tag{35}$$

with

$$C = \frac{(2(n+1)\kappa - n - 1)!}{((n+1)\kappa - 1)!^2} \pi^n 2^{2(n+1)(1-\kappa)-1}. \quad \square$$

The following corollary is immediate from Theorem 11.

**COROLLARY 13.** *Let  $\gamma_1, \gamma_0 \in \Gamma$  be two primitive loxodromic elements having the same axis. Then  $\Theta_{\gamma_0, \kappa} = \Theta_{\gamma_1, \kappa}$ .*

The integral (35) is well-defined and is called the *period of  $f$  over the closed geodesic  $[\gamma_0]$* . The reason for this definition is the following result.

**THEOREM 14.** *Let  $\tilde{f}$  be the lift of the cusp form  $f(z) \in S_{2(n+1)\kappa}(\Gamma)$  to  $G$ . Then for any lift  $[\gamma_0]_w$  of the closed geodesic  $[\gamma_0]$  to  $G$  we have*

$$(f, \Theta_{\gamma_0, \kappa}) = e^{-2(n+1)\kappa i \psi} \alpha^{-(n+1)\kappa} C \int_{[\gamma_0]_w} \tilde{f} dt.$$

Here  $w = \begin{pmatrix} u_{n-1} & 0 & 0 \\ 0 & e^{-i\psi} & 0 \\ 0 & 0 & e^{-i\psi} \end{pmatrix} \in W$  with  $\det u_{n-1} = e^{2i\psi}$ , and  $C$  is the constant from Theorem 11.



*Proof.* We make a change of variables  $g$  used in §2.1 which maps the ‘horizontal’ geodesic to the axis of  $\gamma_0$  in such a way that  $g(0) = z_0$ . According to §2.1 all lifts of the segment of the geodesic  $[z_0, \gamma_0 z_0]$  are given by  $\{gwa_t \mid w \in W, 0 \leq t \leq s\}$ , and all lifts of the corresponding segment of the ‘horizontal’ geodesic by  $\{wa_t \mid w \in W, 0 \leq t \leq s\}$ . Using local coordinates (31) on  $G$  we obtain

$$\begin{aligned} & \int_{z_0}^{\gamma_0 z_0} f(z)(\langle z, X \rangle \langle z, Y \rangle)^{(n+1)\kappa} dt \\ &= \bar{\alpha}^{(n+1)\kappa} \int_0^{g^{-1}\gamma_0 g(0)} f(gx)(1-x_n^2)^{(n+1)\kappa} j(g, x)^{2(n+1)\kappa} dt \\ &= \bar{\alpha}^{(n+1)\kappa} \int_{[g^{-1}\gamma_0 g]_w} f(gx)(e^{-i\psi} \zeta_x)^{2(n+1)\kappa} j(g, x)^{2(n+1)\kappa} dt \\ &= \bar{\alpha}^{(n+1)\kappa} e^{-2(n+1)\kappa i\psi} \int_{[\gamma_0]_w} f(z) \zeta_z^{2(n+1)\kappa} dt. \end{aligned}$$

The last two integrals are in  $G$  over the lifts  $[g^{-1}\gamma_0 g]_w$  and  $[\gamma_0]_w$ , and  $x, \zeta_x$  and  $z, \zeta_z$  are evaluated at the corresponding value of the parameter  $t$ :  $x_n = (0, \dots, x_n(t)) = \varphi_t(0)$ ,  $x_n(t) = \tanh t$ ,  $\zeta_x = e^{i\psi} / \cosh t$ . We used the cocycle identity to obtain the last equality:

$$\zeta_z = j(g, x)\zeta_x.$$

The required formula now follows from Theorem 11. Notice that since  $\tilde{f} = f(z)\zeta_z^{2(n+1)\kappa}$  is  $\Gamma$ -invariant, the integral is in  $\Gamma \backslash G$ .  $\square$

