HIGHER COHOMOLOGY FOR ABELIAN GROUPS OF TORAL AUTOMORPHISMS

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ABSTRACT. We give a complete description of smooth untwisted cohomology with coefficients in $\mathbb{R}^\ell$ for $\mathbb{Z}^k$-actions by hyperbolic automorphisms of a torus. For $1 \le n \le k-1$ the n^{th} cohomology trivializes, i.e. every cocycle is cohomologous to a constant cocycle via a smooth coboundary. For $n = k$ a counterpart of the classical Livshitz Theorem holds: the cohomology class of a smooth k-cocycle is determined by periodic data.

0. Introduction. Let us consider an action α of $\mathbb{Z}_+^k$ on a compact differentiable manifold M generated by commuting C^∞ maps $F_1, \dots, F_k$. We will use terms C^∞ and "smooth" interchangeably. Later we will mention other classes of cocycles such as C^1, Hölder and real-analytic. Let $1 \le n \le k$. A n-cochain on M with values in $\mathbb{R}^\ell$ ($\ell \ge 1$) is a function

$$\varphi : M \times (\mathbb{Z}_+^k)^n \to \mathbb{R}^\ell$$

that is multi-linear and skew-symmetric in the last n variables, and C^∞ in the first variable. Since every such function is determined by its coefficients, such a φ can be

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viewed as a smooth vector-function

$$\varphi : M \to (\mathbb{R}^\ell)^{\binom{k}{n}}$$

whose components are indexed by $i_1 < \dots < i_n, \quad i_1, \dots, i_n \in \{1, \dots, k\}$. The coboundary operator $\mathcal{D}$ is given by the formula

$$(\mathcal{D}\varphi)_{i_1,\dots,i_{n+1}}(x) = \sum_{j=1}^{n+1} (-1)^{j+1} \Delta_{i_j} \varphi_{i_1,\dots,\hat{i_j},\dots,i_{n+1}}(x), \tag{0.1}$$

where the operators Δ_i are defined on functions $\psi : M \to \mathbb{R}^\ell$ by

$$\Delta_i \psi = \psi \circ F_i - \psi. \tag{0.2}$$

The cohomology of this cochain complex is called the *smooth cohomology* of the action α.

Similarly, we define smooth cohomology for an action of $\mathbb{R}^k$. In that case, which is somewhat easier to visualize geometrically, n-cochains are smooth fields of differential n-forms, cocycles correspond to the fields of closed forms, and coboundaries are given by the restrictions to the orbit foliation of differentials of smooth globally defined $(n-1)$-forms.

Let $\mathcal{A}$ be $\mathbb{Z}_+^k$, $\mathbb{Z}^k$ or $\mathbb{R}^k$, and α be an $\mathcal{A}$-action on M. Let us consider a closed orbit $\mathcal{C}$ of α. Restrictions of cochains, cocycles and coboundaries to $\mathcal{C}$ form cochains, cocycles and coboundaries for the transitive action on the orbit. Thus, the cohomology class of the restriction of a smooth cocycle φ to $\mathcal{C}$ is an obvious cohomology invariant of φ. This invariant is conveniently described as follows. The orbit $\mathcal{C}$ carries the unique normalized α-invariant measure $\sigma_{\mathcal{C}}$. Integrating φ with respect to this measure produces a n-cocycle independent on the first variable which determines an element $[\varphi]_{\mathcal{C}} \in H^n(\mathcal{A}; \mathbb{R}^\ell)$.

Let $\mathcal{P}(\alpha)$ denote the set of all closed orbits of the action α. We will say that the action α satisfies the C^∞ *Livshitz property for n-cocycles* if $\{[\varphi]_{\mathcal{C}} \mid \mathcal{C} \in \mathcal{P}(\alpha)\}$ form a complete system of cohomology invariants for smooth n-cocycles, i.e. if smooth n-cocycles φ and ψ are such that for every closed orbit $\mathcal{C}$

$$[\varphi]_{\mathcal{C}} = [\psi]_{\mathcal{C}},$$

then there exists a smooth $(n-1)$-cochain Φ such that

$$\psi = \varphi + \mathcal{D}\Phi.$$

Replacing C^∞ in the above discussion by Hölder, C^1 and real-analytic, we define corresponding cohomology of the action and appropriate versions of the Livshitz property. The situation of intermediate regularity between C^1 and C^∞ is usually technically somewhat more complicated, and we will not discuss it in the present paper. On the other hand, although classes $[\varphi]_{\mathcal{C}}$ can be defined already for continuous cocycles, continuous cohomology of most actions is quite pathological and, in particular, Livshitz property never takes place in the continuous category [6].

Integrating the cocycle φ with respect to any α-invariant Borel probability measure μ also produces a cohomology invariant $[\varphi]_\mu \in H^n(\mathcal{A};\mathbb{R}^\ell)$. In particular, Livshitz property implies that all cohomology classes $[\varphi]_\mu$ are determined by the classes $[\varphi]_\mathcal{C}$. For $\mathcal{A} = \mathbb{Z}_+$, $\mathbb{Z}$ or $\mathbb{R}$ the first cohomology is the only non-trivial one, so in these cases it is natural to call the Livshitz property for 1-cocycles simply the *Livshitz property.*

The reason for calling the situation when cohomology of an action is determined by the periodic data Livshitz property is the pioneering work by A. Livshitz [11] from the early seventies who considered the first Hölder and C^1 cohomology for Anosov diffeomorphisms ($\mathbb{Z}$-actions) and flows ($\mathbb{R}$-actions). He established the property for those cases and also noticed that in the algebraic situations such as hyperbolic toral automorphisms the C^∞-property holds as well. The C^∞-result for some geodesic flows was proved by V. Guillemin and D. Kazhdan [2], [3] in the late seventies, and for arbitrary Anosov flows and diffeomorphisms by R. de la Llave, J. Marko and R. Moriyon [13]; de la Llave later proved the real-analytic version of the result [12]. See also [4] and [5].

Away from the hyperbolic case, the situation even for the first cohomology of a $\mathbb{Z}$- or $\mathbb{R}$-action is much more complicated. An important result in that direction was obtained by W. Veech in [14] who established the C^∞ Livshitz property for partially hyperbolic toral endomorphisms. In fact, Veech works in finite differentiability and estimates the loss of differentiability for the solution of the coboundary equation. In particular, he constructs examples showing that the C^1 Livshitz property does not hold in general in this case. There are examples of partially hyperbolic systems of an algebraic nature, for which even the C^∞ Livshitz property does not hold. In this paper we are using a version of Veech's method to study cohomology of higher rank groups of hyperbolic toral automorphisms. Let us point out that the original Livshitz method does not work in the case of higher cohomology at all. On the other hand, A. Katok and R. Spatzier [7, Theorem 2.10] adapted the original Livshitz method to establish Hölder, C^1 and C^∞ Livshitz property for 1-cocycles for $\mathbb{R}^k$ Anosov actions. Their result, however, is only a preliminary to a much stronger result for $k \geq 2$. To put their result in a context appropriate to our discussion, we introduce the following notion.

We shall say that C^∞ (C^1 etc.) n-cohomology for an $\mathcal{A}$-action α *trivializes* if any n-cocycle in the corresponding category is cohomologous to a constant cocycle. This implies, in particular, that cohomology classes $[\varphi]_\mathcal{C}$ are equal for all closed orbits $\mathcal{C}$; the same is true for the classes $[\varphi]_\mu$. Thus, for $\mathbb{Z}$- or $\mathbb{R}$-actions a necessary condition for trivialization is unique ergodicity; indeed, trivialization of C^∞ cohomology happens for translations of the torus satisfying Diophantine conditions. On the other hand, it never happens for hyperbolic systems.

In the sharp contrast to that Katok and Spatzier proved that for a large class of partially hyperbolic algebraic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $k \geq 2$, which they call *standard* actions (see [7, Section 2.2], [8, Section 3], [9, Sections 3 and 6] for exact definitions; notice that in the widely distributed preprint version of [7] and [8], "Differential rigidity of hyperbolic abelian actions" the term *principal* was used instead of *standard*), the C^∞ 1-cohomology trivializes. Then using their version of Livshitz Theorem they extend this result in the case of standard Anosov actions to Hölder and C^1 cohomology. Although [7] deals only with invertible case, corresponding results are also true for standard actions of $\mathbb{Z}_+^k$ by toral endomorphisms.

The contrast between Livshitz and Katok-Spatzier results can be explained if one remembers that for the $\mathbb{Z}_+$ or $\mathbb{R}$ action the first cohomology is also the highest. For the highest cohomology every cochain is a cocycle, and hence there is one-to-one correspondence between cocycles and $\mathbb{R}^\ell$-valued functions on M. In particular, the classes $[\varphi]_\mathcal{C}$ are "independent" since for any finite set of closed orbits one can find a C^∞ function with prescribed averages over those orbits. Thus a "true" generalization of Livshitz Theorem should deal with the highest cohomology of an action.

Although in this paper we deal exclusively with actions by automorphisms and endomorphisms of the torus, in order to keep the general prospective we recall that the standard partially hyperbolic actions of $\mathbb{R}^k$ all come from the following construction. Let G be a connected Lie group, $A \subset G$ a closed abelian subgroup which is isomorphic to $\mathbb{R}^k$, S a compact subgroup of the centralizer $Z(A)$ of A, and Γ a cocompact lattice in G. Then A acts by left translations on the compact space $M = S\backslash G/\Gamma$. This class includes suspensions of actions by automorphisms of tori and nilmanifolds which we call the standard $\mathbb{Z}^k$ actions. Furthermore, the standard actions of the semigroup $\mathbb{Z}_+^k$ are partially hyperbolic actions by endomorphisms of a torus or a nilmanifold.

Conjecture. *Let α be a standard partially hyperbolic action of $\mathbb{Z}_+^k$, $\mathbb{Z}^k$ or $\mathbb{R}^k$, $k \geq 2$. Then C^∞ n-cohomology of α trivializes for $1 \leq n \leq k-1$, and α satisfies C^∞ Livshitz property for k-cocycles. If α is a standard Anosov action the same is true in C^1 and Hölder cases.*

For $n = 1$ this statement is contained in [7, Theorem 2.9] for the Anosov case and in [8, Theorem 3.6] for the partially hyperbolic case.

A positive solution of this conjecture for the highest cohomology in the Weyl chamber flow case [7, Example 2.6] would provide a crucial step in the construction of the spanning sets for cusp forms on some locally symmetric spaces of higher rank. Those cusp forms are generalizations of relative Poincaré series associated with closed geodesics [10] which in this case are associated with maximal compact flats.

The highest cohomology case, however, looks the most difficult. On the other hand, in the case of intermediate cohomology, a combination of decay estimates for matrix coefficients [7, Section 3] with the method of constructing solutions developed in the present paper, is likely to work.