8. *Cohomological equation for cusp forms: the vanishing result*

8.1. *Three-dimensional subalgebra.* Now we consider  $L^2(\Gamma \backslash G)$  with the inner product (7). The infinitesimal generators of the frame flow (28) and the one-parameter subgroup

$$m_\psi = \left\{ \begin{pmatrix} 1_{n-1} & 0 & 0 \\ 0 & e^{i\psi} & 0 \\ 0 & 0 & e^{-i\psi} \end{pmatrix} \mid \psi \in \mathbb{R}/2\pi\mathbb{Z} \right\} \subset K,$$

belong to the Lie algebra  $\mathfrak{g} = \mathfrak{su}(n, 1)$ . The corresponding left-invariant differential operators

$$\mathfrak{D}F(g) = \frac{d}{dt}F(g \cdot a_t)|_{t=0} \quad \text{and} \quad \frac{\partial}{\partial \psi}F(g) = \frac{\partial}{\partial \psi}F(g \cdot m_\psi)|_{\psi=0}$$

are defined on a dense set of functions in  $L^2(\Gamma \backslash G)$ , differentiable along the orbits of the corresponding flows, and are given by the matrices

$$\mathfrak{D} = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial \psi} = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Complemented by the third differential operator

$$\mathfrak{D}' = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix},$$

they generate a three-dimensional Lie subalgebra of  $\mathfrak{g}$  with the commutation relations

$$\left[ \frac{\partial}{\partial \psi}, \mathfrak{D} \right] = 2\mathfrak{D}', \quad \left[ \frac{\partial}{\partial \psi}, \mathfrak{D}' \right] = -2\mathfrak{D}, \quad [\mathfrak{D}, \mathfrak{D}'] = -2\frac{\partial}{\partial \psi}.$$

Then

$$\mathfrak{D}^+ = \frac{\mathfrak{D} - i\mathfrak{D}'}{2}, \quad \mathfrak{D}^- = \frac{\mathfrak{D} + i\mathfrak{D}'}{2} \quad \text{and} \quad \Psi = -i\frac{\partial}{\partial \psi}$$

belong to the complexification of  $\mathfrak{g}$ ,  $\mathfrak{g}^c = \mathfrak{sl}(n+1, \mathbb{C})$ , and have the following commutation relations:

$$[\Psi, \mathfrak{D}^+] = 2\mathfrak{D}^+, \quad [\Psi, \mathfrak{D}^-] = -2\mathfrak{D}^-, \quad [\mathfrak{D}^+, \mathfrak{D}^-] = \Psi. \tag{36}$$

They generate a three-dimensional real Lie algebra isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ . The properties of these operators are exactly the same as in the case  $n = 1$  (cf. [9, §3; 13, §2]).

PROPOSITION 15. *The differential operator  $\mathfrak{D}$  is skew-self-adjoint:  $\mathfrak{D}^* = -\mathfrak{D}$ , or equivalently,  $(\mathfrak{D}F, H) = -(F, \mathfrak{D}H)$  for  $F, H \in L^2(\Gamma \backslash G) \cap \text{domain } \mathfrak{D}$ .*

A standard Fourier analysis argument shows that the space  $L^2(\Gamma \backslash G)$  can be decomposed into a direct sum of orthogonal subspaces  $\bigoplus_{-\infty}^{\infty} H_m$  such that

$$H_m = \{F \in L^2(\Gamma \backslash G) \mid \Psi F = mF\}. \tag{37}$$

Notice that the lift of a cusp form  $f \in S_{2(n+1)\kappa}(\Gamma)$  to the group  $G$  belongs to the space  $H_{2(n+1)\kappa}$ .

PROPOSITION 16.

- (1) *If  $F \in H_m$ , then  $\mathfrak{D}^+ F \in H_{m+2}$ , and  $\mathfrak{D}^- F \in H_{m-2}$ ;*
- (2)  *$(\mathfrak{D}^+)^* = -\mathfrak{D}^-$ ;  $(\mathfrak{D}^-)^* = -\mathfrak{D}^+$ .*

8.2. *Proof of Theorem 1.* Suppose there is a cusp form  $f \in S_{2(n+1)\kappa}(\Gamma)$ , such that  $(f, \Theta_{\gamma_0, \kappa}) = 0$  for all loxodromic elements  $\gamma_0 \in \Gamma$ . First we show that  $f(z)\zeta^{2(n+1)\kappa} \in \mathcal{BL}(\Gamma \backslash G)$ . If  $\Gamma \backslash G$  is compact, it follows from Remarks 1 and 2 following Definition 3. Alternatively,  $\Gamma \backslash G/K$  has a finite number of cusps [6], and it is sufficient to show that  $f(z)$  and its first derivatives vanish at each cusp  $\sigma$ .