In this paper we prove this conjecture for arbitrary actions by Anosov (hyperbolic) toral automorphisms. At the end of Section 3 we also outline the strategy for extending our results to the more general case of actions by partially hyperbolic toral automorphisms.

Theorem 1. *Let α be an action of $\mathbb{Z}^k$ by hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ $(\ell \geq 1)$ that vanishes on all periodic orbits of $\mathbb{Z}^k$, i.e. $[\varphi]_\mathcal{C} = 0$ for each $\mathcal{C} \in \mathcal{P}(\alpha)$. Then for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^k$*

$$\varphi(x,t) = \mathcal{D}\Phi(x,t), \tag{0.3}$$

where Φ is a C^∞ $(k-1)$-cochain.

Theorem 2. *Let α be an action of $\mathbb{Z}^k$ by hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ n-cocycle over α with values in $\mathbb{R}^\ell$ $(\ell \geq 1)$ and $1 \leq n \leq k-1$. Then φ is C^∞-cohomologous to a constant cocycle ψ, i.e. for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^n$*

$$\varphi(x,t) = \psi(t) + \mathcal{D}\Phi(x,t), \tag{0.4}$$

where Φ is a C^∞ $(n-1)$-cochain.

Theorem 3. *Let α be a faithful action of $\mathbb{Z}^k_+$ by partially hyperbolic endomorphisms of $\mathbb{T}^N$. Then the linear combinations of invariant δ-measures concentrated on periodic orbits of the action α are dense in the space of all invariant pseudomeasures in the weak-* topology of pseudomeasures, i.e. the dual space to the space of functions on $\mathbb{T}^N$ with absolutely convergent Fourier series.*

An outline of the paper is as follows. The main tool is to pass to a dual problem. The dual to a (vector-)function on the torus is a collection of its Fourier coefficients, i.e. a (vector-)function on the lattice $\mathbb{Z}^N$. Thus we associate to a cochain on the torus a dual cochain on the lattice $\mathbb{Z}^N$. The dual coboundary equation reduces to the equations for the transitive action on each orbit $\mathcal{O}$ of the dual action. Smoothness of the original cocycle corresponds to a super-polynomial decay of its Fourier coefficients, in particular, it ensures that the dual cocycle is ℓ^1. In §1 we show that in the most interesting case $n = k$ the vanishing of a cocycle on all periodic orbits of the given action implies that the sum of its Fourier coefficients over any dual orbit $\mathcal{O}$ vanishes (Corollary 1.4). The statement is valid in greater generality than Theorem 1 requires; it is true for actions by commuting ergodic toral endomorphisms. Aside from Theorem 3, which is of independent interest, these considerations provide the basis for treatment of the cohomology problem in this more general case. We will come back to that in a later paper. The key result here is Theorem 1.3 and the method is a generalization of Veech [14] to the case of several commuting toral endomorphisms. Theorem 3 (Theorem 1.2 of the text) is also deduced from Theorem 1.3. In §2 we discuss elementary properties of ℓ^1 cohomology of $\mathbb{Z}^k$ and construct a solution of the coboundary equation on a dual orbit. There is an obvious obstruction for solving this equation in functions decaying at infinity, namely, the average value of the dual cocycle over the orbit $\mathcal{O}$, i.e. the sum of the Fourier coefficients of the original cocycle over $\mathcal{O}$ (Proposition 2.2). In Proposition 2.3 we show the vanishing of this obstruction for n-cocycles for $1 \leq n \leq k-1$ by a generalization of the argument given for $n = 1$ in [7] . Notice that in the contrast to this result, in the case $n = k$ the vanishing of the dual obstructions is a consequence of the vanishing of the periodic data $[\varphi]_{\mathcal{C}}$ for every closed orbit $\mathcal{C}$ of the original action (Corollary 1.4). In §3 we establish the exponential growth of a dual orbit in terms of the norm of coordinates on the orbit with respect to some initial point with estimates independent on the choice of the orbit (Theorem 3.1). This result is only true for hyperbolic automorphisms. In §4 we deal with solution of the coboundary equation for C^∞ k-cocycles whose dual cocycles have 0 average over each non-trivial dual orbit. We prove that solutions constructed on all dual orbits can be "patched together" to ensure their super-polynomial decay and thus obtain a C^∞-cochain that is a solution of our original coboundary equation (Proposition 4.1).

Theorem 1 now follows immediately from Proposition 4.1 and the previous results. The proof of Theorem 2 is inductive, and the base of the induction corresponds to the case $n = k$ dealt with in Proposition 4.1 and the case $n = 1$.

Let us point out an essential difference between the first and higher cohomology. In the former case a regular solution Φ of the coboundary equation $\varphi = \mathcal{D}\Phi$ is unique up to a constant and its existence is equivalent to the coincidence of two distribution solutions Φ^+ and Φ^- which always exist and which are obtained by integrating or summing the values of the cocycle in the positive and negative direction along a one-parameter subgroup (see §4 and [7, Section 4] for details). For $n \geq 2$ the solution is not unique and when it exists, there is no canonical procedure for constructing it. In the standard de Rham cohomology theory the issue of non-uniqueness is handled by introducing a Hodge structure. This method does not work in the case of cohomology of ergodic group actions. Roughly speaking, the reason is the following. Normalization given by Hodge theory requires choosing a particular solution on each closed orbit, but those solutions cannot be glued together into a global solution. This is already apparent in the Livshitz case ($n = k = 1$) where the solution is unique up to a constant so the normalization on every closed orbit is uniquely determined once it has been fixed on a single orbit. From a different viewpoint, one may say that Hodge theory requires solving second-order differential equations whereas in the case of cohomology of an action one must stick to the first-order equations since no ellipticity is present which would guarantee solvability of higher-order equations. Assuming vanishing of the classes $[\varphi]_{\mathcal{C}}$ we construct a particular solution of the coboundary equation in the case $n = k \geq 2$ which depends on choosing a point on each orbit of the dual action on the group of characters.

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1. Periodic orbits and orbits for the dual action by toral endomorphisms. In this section we generalize results of Veech [14] to the case of several commuting toral endomorphisms. Let $\mathbb{A} = \mathbb{A}(\mathbb{T}^N)$ be the space of absolutely convergent Fourier series on the torus $\mathbb{T}^N$

$$f \sim \sum_{n \in \mathbb{Z}^N} \hat{f}(n) e(n \cdot x),$$

where $e(t) = \exp(2\pi i t)$, with ℓ^1-norm $\|f\|_1 = \sum_{n \in \mathbb{Z}^N} |\hat{f}(n)|$. The dual space to $\mathbb{A}$ is a space of distributions on $\mathbb{T}^N$ denoted by $\mathbb{P} = \mathbb{P}(\mathbb{T}^N)$, equipped with the ℓ^∞-norm $\|\mu\|_\infty = \sup_{n \in \mathbb{Z}^N} |\hat{\mu}(n)|$, where $\hat{\mu}(n) = \mu(e(n \cdot x))$. We shall call these distributions *pseudomeasures.*

Any surjective endomorphism $A : \mathbb{T}^N \to \mathbb{T}^N$ is given by a non-singular integral matrix which we will also denote by A; it induces a map $f \to f \circ A$ of $\mathbb{A}$ given by $f(x) \to f(Ax)$. The dual endomorphism $A^* : \mathbb{Z}^N \to \mathbb{Z}^N$ is given by the transpose matrix ${}^tA$. It induces a dual map on the characters:

$$e(n \cdot x) \to e(A^* n \cdot x).$$

In terms of Fourier coefficients A sends

$$f \sim \sum_{n\in\mathbb{Z}^N} \hat{f}(n)e(n\cdot x) \quad \text{to} \quad f\circ A \sim \sum_{m\in\mathbb{Z}^N} (f \hat{\circ} A)(m)e(m\cdot x)$$

where

$$(f \hat{\circ} A)(m) = \begin{cases} \hat{f}(n), & \text{if } m = {}^tAn \text{ for some } n\in\mathbb{Z}^N \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

It is clear that this map is not expanding with respect to ℓ^1-norm, and if A is an automorphism, then it is an isometry. Similarly, it induces a mapping $\mu \to A\mu$ of $\mathbb{P}$ which is non-expanding with respect to the ℓ^∞-norm.

It is a fairly standard exercise in linear algebra to show that the following four conditions are equivalent:

(1) A is ergodic with respect to Lebesgue measure;
(2) The set of periodic points of A coincides with the set of points in $\mathbb{T}^N$ with rational coordinates;
(3) None of the eigenvalues of the matrix A are roots of unity;
(4) A has at least one eigenvalue of absolute value greater than one and has no eigenvectors with rational coordinates.

We will call an endomorphism A satisfying these properties as well as the matrix A *partially hyperbolic.*

Let α be a faithful action by partially hyperbolic commuting endomorphisms of $\mathbb{T}^N$ given by $N\times N$ integer matrices $A_1,\dots,A_k, k\geq 1$, with determinants $\Delta_1,\dots,\Delta_k\neq 0$, and β be the dual action on $\mathbb{Z}^N$ by transpose matrices $B_i = {}^tA_i$.

We write

$$\beta^t m^* = B_1^{t_1}B_2^{t_2}\dots B_k^{t_k}m^*$$

for $t=(t_1,\dots,t_k)$ and $m^*\in\mathbb{Z}^N$.

The following definition is useful while considering the non-invertible case.

Definition 1.1. A *dual semiorbit* of a given vector $m^*\in\mathbb{Z}^N$ is a subset of $\mathbb{Z}^N$:

$$\mathcal{O}^+(m^*) = \{m = \beta^t m^*, t\in\mathbb{Z}_+^k\}.$$

A *dual orbit* of a given vector $m^*\in\mathbb{Z}^N$ is a subset of $\mathbb{Z}^N$:

$$\mathcal{O}(m^*) = \{m\in\mathbb{Z}^N \mid m = \beta^t m^*, t\in\mathbb{Z}^k\}.$$

Dual orbits form an equivalence relation on $\mathbb{Z}^N$.

A pseudomeasure μ is invariant under α if for any $A\in\alpha$, $A\mu=\mu$. For any dual orbit $\mathcal{O}(m^*)$ with the initial point $m^*\in\mathbb{Z}^N$ we construct an α-invariant pseudomeasure $\mu_{\mathcal{O}(m^*)}$ by setting its Fourier coefficients

$$\hat{\mu}_{\mathcal{O}(m^*)}(m) = \begin{cases} 1, & \text{if } m\in\mathcal{O}(m^*) \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Obviously $\hat{\mu}_{\mathcal{O}(m^*)}$ does not depend on the choice of the initial point m^* and hence can be denoted by $\hat{\mu}_{\mathcal{O}}$. Pseudomeasures $\mu_{\mathcal{O}}$ for different dual orbits $\mathcal{O}$ form a basis in the space of all α-invariant pseudomeasure denoted by $\mathbb{IP}(\alpha)$.

To each periodic (finite) orbit $\mathcal{C}$ of α one associates an α-invariant measure $\sigma_{\mathcal{C}}$ concentrated on that orbit:

$$\sigma_{\mathcal{C}} = \frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}}\delta_x.$$

Let us point out that in the non-invertible case not every point whose α-orbit is finite belongs to a periodic orbit. Let $\mathcal{P}(\alpha)$ be the set of all periodic orbits of $\mathbb{T}^N$. We denote by $<\mathcal{P}(\alpha)>$ the $\mathbb{C}$-linear span of $\{\sigma_{\mathcal{C}} \mid \mathcal{C}\in\mathcal{P}(\alpha)\}$.

Let $q>0$ be an integer relatively prime to all determinants $\Delta_1,\dots,\Delta_k$. Notice that B_i are invertible if considered as matrices with entries in $\mathbb{Z}/q\mathbb{Z}$, so $G_q =< B_1,\dots,B_k >$ as a subgroup of $GL(N,\mathbb{Z}/q\mathbb{Z})$ is finite. Fix $m^*\in\mathbb{Z}^N$ and consider it as an element of $(\mathbb{Z}/q\mathbb{Z})^N$. Then the factor group $G_q/C(m^*)$ where $C(m^*)$ is the stabilizer of m^* in G_q, is finite, and the representatives can be always chosen of the form

$$B_1{}^{\tau_1}\dots B_k{}^{\tau_k} \quad \text{with} \quad \tau_1,\dots\tau_k\geq 0. \tag{1.3}$$

We pick a fundamental domain for $G_q/C(m^*)$ satisfying (1.3) and denote it by $\mathcal{F}$.

Let $\Gamma(q)$ be the set of points in the torus $\mathbb{T}^N$ whose coordinates are rational numbers whose denominators are divisors of q, i.e. the image in $\mathbb{T}^N$ of $q^{-1}\mathbb{Z}^N$ under the canonical projection. Then $\Gamma(q)\subset\mathcal{P}(\alpha)$. We can define now a complex valued measure, $\mu = \mu(m^*,q)$ by

$$\mu=\mu(m^*,q)=q^{-N}\sum_{x\in\Gamma(q)}\Big(\sum_{B_1^{j_1}\dots B_k^{j_k}\in\mathcal{F}} e(-B_1^{j_1}\dots B_k^{j_k}m^*\cdot x)\Big)\delta_x. \tag{1.4}$$

Notice that its Fourier coefficients are given by the formula

$$\hat{\mu}(m)=\begin{cases} 1, & \text{if } m\equiv B_1^{j_1}\dots B_k^{j_k}m^* \pmod q \text{ for some } B_1^{j_1}\dots B_k^{j_k}\in\mathcal{F}\\ 0, & \text{otherwise.}\end{cases} \tag{1.5}$$

Theorem 1.2. *Let α be an action by commuting partially hyperbolic endomorphisms of $\mathbb{T}^N$ given by $N\times N$ matrices $A_1,\dots,A_k$ with determinants $\Delta_1,\dots,\Delta_k$, and let $m^*\in\mathbb{Z}^N$. If $p^{(n)}$ is the product of the first n primes numbers relatively prime to $\Delta_1,\dots,\Delta_k$, then in the notations of (1.4) and (1.2),*

$$\lim_{n\to\infty}\mu(m^*,p^{(n)})=\mu_{\mathcal{O}(m^*)}$$

in the weak- topology of $\mathbb{P}(\mathbb{T}^N)$.*

This result is a consequence of the following theorem.

Theorem 1.3. *Let $B_1, \dots, B_k \in GL(N, \mathbb{Z})$ be commuting partially hyperbolic matrices with determinants $\Delta_1, \dots, \Delta_k$, $p^{(n)}$ the product of the first n primes numbers relatively prime to $\Delta_1, \dots, \Delta_k$. If $m^*, m \in \mathbb{Z}^N$ and there are k sequences $\{j_i^{(n)}, 1 \le i \le k\}$ of integers such that*

$$B_1^{j_1^{(n)}} \dots B_k^{j_k^{(n)}} m^* \equiv m \pmod{p^{(n)}} \tag{1.6}$$

then there exists a vector $(j_1^{(0)}, \dots, j_k^{(0)}) \in \mathbb{Z}^k$ such that

$$B_1^{j_1^{(0)}} \dots B_k^{j_k^{(0)}} m^* = m$$

and for each $0 < r \in \mathbb{Z}$ and $1 \le i \le k$

$$\lim_{n \to \infty} j_i^{(n)} \equiv j_i^{(0)} \pmod{r}.$$

Proof. Following Veech [14] we let $\Lambda(m^*)$ be the smallest subgroup of $\mathbb{Z}^N$ containing the semiorbit $\mathcal{O}^+(m^*)$, $\Omega(m^*)$ be the $\mathbb{Q}$-span of $\Lambda(m^*)$ and $\Lambda^*(m^*) = \Omega(m^*) \cap \mathbb{Z}^N$. Then $\Lambda^*(m^*)$ is a β-invariant subpace and without loss of generality we may assume that $\Lambda^*(m^*) = \mathbb{Z}^N$ and m^* is a cyclic vector of the action β on $\mathbb{Z}^N$.

The transpose matrices ${}^tB_1, \dots, {}^tB_k$ commute, hence they have a common eigenvector $\xi = (\xi_1, \dots, \xi_N)$ such that ${}^tB_i \xi = \lambda_i \xi$ for $1 \le i \le k$. Let $K = \mathbb{Q}(\lambda_1)$. Then by the Cramer's Rule, all $\xi_i \in K$, and by solving the equation

$$({}^tB_i - \lambda_i)\xi = 0$$

for $i = 2, \dots, k$, we obtain that all $\lambda_i \in K$. Moreover, there exists a cofinite set of non-archimedian prime divisors of K, $\mathcal{S}_0$, such that $\xi_j \in \mathcal{O}^*(\mathcal{S}_0)$ for $1 \le j \le N$ and $\lambda_i \in \mathcal{O}^*(\mathcal{S}_0)$ for $1 \le i \le k$, where $\mathcal{O}^*(\mathcal{S}_0)$ is the group of units of the ring of integers $\mathcal{O}(\mathcal{S}_0)$ in K with respect to the set $\mathcal{S}_0$. (For all necessary references see, e.g. [15, Chapter 5]). Choose a non-singular matrix C having ξ as the last row and other integer entries. Then $C \in GL(N, \mathcal{O}(\mathcal{S}_0))$. Let $\zeta_0 = Cm^*, \zeta = Cm$.

Let p be a rational prime and assume that the congruence

$$B_1^{j_1} \dots B_k^{j_k} m^* \equiv m \pmod{p}$$

admits a solution $j = (j_1, \dots, j_k) \in \mathbb{Z}^N$. Then for the same j we have

$$CB_1^{j_1} \dots B_k^{j_k} C^{-1} \zeta_0 \equiv \zeta \pmod{p\mathcal{O}(\mathcal{S}_0)^N}.$$

The last row of the matrix $CB_1^{j_1} \dots B_k^{j_k} C^{-1}$ will have the form $(0, \dots, 0, \lambda_1^{j_1} \dots \lambda_k^{j_k})$ and we obtain the following congruence

$$\lambda_1^{j_1} \dots \lambda_k^{j_k} a \equiv b \pmod{p\mathcal{O}(\mathcal{S}_0)},$$

where a is the N-th component of ζ_0 and b is N-th component of ζ. Since m^* is a cyclic vector, $a \neq 0$. Condition (1.6) implies that there exists a cofinite set of non-archimedian prime divisors of K, $\mathcal{S} \subset \mathcal{S}_0$ such that $a, b \in \mathcal{O}^*(\mathcal{S})$ and

$$\lambda_1^{j_1} \dots \lambda_k^{j_k} a \equiv b \pmod{\pi(P)}, \tag{1.7}$$

admits a solution $j = (j_1, \dots, j_k)$ for each finite set of $P \in \mathcal{S}$. Since matrices $B_1, \dots, B_k \in GL(N, \mathbb{Z})$ and have no eigenvalues that are roots of unity, neither of $\lambda_1, \dots, \lambda_k$ is a root of unity. The rest of the proof verbatim follows the scheme of Veech [14]. It makes use of Chevalley's Theorem [1, 14] in order to interpret (1.7) as a limit in the profinite topology of the group $\mathcal{O}^*(\mathcal{S})$ and the fact that the set

$$\{\lambda_1^{j_1} \dots \lambda_k^{j_k} a \mid (j_1 \dots j_k) \in \mathbb{Z}^k\}$$

which is a sublattice in $\mathcal{O}^*(\mathcal{S}) \sim \mathbb{Z}_m \times \mathbb{Z}^s$, is closed in the profinite topology on $\mathcal{O}^*(\mathcal{S})$. $\square$

Theorem 1.2 can be applied to k-cocycles over a $\mathbb{Z}_k^+$-action α by toral endomorphisms in the following way. A k-cocycle $\varphi(x, t)$ is uniquely determined by a function

$$\varphi(x) : \mathbb{T}^N \to \mathbb{R}^\ell \quad (\ell \geq 1)$$

(see Introduction for general definitions). Recall that a k-cocycle vanishes on a periodic orbit $\mathcal{C}$ if

$$[\varphi]_{\mathcal{C}} = \int_{\mathbb{T}^N} \varphi d\sigma_{\mathcal{C}} = 0. \tag{1.8}$$

Corollary 1.4. *Let α be an action by commuting ergodic endomorphisms of $\mathbb{T}^N$ given by $N \times N$ matrices $A_1, \dots, A_k$ with determinants $\Delta_1, \dots, \Delta_k$. Let φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ $(\ell \geq 1)$ that vanishes on all periodic orbits of α. Then for any dual orbit $\mathcal{O}(m^*)$*

$$\sum_{m \in \mathcal{O}(m^*)} \hat{\varphi}(m) = 0.$$

Proof. Since φ vanishes on all periodic orbits of α, for any $n \geq 1$,

$$\int_{\mathbb{T}^N} \varphi d\mu(m^*, p^{(n)}) = 0,$$

where $\mu(m^*, \cdot)$ was defined in (1.4). By Theorem 1.2

$$\mu_{\mathcal{O}(m^*)}(\varphi) = 0.$$

But

$$\mu_{\mathcal{O}(m^*)}(\varphi) = \sum_{m \in \mathbb{Z}^N} \hat{\varphi}(m) \hat{\mu}_{\mathcal{O}(m^*)}(m) = \sum_{m \in \mathcal{O}(m^*)} \hat{\varphi}(m).$$