Let  $R$  be a partial Cayley transform [22, Ch. 4] mapping biholomorphically a Siegel domain

$$S = \{(w, u) = (w, u_1, \dots, u_{n-1}) \in \mathbb{C}^n \mid \text{Im } w - |u_1|^2 - \dots - |u_{n-1}|^2 > 0\}$$

to  $B^n = \{\sum_{i=1}^n |z_i|^2 < 1\}$  in such a way that  $R(\infty) = \sigma$ . By [6]  $\Gamma_\sigma = \{\gamma \in \Gamma \mid \gamma(\sigma) = \sigma\} \neq \emptyset$ . Then  $R^{-1}\Gamma_\sigma R$  contains ‘parallel translations’  $T_m : (w, u) \rightarrow (w + m, u)$ , ( $m \in \mathbb{Z}$ ), with  $j(T_m, (w, u)) = 1$ , and  $\Phi(w, u) = (f|_R)(w, u) = f(R(w, u))j(R, (w, u))^{2(n+1)\kappa}$  is invariant under  $T_m$ :  $\Phi(w + m, u) = \Phi(w, u)$ . Then  $\Phi$  has a Fourier–Jacobi expansion [22, Ch. 3, §5, 2, Ch. 11]

$$\Phi(w, u) = \sum_{m \in \mathbb{Z}} \psi_m(u) e^{2\pi i m w},$$

and since  $f$  is a cusp form, by Satake’s theorem [2, Ch. 11, §5]  $\psi_m(u) = 0$  for  $m \leq 0$ , and the claim follows.

By Theorem 14 the function  $f(z)\zeta^{2(n+1)\kappa}$  satisfies Theorem 8, and its application guarantees us existence of a Lipschitz function  $F : \Gamma \backslash G \rightarrow \mathbb{C}$  such that

$$\mathfrak{D}F = f(z)\zeta^{2(n+1)\kappa}. \tag{38}$$

LEMMA 17. *Let  $F$  be the function obtained from the Theorem 8 for  $f(z)\zeta^{2(n+1)\kappa}$ , where  $f(z) \in S_{2(n+1)\kappa}(\Gamma)$ . Then  $F \in L^2(\Gamma \backslash G)$ .*

*Proof.* The uniform boundedness of  $|F|$  on  $\Gamma \backslash G$  of finite volume would imply the required result. If  $\Gamma \backslash G$  is compact, it follows from the Lipschitz condition. If  $\Gamma \backslash G$  is not compact, it is sufficient to show that  $|F|$  is bounded at each cusp  $\sigma$ . The proof is similar to the proof for Fuchsian groups in [13]. It is based on an application of a partial Cayley transform  $R$  described above. Let  $I$  be the geodesic in  $\mathcal{S}$  given by  $\text{Re } w = 0, u = 0$ . Then  $R(I)$  is the geodesic in  $\Gamma \backslash B^n$  going to the cusp  $\sigma$ . On  $I$  we have  $|\Phi(w, u)| = O(e^{-2\pi y})$ , where  $y = \text{Im } w$ . Then on  $R(I)$ , considered as an orbit of  $\tilde{\varphi}_t$ ,  $|F(\tilde{\varphi}_s(g)) - F(g)|$  may be estimated using (38) for any  $s > 0$  by the integral  $\int_{y_0}^\infty |\Phi(w, u)|y^{(n+1)\kappa} dy/y$  over  $I$ , which is finite. The finiteness of the volume implies that for any  $\epsilon > 0$  there exists a neighborhood of the cusp  $U(\sigma)$  such that  $d((w, u), R(I)) < \epsilon$ , and the uniform boundedness of  $|F|$  now follows from the Lipschitz condition.  $\square$