Thus if the condition (1.8) holds for any $\mathcal{C} \in \mathcal{P}(\alpha)$, for any dual orbit $\mathcal{O}(m^*)$

$$\sum_{m \in \mathcal{O}(m^*)} \hat{\varphi}(m) = 0.$$

$\square$

In particular, since $\mathcal{O}(0) = 0$ we obtain

Corollary 1.5. *Let α be an action by commuting ergodic endomorphisms of $\mathbb{T}^N$ given by $N \times N$ matrices $A_1, \dots, A_k$ with determinants $\Delta_1, \dots, \Delta_k$. Let φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ ($\ell \geq 1$) that vanishes on all periodic orbits of α. Then $\hat{\varphi}(0) = \int_{\mathbb{T}^N} \varphi d\lambda = 0$, where λ is the Lebesgue measure on $\mathbb{T}^N$.*

2. The ℓ^1 n-cohomology of $\mathbb{Z}^k$. Now we proceed to the consideration of the dual cohomology problem which appears in the space of Fourier coefficients. The case of the highest cohomology plays a special role since, together with the first cohomology, it is a base of the induction in the general case. So we begin by discussing the case $n = k$. Let φ be a C^∞ k-cocycle over a $\mathbb{Z}^+_k$-action α by ergodic toral endomorphisms. Let us fix an initial point $m^* \in \mathbb{Z}^N$. The dual orbit $\mathcal{O}(m^*)$ can be identified with a subset of $\mathbb{Z}^k$ via the correspondence

$$m = B_1^{m_1} \dots B_k^{m_k} m^* \to (m_1, \dots, m_k).$$

If $m^* = 0$ we obtain a trivial orbit. Throughout this section we assume that $m^* \neq 0$. Since all elements are ergodic, $\mathcal{O}(m^*)$ has rank k. Since the original cocycle φ is C^∞, its Fourier coefficients decay *super-polynomially*, i.e.

$$\forall j \in \mathbb{Z}_+ \quad \exists C(j) \quad \text{such that} \quad \forall m \in \mathbb{Z}^N \quad \mid \hat{\varphi}(m) \mid \leq C(j) \|m\|^{-j},$$

where $\| \cdot \|$ is any norm in $\mathbb{R}^N$. Hence on each dual orbit $\mathcal{O}(m^*)$ we have

$$\sum_{m \in \mathcal{O}(m^*)} \mid \hat{\varphi}(m) \mid < \infty.$$

If we restrict $\hat{\varphi}$ to $\mathcal{O}(m^*)$, i.e. assign $\hat{\varphi}(m) = 0$ for $m \notin \mathcal{O}(m^*)$ we obtain a (untwisted) ℓ^1 k-cocycle on $\mathbb{Z}^k$ dual to φ,

$$\hat{\varphi} : \mathbb{Z}^k \to \mathbb{R}^\ell, \quad \sum_{m \in \mathbb{Z}^k} \mid \hat{\varphi}(m) \mid < \infty.$$

We shall ignore its dependence on m^* for the moment.

A $(k-1)$-cochain Φ over α can be identified with a vector function

$$\Phi : \mathbb{T}^N \to (\mathbb{R}^\ell)^k, \quad \Phi = (\Phi_1, \dots, \Phi_k).$$

The coboundary operator is given by the formula

$$\mathcal{D}\Phi = \sum_{i=1}^k (-1)^{i+1} \Delta_i \Phi_i,$$

where

$$\Delta_i \varphi = \varphi \circ A_i - \varphi. \tag{2.1}$$

According to the formula (1.1) A_i acts on Fourier coefficients corresponding to the points of the dual orbit exactly the way the i-th left coordinate shift

$$\sigma_i(m_1,\dots,m_k)=(m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_k).$$

acts on the dual cocycle $\hat{\varphi}$:

$$(\varphi \hat{\circ} A_i)=\hat{\varphi}\circ\sigma_i.$$

Consequently, if we identify a dual $(k-1)$-cochain with a vector-function

$$\hat{\Phi}:\mathbb{Z}^k\to(\mathbb{R}^\ell)^k,\quad \hat{\Phi}=(\hat{\Phi}_1,\dots,\hat{\Phi}_k),$$

the coboundary operator for dual cochains will be given by the formula

$$\mathcal{D}\hat{\Phi}=\sum_{i=1}^{k}(-1)^{i+1}\Delta_i\hat{\Phi}_i,$$

where

$$\Delta_i\hat{\varphi}=\hat{\varphi}\circ\sigma_i-\hat{\varphi}. \tag{2.2}$$

We shall discuss here elementary properties of untwisted ℓ^1-k-cocycles on $\mathbb{Z}^k$. Now let

$$(m_1,\dots,m_k)=m\in\mathbb{Z}^k.$$

If $\hat{\varphi}$ is a ℓ^1-k-cocycle on $\mathbb{Z}^k$, it vanishes at ∞, i.e. $\mid\hat{\varphi}(m)\mid\to 0$ as $\|m\|\to\infty$.

We use the following notations:

For $\hat{\varphi}\in\ell^1(\mathbb{Z}^k)$ let

$$\overline{\hat{\varphi}}=\sum_{m\in\mathbb{Z}^k}\hat{\varphi}(m).$$

For $\hat{\varphi}\in\ell^1(\mathbb{Z}^k),\quad i=1,\dots,k$

$$(\Sigma_i\hat{\varphi})(m_1,\dots,m_k)=\sum_{j=-\infty}^{\infty}\hat{\varphi}(m_1,\dots,m_{i-1},j,m_{i+1},\dots,m_k),$$

$$(\Sigma_i^-\hat{\varphi})(m_1,\dots,m_k)=\sum_{j=-\infty}^{m_i-1}\hat{\varphi}(m_1,\dots,m_{i-1},j,m_{i+1},\dots,m_k),$$

$$(\Sigma_i^+\hat{\varphi})(m_1,\dots,m_k)=-\sum_{j=m_i}^{\infty}\hat{\varphi}(m_1,\dots,m_{i-1},j,m_{i+1},\dots,m_k).$$

Obviously $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$ (as operators on functions). Thus $\Sigma_i^-\hat{\varphi}=\Sigma_i^+\hat{\varphi}$ if and only if $\Sigma_i\hat{\varphi}\equiv 0$. Note that the operators Σ_i, Σ_i^-, Σ_i^+ do not preserve the ℓ^1-condition.

Lemma 2.1. *$\Sigma_i^-\hat{\varphi}$ and $\Sigma_i^+\hat{\varphi}$ vanish af ∞ if and only if they coincide.*

Proof. $(\Sigma_i^-\hat{\varphi})(m_1,\dots,m_i,\dots,m_k)$ converges to $(\Sigma_i\hat{\varphi})(m_1,\dots,m_k)$ as $m_i\to\infty$, and similarly for Σ_i^+. □

Furthermore, we have

$$\Sigma_i^+\Delta_i=\Sigma_i^-\Delta_i=\mathrm{Id}\quad\text{and}\quad\Delta_i\Sigma_i^+=\Delta_i\Sigma_i^-=\mathrm{Id}\tag{2.3}$$

as operators on ℓ^1-functions.

Let $(\delta_i\hat{\varphi})(m_1,\dots,m_k)=\delta(m_i)\hat{\varphi}(m_1,\dots,m_k)$, where

$$\delta(n)=\begin{cases}1, & \text{if } n=0\\ 0, & \text{otherwise}\end{cases}$$

is the usual δ-function. Operators $\Sigma_1,\dots,\Sigma_k$ commute, hence we can define composition operators

$$\Sigma_{i_1,\dots,i_j}=\Sigma_{i_1}\dots\Sigma_{i_j}$$

for any set of pairwise distinct indices. Similarly, $\delta_1,\dots,\delta_k$ commute and we define

$$\delta_{i_1,\dots,i_j}=\delta_{i_1}\dots\delta_{i_j}.$$

Operator $\Sigma_1\dots\Sigma_k$ associates to every ℓ^1-function $\hat{\varphi}$ the constant function equal to $\overline{\hat{\varphi}}$.

Proposition 2.2. *An ℓ^1 k-cocycle $\hat{\varphi}$ satisfies*

$$\hat{\varphi}=\mathcal{D}\hat{\Phi},\tag{2.4}$$

where $\hat{\Phi}$ vanishes at ∞ ($\hat{\Phi}$ may not be ℓ^1 itself) if and only if $\overline{\hat{\varphi}}=0$. If $\overline{\hat{\varphi}}=0$ a solution of (2.4) $\hat{\Phi}=(\hat{\Phi}_1,\dots,\hat{\Phi}_k)$ is given in the form (2.5) below.

Proof. Let $\hat{\varphi}\in\ell^1(\mathbb{Z}^k)$. Then the function $\delta_1\Sigma_1\hat{\varphi}\in\ell^1(\mathbb{Z}^k)$. An easy calculation shows that $\Sigma_1(\hat{\varphi}-\delta_1\Sigma_1\hat{\varphi})\equiv 0$. Let

$$\hat{\Phi}_1(\hat{\varphi})=\Sigma_1^-(\hat{\varphi}-\delta_1\Sigma_1\hat{\varphi}).$$

By Lemma 2.1 $\hat{\Phi}_1(\hat{\varphi})$ vanishes at ∞, and by (2.3)

$$\hat{\varphi}-\delta_1\Sigma_1\hat{\varphi}=\Delta_1\hat{\Phi}_1(\hat{\varphi}).$$

Since the function $\delta_i\Sigma_i\hat{\varphi}$ vanishes outside the hyperplane $m_i=0$ one can proceed by induction. Thus

$$\begin{aligned}\hat{\varphi}-\delta_{1,\dots,k}\Sigma_{1,\dots,k}\hat{\varphi}&=\sum_{j=1}^{k}(\delta_{1,\dots,j-1}\Sigma_{1,\dots,j-1}\hat{\varphi}-\delta_{1,\dots,j}\Sigma_{1,\dots,j}\hat{\varphi})\\&=\sum_{j=1}^{k}\delta_{1,\dots,j-1}(\Sigma_{1,\dots,j-1}\hat{\varphi}-\delta_j\Sigma_j(\Sigma_{1,\dots,j-1}\hat{\varphi})\\&=\sum_{j=1}^{k}(-1)^{j+1}\Delta_j\hat{\Phi}_j(\hat{\varphi}),\end{aligned}$$

where

$$\hat{\Phi}_j(\hat{\varphi}) = (-1)^{j+1}\Sigma_j^- \delta_1 \dots \delta_{j-1}(\Sigma_{1,\dots,j-1}\hat{\varphi} - \delta_j \Sigma_j(\Sigma_{1,\dots,j-1}\hat{\varphi})). \tag{2.5}$$

$\hat{\Phi}_j$ vanishes at ∞ since

$$\Sigma_j(\delta_1, \dots, \delta_{j-1}(\Sigma_{1,\dots,j-1}\hat{\varphi} - \delta_j \Sigma_j(\Sigma_{1,\dots,j-1}\hat{\varphi}))) \equiv 0.$$

Thus, $\hat{\Phi} = (\hat{\Phi}_1, \dots, \hat{\Phi}_k)$ is a solution of (2.4) if and only if $\overline{\hat{\varphi}} = \Sigma_{1,\dots,k} = 0$. In the latter case formula (2.5) gives a solution. $\square$

The map $\hat{\varphi} \to (\hat{\Phi}_1(\hat{\varphi}), \dots, \hat{\Phi}_k(\hat{\varphi}))$ is linear. It is not bounded in ℓ^1-norm; in fact, $\hat{\Phi}$ may not be ℓ^1. However, if $\hat{\varphi}$ decreases at ∞ fast enough so do $\hat{\Phi}_i$s. The last observation will be used in the next section.

Now let φ be a C^∞ n-cochain on $\mathbb{T}^N$ with $1 \le n \le k-1$. As before, the restriction of its Fourier coefficients $\hat{\varphi}$ to a dual orbit $\mathcal{O}(m^*)$ gives us an ℓ^1 n-cochain on $\mathbb{Z}^k$. It can be viewed as a vector-function

$$\hat{\varphi} : \mathbb{Z}^k \to (\mathbb{R}^\ell)^{\binom{k}{n}}$$

whose components $\hat{\varphi}_{i_1\dots i_n}$ are indexed by $i_1 < \dots < i_n$, $\quad i_1, \dots, i_n \in \{1, \dots, k\}$. The cocycle equation $\mathcal{D}\varphi = 0$ gives us the following equations for components of φ:

$$\sum_{j=1}^{n+1} (-1)^{j+1} \Delta_{i_j} \varphi_{i_1 \dots \hat{i_j} \dots i_{n+1}} = 0, \tag{2.6}$$

where Δ_i is given by the formula (2.1). Then on each non-trivial dual orbit the components of the dual cocycle $\hat{\varphi}$ satisfy to the same equation

$$\sum_{j=1}^{n+1} (-1)^{j+1} \Delta_{i_j} \hat{\varphi}_{i_1 \dots \hat{i_j} \dots i_{n+1}} = 0 \tag{2.7}$$

with Δ_i given by the formula (2.2).

The following Proposition is the generalization to the intermediate cohomology case of the key argument for the first cohomology of the higher rank abelian actions which first appeared in [7]. It demonstrates why for smooth cocycles the dual obstructions vanish. We are going to use elementary facts about n-cocycles analogous to those proved for k-cocycles.

Proposition 2.3. *If $\hat{\varphi}$ is an ℓ^1 n-cocycle on $\mathbb{Z}^k$ with $1 \le n \le k-1$, then for each component we have*

$$\Sigma_{i_1,\dots,i_n} \hat{\varphi}_{i_1 \dots i_n} = 0.$$

Proof. It is sufficient to consider a component $\hat{\varphi}_{i_1\dots i_n}$ with $i_1 > 1$. From (2.7) we have

$$\Delta_1 \hat{\varphi}_{i_1 \dots i_n} = \sum_{j=1}^{n} (-1)^{j+1} \Delta_{i_j} \hat{\varphi}_{1 i_1 \dots \hat{i_j} \dots i_n}.$$

Applying $\Sigma_{i_1,\ldots,i_n} = \Sigma_{i_1}\ldots\Sigma_{i_n}$ we obtain

$$\Delta_1\Sigma_{i_1}\ldots\Sigma_{i_n}\hat{\varphi}_{i_1\ldots i_n} = \Sigma_{i_1,\ldots,i_n}\Delta_1\hat{\varphi}_{i_1\ldots i_n} =$$
$$\Sigma_{i_1,\ldots,i_n}\sum_{j=1}^{n}(-1)^{j+1}\Delta_{i_j}\hat{\varphi}_{1i_1\ldots\hat{i_j}\ldots i_n} = 0.$$

since $\Sigma_{i_1},\ldots,\Sigma_{i_n}$ commute and $\Sigma_i\Delta_i = 0$. Therefore for each point $(c_1, c_2, \ldots, c_k) \in \mathbb{Z}^k$ we have

$$\Sigma_{i_1}\ldots\Sigma_{i_n}\hat{\varphi}_{i_1\ldots i_n}(m_1, c_2, \ldots, c_k) = \Sigma_{i_1}\ldots\Sigma_{i_n}\hat{\varphi}_{i_1\ldots i_n}(c_1, c_2, \ldots, c_k) = C$$

for each $m_1 \in \mathbb{Z}$. But applying Σ_1 and using that $\hat{\varphi}$ is an ℓ^1 cocycle, we see that

$$\Sigma_1\Sigma_{i_1}\ldots\Sigma_{i_n}\hat{\varphi}_{i_1\ldots i_n}(c_1,\ldots,c_k) < \infty$$

which implies that $C = \Sigma_{i_1}\ldots\Sigma_{i_n}\hat{\varphi}_{i_1\ldots i_n}(c_1, c_2, \ldots, c_k) = 0$. $\square$

3. Growth estimates for the orbits of the dual action. The following theorem gives us estimates on the growth of a given dual orbit in terms of the norm of coordinates on the orbit.

Theorem 3.1. *Let α be an action by commuting hyperbolic automorphisms of $\mathbb{T}^N$, β be the dual action, and $\mathcal{O}$ a non-trivial dual orbit in $\mathbb{Z}^N$. Then there exists an initial point $m^* \in \mathcal{O}$ such that for some constants $a, b, C_5, C_6 > 0$*

$$C_5\|m^*\|\exp(b\|t\|) \le \|\beta^t m^*\| \le C_6\|m^*\|\exp(a\|t\|).$$

Proof. Let $B_1,\ldots,B_k \in SL(N,\mathbb{Z})$ be generators for the dual action β. Since β is an action by automorphisms, $\mathcal{O} \approx \mathbb{Z}^k$.

Since $B_1,\ldots,B_k$ are commuting real matrices, the space $\mathbb{R}^N$ can be decomposed into a direct sum of subspaces invariant under all B_j:

$$\mathbb{R}^N = \mathbb{I}_1 \oplus \cdots \oplus \mathbb{I}_r, \tag{3.1}$$

such that the minimal polynomial of B_j on $\mathbb{I}_i$ is a power of an irreducible polynomial $q_{ij}(x)$ over $\mathbb{R}$. According to this decomposition matrices $B_1,\ldots,B_k$ can be simultaneously brought to the following form with square blocks along the diagonal of sizes $N_1,\ldots,N_r$, $N_1 + \cdots + N_r = N$:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Lambda_{r1} \end{pmatrix}, \ldots, \Lambda_k = \begin{pmatrix} \Lambda_{1k} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Lambda_{rk} \end{pmatrix}. \tag{3.2}$$

If for a given $1 \le i \le r$ the minimal polynomials of all B_j on $\mathbb{I}_i$ are powers of linear polynomials: $q_{ij}(x) = (x - \lambda_{ij})^{N_i}$, all blocks Λ_{ij} $(1 \le j \le k)$ can be simultaneously brought to an upper-triangular form with λ_{ij} on the diagonal. If for at least one $j, 1 \le j \le k$, the minimal polynomial of B_j is a power of an irreducible quadratic polynomial with complex conjugate roots $(\lambda_{ij}, \overline{\lambda_{ij}})$, then the blocks Λ_{ij} $(1 \le j \le k)$ can be simultaneously brought to the following form:

$$\begin{pmatrix} |\lambda_{ij}| \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} & \dots & \star \\ \vdots & \ddots & \vdots \\ 0 & \dots & |\lambda_{ij}| \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} \end{pmatrix}.$$

Notice that for some j, θ_{ij} may be equal to 0 or π, so the 2×2 blocks on the diagonal will correspond to a pair of equal real eigenvalues. Each block Λ_{ij} can be canonically represented as a product

$$\Lambda_{ij} = S_{ij} U_{ij} \tag{3.3}$$

where S_{ij} is semisimple and U_{ij} is unipotent (identity plus nilpotent) which commute. The unipotent part U_{ij} is upper-triangular, and the semisimple part S_{ij} is either a diagonal matrix with $\lambda_{ij} \in \mathbb{R}$ on the diagonal, or is of the form

$$\begin{pmatrix} |\lambda_{ij}| \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & |\lambda_{ij}| \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} \end{pmatrix}.$$

We shall refer to the invariant subspaces $\mathbb{I}_i$ of being *of the first* or *of the second kind* depending on the form of the semisimple parts S_{ij} $(1 \le j \le k)$ described above.

Unipotent matrices U_{ij} have the following important property which will play a crucial role in the argument:

$$C^{-1} \|t_j\|^{-N_i} \le \|U_{ij}^{t_j}\| \le C \|t_j\|^{N_i}. \tag{3.4}$$

for some constant $C > 0$ independent on the choice of the orbit.

Let $x = (x_{11}, \dots, x_{1N_1}, \dots, x_{r1}, \dots, x_{rN_r})$ be coordinates in $\mathbb{R}^N$ in which matrices have the form (3.2), where $x_i = (x_{i1}, \dots, x_{iN_i}) \in \mathbb{I}_i$. Let $x^* = (x_1^*, \dots, x_r^*)$ with $x_i^* = (x_{1i}^* \dots, x_{iN_i}^*)$ be an initial point in the lattice $\mathbb{Z}^N$, then the orbit of this point is given by

$$\mathcal{O}(x^*) = \{\beta^t(x_1^*, \dots, x_r^*) = (\prod_{j=1}^k \Lambda_{1j}^{t_j} x_1^*, \dots, \prod_{j=1}^k \Lambda_{rj}^{t_j} x_r^* \mid t \in \mathbb{Z}^k\}. \tag{3.5}$$

First, let us notice that for a unipotent matrix U_{ij} its logarithm is uniquely defined, hence it can be included into a 1-parameter subgroup by $U_{ij}^t = \exp(t \log U_{ij}), t \in \mathbb{R}$. For a semisimple matrix S_{ij} its square can be also included into a 1-parameter subgroup S_{ij}^{2t} since the only obstacle for that is an odd number of equal negative eigenvalues. Then we put $\Lambda_{ij}^{2t} = S_{ij}^{2t} U_{ij}^{2t}$. Taking products of block matrices with blocks of this form, we obtain a finite index subgroup $H \subset \mathbb{Z}^k$ such that both H-action and its semisimple component can be included into $\mathbb{R}^k$-actions, the latter being the semisimple component of the former, whose semisimple and unipotent components commute. If we prove the estimates in Theorem 3.1 for $t \in H$, the estimates for the whole group will follow with, possibly, different constants. Hence, without loss of generality, we may assume that both the $\mathbb{Z}^k$-action β and its semisimple component β_s can be included into $\mathbb{R}^k$- actions which we also denote by β and β_s. The $\mathbb{R}^k$-orbit of x^*, $\mathcal{O}_{\mathbb{R}}(x^*)$ is given by the same formula as (3.5) only with $t \in \mathbb{R}^k$, and $\mathcal{O}(x^*) = \mathcal{O}_{\mathbb{R}}(x^*) \cap \mathbb{Z}^N$. Notice that not all $\mathbb{R}^k$-orbits contain integral points, but if an $\mathbb{R}^k$-orbit contains one integral point, it contains the whole $\mathbb{Z}^k$-orbit.