We decompose  $F$  according to (37) and rewrite (38) as the following system:

$$\begin{aligned} \mathfrak{D}^- F_{2(n+1)\kappa+2} + \mathfrak{D}^+ F_{2(n+1)\kappa-2} &= f(z)\zeta^{2(n+1)\kappa} \\ \mathfrak{D}^- F_{j+2} + \mathfrak{D}^+ F_{j-2} &= 0, \quad \text{for all } j \neq 2(n+1)\kappa. \end{aligned} \tag{39}$$

The argument of Guillemin and Kazhdan [9, Theorem 3.6] for negatively curved surfaces is applicable to this situation since it depends only on the commutation relations (36) and the fact which immediately follows from it: for  $F_m \in H_m$

$$\|\mathfrak{D}^+ F_m\|^2 = \|\mathfrak{D}^- F_m\|^2 + m\|F_m\|^2.$$

PROPOSITION 18. *Let  $F$  be a solution of (39). Then  $F_j = 0$  for  $j \geq 2(n+1)\kappa$ .*

Thus  $F_{2(n+1)\kappa+2} = 0$  and the first equation of (39) has the form

$$\mathfrak{D}^+ F_{2(n+1)\kappa-2} = f(z)\zeta^{2(n+1)\kappa}.$$

PROPOSITION 19. *If  $\tilde{f}(g) = f(z)\zeta^{2(n+1)\kappa}$  is the lift of a holomorphic cusp form  $f \in S_{2(n+1)\kappa}$  to  $G$ , then  $\mathfrak{D}^- f(z)\zeta^{2(n+1)\kappa} = 0$ .*

*Proof.* It follows immediately from the fact that has been, apparently, first pointed out in [7] (see also [1, §5, 3, p. 203]) that for any  $Y \in \mathfrak{p}^-$

$$Y \tilde{f}(g) = j(g, 0)^{2(n+1)\kappa} (\tilde{Y} f)(z).$$

Here  $\tilde{Y}$  is the linear combination of the partial derivatives  $\partial/\partial \bar{z}_j$  where  $z_j$  are coordinates on the bounded domain  $B^n$  and  $z = g(0) = (z_1, \dots, z_n)$ . For  $Y = \mathfrak{D}^- \in \mathfrak{p}^-$  this can be easily checked by a direct differentiation along the orbit of the frame flow  $\tilde{\varphi}_t(g)$  using the

decomposition  $g = g_0 \cdot m_\psi$  corresponding to the local coordinates (31) with coefficients of

$$g_0 = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix}$$

independent on  $\psi$ . Then

$$\begin{aligned} \mathfrak{D}(f(z, \bar{z})\zeta^m) &= \left( \sum (a_{jn}d - c_nb_j) \frac{\partial f}{\partial z_j} - mc_n df(z, \bar{z}) \right) \zeta^{m+2} \\ &+ \left( \sum \frac{\overline{(a_{jn}d - c_nb_j)}}{(d\bar{d})^2} \frac{\partial f}{\partial \bar{z}_j} \right) \zeta^{m-2}. \end{aligned}$$

Since  $f(z)$  is holomorphic we have

$$\mathfrak{D}^- f(z)\zeta^{2(n+1)\kappa} = \left( \sum \frac{\overline{(a_{jn}d - c_nb_j)}}{(d\bar{d})^2} \frac{\partial f}{\partial \bar{z}_j} \right) \zeta^{2(n+1)\kappa-2} = 0. \quad \square$$

The end of the proof goes exactly as in [13]. By Proposition 19

$$\mathfrak{D}^- \mathfrak{D}^+ F_{2(n+1)\kappa-2} = \mathfrak{D}^- f(z)\zeta^{2(n+1)\kappa} = 0.$$

Therefore

$$0 = (F_{2(n+1)\kappa-2}, \mathfrak{D}^- \mathfrak{D}^+ F_{2(n+1)\kappa-2}) = -\|\mathfrak{D}^+ F_{2(n+1)\kappa-2}\|^2,$$

hence  $f(z)\zeta^{2(n+1)\kappa-2} = \mathfrak{D}^+ F_{2(n+1)\kappa-2} = 0$ . Since  $\zeta \neq 0$ ,  $f(z) = 0$ .  $\square$

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