Let us define linear functionals in $\mathbb{R}^k$

$$\chi_i(t) = \sum_{j=1}^{k} t_j \ln |\lambda_{ij}|$$

for $1 \leq i \leq r$. The hyperbolicity condition is equivalent to the fact that all χ_i are different from 0. However, the following Lemmas 3.2 and 3.3 are true for the actions by partially hyperbolic automorphisms as well.

Lemma 3.2.

$$N_1\chi_1(t) + N_2\chi_2(t) + \cdots + N_r\chi_r(t) = 0. \tag{3.6}$$

Proof. We have

$$N_i\chi_i(t) = N_i \sum_{j=1}^{k} t_j \ln |\lambda_{ij}| = \sum_{j=1}^{k} t_j \ln |\det \Lambda_{ij}|.$$

Hence

$$\sum_{i=1}^{r} N_i\chi_i(t) = \sum_{i=1}^{r}\sum_{j=1}^{k} t_j \ln |\det \Lambda_{ij}| = \sum_{j=1}^{k} t_j \ln \prod_{i=1}^{r} |\det \Lambda_{ij}| = \sum_{j=1}^{k} t_j \ln |\det B_j| = 0,$$

since $|\det B_j| = 1$. □

Lemma 3.3.

(1) *The number of linearly independent linear functionals among* $\chi_1(t), \ldots, \chi_r(t)$ *is equal to* k.

(2) *The function* $\max \chi_i(t)$ *is a norm in* $\mathbb{R}^k$, *i.e. there exist constants* $a, b > 0$ *such that*

$$b\|t\| \leq \max \chi_i(t) \leq a\|t\|.$$

Proof. Since linear functionals $\chi_1(t), \dots, \chi_r(t)$ are in k variables, the number of linearly independent among them is not greater than k. Suppose it is less than k. Then there is a point $t \neq 0$ such that $\chi_i(t) = 0$ for all $1 \le i \le r$. If all t_i are rational, there exists an integer n such that all nt_i are integers. Then

$$B_1^{nt_1} \dots B_k^{nt_k}$$

has all eigenvalues of absolute value 1, which contradicts the ergodicity of the action unless $t = 0$. If some of t_i are irrational, for any $\epsilon > 0$ there exists $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$ such that $A = B_1^{n_1} \dots B_k^{n_k}, A^2, \dots, A^N$ have all their eigenvalues ϵ-close to 1, and hence their traces ϵ-close to N. Since all traces must be integers, they all are equal to N. Let eigenvalues of A be equal to $\lambda_1, \dots, \lambda_N$. Then eigenvalues of A^i are equal to $\lambda_1^i, \dots, \lambda_N^i$, and we have the following system of equations:

$$\begin{cases} \lambda_1 + \dots + \lambda_N = N \\ \dots\dots\dots\dots\dots\dots \\ \lambda_1^N + \dots + \lambda_N^N = N. \end{cases}$$

It follows that all symmetric functions in $\lambda_1, \dots, \lambda_N$, which appear as coefficients of the polynomial $(x - \lambda_1) \dots (x - \lambda_N)$ are the same as for $\lambda_1 = \lambda_2 = \dots = \lambda_N = 1$. This proves part (1). It follows from Lemma 3.2 that for any point $t \neq 0$ there exists at least one χ_i such that $\chi_i(t) > 0$, hence $\max \chi_i(t) > 0$. Since the functions $\chi_i(t)$ are linear, $\max \chi_i(t)$ is a norm in $\mathbb{R}^k$, hence equivalent to the $\| \cdot \|$, so part (2) follows. $\square$

Remark. It follows immediately from Lemmas 3.3 and 3.2 that the number r in the decomposition (3.1) is greater than the rank k of the abelian action, and $k \le N - 1$.

If $(x_1^*, \dots, x_r^*) \in \mathcal{O}$ is an initial point, then

$$\beta^t(x_1^*, \dots, x_r^*) = (\prod_{j=1}^k \Lambda_{1j}^{t_j} x_1^*, \dots, \prod_{j=1}^k \Lambda_{mj}^{t_j} x_r^*).$$

Using the decomposition (3.3) we can write $\beta^t = u^t \beta_s^t$ as a product of an unipotent and semisimple actions. It follows from partial hyperbolicity that for any integral point $m^* \in \mathbb{Z}^N$, $m^* \neq 0$, $\min_{t \in \mathbb{Z}^k} \|\beta^t(m^*)\| > 0$, so we arrive to the following definition:

Definition 3.4. A $\mathbb{R}^k$-orbit $\mathcal{O}_{\mathbb{R}}(x^*)$ is called *admissible* if $\min_{t \in \mathbb{R}^k} \|\beta^t(x^*)\| > 0$.

Since we are interested in an estimate on $\mathbb{Z}^k$-orbits, we shall study only admissible $\mathbb{R}^k$-orbits thereafter. We choose the initial point $x^* \in \mathcal{O}_{\mathbb{R}}$ such that

$$\min_{t \in \mathbb{R}^k} \|\beta^t(x^*)\| = \|x^*\| \tag{3.7}$$

is assumed at $t = 0$. Since $\mathcal{O}_{\mathbb{R}}(x^*)$ is admissible, $\|x^*\| > 0$. Our goal now is to prove that there exist constants $C_1, C_2 > 0$ independent on the choice of the orbit such that

$$C_2 \|x^*\| \exp(b\|t\|) \le \|\beta^t x^*\| \le C_1 \|x^*\| \exp(a\|t\|). \tag{3.8}$$

The following argument shows that the estimate on $\mathcal{O} = \mathcal{O}_{\mathbb{R}} \cap \mathbb{Z}^N$ follows. Since $\mathcal{O}_{\mathbb{R}}(x^*)$ contains an integral point, say m, $x^* = \beta^\tau m$ for some $\tau \in \mathbb{R}^k$. We can write $\tau = [\tau] + \{\tau\}$ where

$$\{\tau\} \in \{s = (s_1, \dots, s_k) \mid 0 \le s_i \le 1\} = \Delta,$$

the fundamental domain for $\mathbb{R}^k/\mathbb{Z}^k$. But then $\beta^{[\tau]} m = m^*$ is also in $\mathcal{O} \cap \mathbb{Z}^N$, and $x^* = \beta^{\{\tau\}} m^*$. It follows from the compactness of Δ that there exist constants $C_3, C_4 > 0$ independent on the choice of the orbit such that

$$C_3 \|x^*\| \le \|m^*\| \le C_4 \|x^*\|, \tag{3.9}$$

and the estimates

$$C_5 \|m^*\| \exp(b\|t\|) \le \|\beta^t m^*\| \le C_6 \|m^*\| \exp(a\|t\|).$$

follow for some constants $C_5, C_6 > 0$ independent on the choice of the orbit.

We first prove (3.8) in the **leading special case** when the matrices $B_1, \dots, B_k$ are simultaneously diagonalizable over $\mathbb{C}$, i.e. when β^t is semisimple. This happens, for example, if the action is irreducible. Since all norms in $\mathbb{R}^N$ are equivalent, it is sufficient to prove (3.8) for a particular one. It is convenient to consider the following norm: $\|x\| = \|(x_1, \dots, x_r)\| = \sum_{i=1}^r \|x_i\|$, where

$$\|x_i\| = \|(x_{i1}, \dots, x_{iN_i})\| = |x_{i1}| + \dots + |x_{iN_i}|$$

if $\mathbb{I}_i$ is of the first kind, and

$$\|x_i\| = \|(x_{i1}, \dots, x_{iN_i})\| = \sqrt{x_{i1}^2 + x_{i2}^2} + \dots + \sqrt{x_{i(N_i-1)}^2 + x_{iN_i}^2}$$

if $\mathbb{I}_i$ is of the second kind.

Let us consider a function $\omega : \mathbb{R}^k \to \mathbb{R}$ given by the formula

$$\omega(t) = \sum_{i=1}^r \|x_i^*\| \exp \chi_i(t)$$

Then

$$\omega(t) = \sum_{i=1}^r \prod_{j=1}^k |\lambda_{ij}|^{t_j} \|x_i^*\| = \sum_{i=1}^r \| \prod_{j=1}^k \Lambda_{ij}^{t_j} x_i^* \| = \|\beta^t (x_1^*, \dots, x_r^*)\|$$

and an estimate from above follows immediately from Lemma 3.3(2) for any choice of the initial point $x^* \in \mathcal{O}_{\mathbb{R}}$:

$$\omega(t) \le \|x^*\| \max \exp \chi_i(t) \le \|x^*\| \exp(a\|t\|). \tag{3.10}$$

On the other hand, we have

$$\omega(t) \ge \min \|x_i^*\| \max \exp \chi_i(t) \ge \min \|x_i^*\| \exp(b\|t\|) \ge 0,$$

and since $\min \|x_i^*\|$ may as well be equal to 0, we only get a trivial estimate this way.

In order to obtain a non-trivial estimate from below, we need the following definition.

Definition 3.5. Let $A \subset \{1, \dots, r\}$. The subsystem $\{\chi_i \mid i \in A\}$ is called *sufficient* if for any point $t \in \mathbb{R}^k$ there is $j \in A$ such that $\chi_j(t) > 0$.

If $\{\chi_i \mid i \in A\}$ is a sufficient subsystem of linear functionals in $\mathbb{R}^k$, the argument of Lemma 3.3(2) shows that $\max_{i \in A} \chi_i(t)$ is a norm in $\mathbb{R}^k$ and the same estimates as in Lemma 3.3(2) hold. Notice that the following lemma is the only place in the proof where hyperbolicity is required.

Lemma 3.6. *Let $A = \{i \in \{1, \dots, r\} \mid \|x_i^*\| > 0\}$. Then $\{\chi_i \mid i \in A\}$ is a sufficient subsystem.*

Proof. By Lemma 3.3(1) there are k linearly independent linear functionals among $\chi_1, \dots, \chi_r$; without loss of generality we may assume they are $\chi_1, \dots, \chi_k$. Now we make a linear change of variables

$$\begin{aligned}
&\chi_1(t_1, \dots, t_k) = \chi_1 \\
&\chi_2(t_1, \dots, t_k) = \chi_2 \\
&\dots \\
&\chi_k(t_1, \dots, t_k) = \chi_k
\end{aligned}$$

Then

$$\begin{aligned}
&\chi_{k+1}(t_1, \dots, t_k) = a_1^{(k+1)} \chi_1 + \dots + a_k^{(k+1)} \chi_k \\
&\dots \\
&\chi_r(t_1, \dots, t_k) = a_1^{(r)} \chi_1 + \dots + a_k^{(r)} \chi_k,
\end{aligned}$$

so that

$$\omega(t) = \phi(\chi) = \sum_{i=1}^{k} \|x_i^*\| \exp(\chi_i) + \sum_{j=k+1}^{r} \|x_j^*\| \exp(a_1^{(j)} \chi_1 + \dots + a_k^{(j)} \chi_k).$$

By (3.7) we have $\min_{t \in \mathbb{R}^k} \omega(t) = \omega(0)$, hence

$$\frac{\partial \phi}{\partial \chi_i}|_{\chi=0} = 0 \quad \text{for } 1 \le i \le k.$$

From these equations we obtain for $1 \le i \le k$

$$\|x_i^*\| = - \sum_{j=k+1}^{r} a_i^{(j)} \|x_j^*\|.$$

Notice that the constants $a_i^{(j)}$, $1 \le i \le k$, $k+1 \le j \le r$ depend only on the linear functionals $\chi_1, \dots, \chi_r$, i.e. on the given action β.

Let $I = A \cap \{1, \dots, k\}$ and $J = A \cap \{k+1, \dots, r\}$. Suppose that the subsystem is not sufficient, i.e. there exists $t = (t_1, \dots, t_k)$ such that $\chi_i(t) \le 0$ for all $i \in I$ and $\chi_j(t) \le 0$ for all $j \in J$. In χ-variables we have: $\chi_i \le 0$ for all $i \in I$ and $\sum_{i=1}^k a_i^{(j)} \chi_i \le 0$ for all $j \in J$.

Then, on one hand

$$\sum_{j=k+1}^{r} \|x_j^*\| \sum_{i=1}^{k} a_i^{(j)} \chi_i = \sum_{j=k+1}^{r} \|x_j^*\| (\sum_{i \notin I} a_i^{(j)} \chi_i + \sum_{i \in I} a_i^{(j)} \chi_i) =$$
$$\sum_{i \notin I} (\sum_{j=k+1}^{r} \|x_j^*\| a_i^{(j)}) \chi_i + \sum_{i \in I} (\sum_{j=k+1}^{r} \|x_j^*\| a_i^{(j)}) \chi_i = \sum_{i \in I} (\sum_{j=k+1}^{r} \|x_j^*\| a_i^{(j)}) \chi_i =$$
$$- \sum_{i \in I} \|x_i^*\| \chi_i = - \sum_{i \in I} \|x_i^*\| \chi_i(t) \ge 0.$$

On the other hand,

$$\sum_{j=k+1}^{r} \|x_j^*\| \sum_{i=1}^{k} a_i^{(j)} \chi_i = \sum_{j \notin J} \|x_j^*\| \sum_{i=1}^{k} a_i^{(j)} \chi_i + \sum_{j \in J} \|x_j^*\| \sum_{i=1}^{k} a_i^{(j)} \chi_i =$$
$$\sum_{j \in J} \|x_j^*\| \sum_{i=1}^{k} a_i^{(j)} \chi_i = \sum_{j \in J} \|x_j^*\| \chi_j(t) \le 0$$

Hence for this particular t, $\chi_i(t) = 0$ for all $i \in A$. Then since the matrices β^t are semisimple, we obtain $\|\beta^t(x^*)\| = \|x^*\|$ which contradicts hyperbolicity of the action. $\square$

Let us assign to any admissible $\mathbb{R}^k$-orbit its initial point $x^* \in \mathbb{R}^N$ (3.7) normalized by its ℓ^1-norm:

$$\lambda = (\lambda_1, \dots, \lambda_r) = \frac{x^*}{\sum_{i=1}^r \|x_i^*\|}.$$

The moduli space Λ for the collection of all admissible $\mathbb{R}^k$-orbits is described by the following equations

$$\begin{aligned} &\sum_{i=1}^{r} \|\lambda_i\| = 1 \\ &\|\lambda_i\| = - \sum_{j=k+1}^{r} a_i^{(j)} \|\lambda_j\| \text{ for } 1 \le i \le k \end{aligned} \tag{3.11}$$

and is a compact subset of $\mathbb{R}^N$. Applying Lemma 3.6 and a remark preceding it to each $\lambda_0 \in \Lambda$ we obtain the following estimate from below

$$\omega(t) \ge \frac{1}{2} \min_{i \in A} \|\lambda_i\| \exp(b\|t\|)$$

which also holds for all $\lambda \in \Lambda$ close to λ_0. It follows from the compactness of Λ that there exists a constant $C_7 > 0$ independent on the choice of orbit such that

$$\|\beta^t x^*\| \geq C_7 \|x^*\| \exp(b\|t\|). \tag{3.12}$$

Thus we have obtained estimates (3.8) in the semisimple case.

Now let us consider the general case. If $\mathcal{O}$ is an orbit in $\mathbb{Z}^N$, then the corresponding $\mathbb{R}^k$-orbit $\mathcal{O}_{\mathbb{R}}$ is admissible, i.e. there exists a point $x^* \in \mathcal{O}_{\mathbb{R}}$ such that $\|\beta^t x^*\|$ assumes minimum for $t = 0$ and $\|x^*\| > 0$. We decompose β^t into the product of a semisimple and an unipotent action:

$$\beta^t = u^t \beta_s^t.$$

Let $\mathcal{O}_s$ be the $\mathbb{R}^k$-orbit of β_s^t through the point x^*. It has to be also admissible. Otherwise, there exists a sequence $\{t_n\} \in \mathbb{R}^k$ such that $\|\beta_s^{t_n} x^*\|$ decreases exponentially, and since by (3.4)

$$\|\beta^{t_n} x^*\| \leq C\|t_n\|^N \|\beta_s^{t_n} x^*\|$$

we obtain that $\mathcal{O}_{\mathbb{R}}$ is not admissible. So, $\mathcal{O}_s(x^*)$ has a minimum which we denote by q. We have $x^* = \beta_s^{t_0}(q)$. Let $w = \beta^{-t_0}(x^*)$. Then, since x^* is the minimum on $\mathcal{O}_{\mathbb{R}}$, we have $\|w\| \geq \|x^*\|$. On the other hand, $q = \beta_s^{-t_0}(x^*)$, and we have $\|w\| \leq C\|t_0\|^N \|q\|$ by (3.4). So,

$$\|x^*\| \leq C\|q\| \|t_0\|^N.$$

Using the estimates (3.12) for a semisimple orbit we obtain

$$C_7 \exp(b\|t_0\|)\|q\| \leq \|x^*\| \leq C\|q\|\|t_0\|^N,$$

which gives us an estimate by an absolute constant $C_8 > 0$:

$$\|t_0\| < C_8. \tag{3.13}$$

Now, we have $\beta_s^t x^* = \beta_s^{t-t_0} q$, hence by (3.10) and (3.13)

$$\|\beta_s^t x^*\| \leq \exp(a(\|t - t_0\|)\|q\| \leq \exp(a(\|t - t_0\|)\|x^*\| \leq C_9 \exp(a\|t\|)\|x^*\|$$

for some constant $C_9 > 0$. On the other hand,

$$\|\beta_s^t x^*\| \geq C_7 \exp(b(\|t - t_0\|)\|q\| \geq C_{10} \exp(b(\|t - t_0\|)\|x^*\| \geq C_{11} \exp(b(\|t\|)\|x^*\|$$

for some constants $C_{10}, C_{11} > 0$. Now using the estimates

$$C^{-1}\|t\|^{-N}\|\beta_s^t x^*\| \leq \|\beta^t x^*\| \leq C\|t\|^N \|\beta_s^t x^*\|$$

we obtain

$$C_2\|x^*\| \exp(b\|t\|) \leq \|\beta^t x^*\| \leq C_1\|x^*\| \exp(a\|t\|).$$

□

Construction of dual solutions in §2 relies on invertibility of the action. If it is literally carried out to the non-invertible case, i.e. an action by toral endomorphisms, the resulting solution would, in general, not be defined on the torus, but on a solenoid where the natural extension of our action operates.

On the other hand, in the partially hyperbolic invertible case the only difficulty is in obtaining growth estimates like in Theorem 3.1. First notice that due to the presence of eigenvalues of absolute value 1 the crucial estimates from below of Theorem 3.1, which are uniform for all dual orbits, do not hold. Naturally, the estimates from above are completely general. The reason why the uniform estimates from below break down is that the lattice points can be found arbitrary close to the eigenspaces corresponding to the eigenvalues of absolute value 1. Thus exponential growth can be established only for the projection of the vector to the hyperbolic directions. However, since all invariant subspaces come from algebraic equations with integer coefficients, the distance of a lattice point to the neutral subspace satisfies Diophantine conditions. In other words, the slow growth of the norm of the iterate with respect to the norm of the initial vector is offset by the fact that this norm was sufficiently large to begin with. This argument yields a non-uniform estimates from below which are still sufficient to produce a C^∞ solution of the original coboundary equation. In this argument the initial point m^* on a given dual orbit is chosen among those with the projection of minimal norm to the hyperbolic part, i.e. the sum of the root spaces corresponding to non-zero Lyapunov exponents. Details of this argument will appear in a subsequent paper.

4. Proofs of Theorems 1 and 2. We have seen that the average over a dual orbit is an obstruction for solving a coboundary equation on an individual dual orbit (Proposition 2.2). We shall show that the vanishing of this obstruction on *all* dual orbits allows us to obtain a global C^∞ solution of the original coboundary equation. The following Proposition plays a crucial role in the proof of Theorems 1 and 2.

Proposition 4.1. *Let α be an action of $\mathbb{Z}^k$ by hyperbolic automorphisms of $\mathbb{T}^N$, and φ be a C^∞ k-cocycle over α with values in $\mathbb{R}^\ell$ $(\ell \geq 1)$ such that for any non-trivial dual orbit $\mathcal{O}$, $\sum_{m\in\mathcal{O}} \hat{\varphi}(m) = 0$. Then φ is C^∞-cohomologous to a constant cocycle ψ, i.e. for $x \in \mathbb{T}^N$, $t \in (\mathbb{Z}^k)^k$*

$$\varphi(x,t) = \psi(t) + \mathcal{D}\Phi(x,t), \tag{4.1}$$

where Φ is a C^∞ $(k-1)$-cochain.

Proof. First we apply Proposition 2.2 to construct a dual cochain $\hat{\Phi}$ on each non-trivial dual orbit. Since the cocycle φ is C^∞ we have the following estimate on the decay of the dual cocycle $\hat{\varphi}$: for any $B \in \mathbb{Z}_+$ there exists $C = C(B)$ such that

$$|\hat{\varphi}(m)| \leq C\|m\|^{-B}. \tag{4.2}$$

We want to obtain a similar estimate on the decay of each component of the dual cochain $\hat{\Phi}_j$ $(1 \leq j \leq k)$. Each $0 \neq m \in \mathbb{Z}^N$ belongs to some dual orbit $\mathcal{O}(m^*)$ where m^* is chosen by Theorem 3.1: $m = \beta^t m^*$. Let $t = (t_1, \dots, t_k)$. Formula (2.5) shows that $\hat{\Phi}_j(\beta^t m^*) = 0$ if at least one of the coordinates $t_1, \dots, t_{j-1}$ is not equal to 0, hence it is sufficient only to consider the case when $t_1 = \dots = t_{j-1} = 0$. Let $s = (0, \dots, 0, t_j, \dots, t_k)$ be fixed and consider the following half-lattice

$$\mathbb{H}^j = \{r \in \mathbb{Z}^k \mid r = (r_1, \dots, r_{j-1}, r_j, 0, \dots, 0),\ r_j \geq t_j \text{ if } t_j \geq 0,\ r_j < t_j \text{ if } t_j < 0\}.$$

Then again by formula (2.5)

$$|\hat{\Phi}_j(\beta^s m^*)| \leq \sum_{r \in \mathbb{H}^j} |\hat{\varphi}(\beta^{r+s} m^*)|. \tag{4.3}$$

If for $t = r + s$ we put $\|t\| = \sum_{i=1}^k |t_i|$, then $\|r + s\| = \|r\| + \|s\|$. Using both estimates from Theorem 3.1 we obtain:

$$\begin{aligned} \|\beta^s m^*\| &\leq C_6 \|m^*\| \exp(a\|s\|) \\ \|\beta^t m^*\| &\geq C_5 \|m^*\| \exp(b\|s\|) \exp(b\|r\|). \end{aligned}$$

Hence for some constant $C_{12} > 0$

$$\|\beta^s m^*\|^{\frac{b}{a}} \leq C_{12} \|m^*\|^{\frac{b}{a}} \exp(b\|s\|) \leq C_{12} \|m^*\| \exp(b\|s\|)$$

since $\|m^*\| > 1$, so that

$$\|\beta^t m^*\| \geq C_{13} \|\beta^s m^*\|^{\frac{b}{a}} \exp(b\|r\|)$$

for yet another constant $C_{13} > 0$. By (4.2) we have

$$\begin{aligned} |\hat{\varphi}(\beta^t m^*)| &\leq C \|\beta^t m^*\|^{-B} \\ &\leq C C_{13}^{-B} \|\beta^s m^*\|^{-B\frac{b}{a}} \exp(-Bb\|r\|). \end{aligned}$$

Then for some constants $C_{14}, C_{15} > 0$ and $m = \beta^s m^*$ we obtain the desired super-polynomial estimate for $\hat{\Phi}$.

$$|\hat{\Phi}_j(m)| \leq C_{14} \|m\|^{-B\frac{b}{a}} \sum_{r \in \mathbb{H}^j} \exp(-Bb\|r\|) \leq C_{15} \|m\|^{-B\frac{b}{a}}. \tag{4.4}$$

Since by Theorem 3.1 the estimates do not depend on the orbit, we obtain global estimates on the decay of $\hat{\Phi}_j$. Letting $\hat{\Phi}_j(0) = 0$ and using (2.1) and (2.2) we thus obtain a C^∞ $(k-1)$-cochain $\Phi = (\Phi_1, \dots, \Phi_k)$ such that

$$\mathcal{D}\Phi = \varphi - \hat{\varphi}(0),$$

i.e. is a solution of our equation (4.1). $\square$

Proof of Theorem 1. First we apply Corollary 1.4 to conclude that if a C^∞ k-cocycle over α, φ, vanishes on all periodic orbits of α, then for any dual orbit $\mathcal{O}$, including 0, $\sum_{m\in\mathcal{O}}\hat{\varphi}(m) = 0$. Now, by Proposition 4.1. $\mathcal{D}\Phi = \varphi - \hat{\varphi}(0)$, and since $\hat{\varphi}(0) = 0$ (see Corollary 1.5), we obtain a solution of (0.3) $\square$

Proof of Theorem 2. First, let φ be a C^∞ 1-cocycle on $\mathbb{T}^N$: $\varphi = (\varphi_1, \dots, \varphi_k)$ and $\hat{\varphi} = (\hat{\varphi}_1, \dots, \hat{\varphi}_k)$ be a dual cocycle. By Proposition 2.3. $\Sigma_i\hat{\varphi}_i = 0$, and we can write a solution of the dual coboundary equation

$$\hat{\varphi} = \mathcal{D}\hat{\Phi}$$

$\hat{\Phi}(\hat{\varphi}) = \Sigma_i^-\hat{\varphi}_i$. The cocycle equations

$$\Delta_i\hat{\varphi}_j = \Delta_j\hat{\varphi}_i$$

for $i \neq j$ imply that $\Sigma_i^-\hat{\varphi}_i = \Sigma_j^-\hat{\varphi}_j$, hence $\hat{\Phi}$ is well-defined. It is unique up to an additive constant. As before, we construct a solution $\hat{\Phi}$ on each non-trivial dual orbit. Now we recall that the solution of the coboundary equation in the case $n = k$ was constructed inductively (2.5). Hence the estimates (4.2) and (4.3) with $j = 1$ actually give the super-polynomial decay for 1-cocycles. Thus we obtain a C^∞ solution of the equation

$$\mathcal{D}\Phi = \varphi - \hat{\varphi}(0).$$

We are going to proceed by induction on k. Our hypothesis holds for the highest cocycles for which their dual cocycles have 0 average over each dual orbit (Proposition 4.1) and for 1-cocycles. These cases will be considered as the base step in our induction argument. Suppose the equation (0.4) has a C^∞ solution for n-cocycles on $\mathbb{Z}^p$, where $2 \le p \le k-1$ and $1 \le n \le p-1$. Let φ be a C^∞ n-cocycle on $\mathbb{T}^N$, $1 \le n \le k-1$, i.e. a vector-function with $\binom{k}{n}$ components satisfying cocycle equations (2.6). The $\binom{k-1}{n}$ components not containing index 1 may be regarded as an n-cocycle of the $\mathbb{Z}^{k-1}$-action by $A_2, \dots, A_k$ since for components with $i_1 > 1$ $(\mathcal{D}\varphi)_{i_1,\dots,i_{n+1}} = 0$ are just part of the cocycle equations for φ. If $n < k-1$, by the induction hypothesis we can find C^∞ solutions for the first $\binom{k-1}{n}$ equations. If $n = k-1$, then by Proposition 2.3 the dual cocycle has 0 average over each non-trivial dual orbit, hence a C^∞ solution of (4.1) can be found by Proposition 4.1. The remaining $\binom{k-1}{n-1}$ components contain index 1. Let $\phi_{i_2,\dots,i_n} = \varphi_{1,i_2,\dots,i_n} - \Delta_1\Phi_{i_2,\dots,i_n}$, where $\Phi_{i_2,\dots,i_n}$ are already obtained from the first $\binom{k-1}{n}$ equations. We need to show that ϕ is a C^∞ $(n-1)$-cocycle of the $\mathbb{Z}^{k-1}$-action by $A_2, \dots, A_k$. For,

$$(\mathcal{D}\phi)_{i_2,\dots,i_{n+1}} = \sum_{j=2}^{n+1}(-1)^j\Delta_{i_j}(\varphi_{1,i_2,\dots,\hat{i_j},\dots,i_{n+1}} - \Delta_1\Phi_{i_2,\dots,\hat{i_j},\dots,i_{n+1}}) =$$

$$\Delta_1\varphi_{i_2,\dots,i_{n+1}} - \Delta_1\sum_{j=2}^{n+1}(-1)^j\Delta_{i_j}\Phi_{i_2,\dots,\hat{i_j},\dots,i_{n+1}} = \Delta_1\varphi_{i_2,\dots,i_{n+1}} - \Delta_1\varphi_{i_2,\dots,i_{n+1}} = 0.$$

We used the cocycle equation for φ:

$$\sum_{j=2}^{n+1}(-1)^j \Delta_{i_j}\varphi_{1,i_2,\dots,\hat{i_j},\dots,i_{n+1}} = \Delta_1\varphi_{i_2,\dots,i_{n+1}}$$

and that

$$\varphi_{i_2,\dots,i_{n+1}} = \sum_{j=2}^{n+1}(-1)^j \Delta_{i_j}\Phi_{i_2,\dots,\hat{i_j},\dots,i_{n+1}} + \hat{\varphi}_{i_2,\dots,i_{n+1}}(0).$$

Notice that $\phi_{i_2,\dots,i_n}$ is C^∞ since $\varphi_{1,i_2,\dots,i_n}$ is, $\Phi_{i_2,\dots,i_n}$ is by the induction hypothesis, and the operator Δ_1 preserves the smoothness. Then by the induction hypothesis we can solve the coboundary equation for $(n-1)$-cocycles of the $\mathbb{Z}^{k-1}$-action:

$$\phi_{i_2,\dots,i_n} = (\mathcal{D}g)_{i_2,\dots,i_n} + \hat{\varphi}_{i_2,\dots,i_n}(0) = \sum_{j=2}^{n}(-1)^{j+1}\Delta_{i_j} g_{i_2,\dots,\hat{i_j},\dots,i_n} + \hat{\varphi}_{i_2,\dots,i_n}(0).$$

But then, if we define components of Φ that have index 1 by the formula

$$\Phi_{1,i_2,\dots,i_{n-1}}(x) = g_{i_2,\dots,i_{n-1}}(x)$$

and the components of the constant cocycle by

$$\psi_{1,i_2,\dots,i_{n-1}} = \hat{\varphi}_{1,i_2,\dots,i_{n-1}}(0)$$

we get

$$\varphi_{1,i_2,\dots,i_n} = \phi_{i_2,\dots,i_n} + \Delta_1\Phi_{i_2,\dots,i_n} = (\mathcal{D}\Phi)_{1,i_2,\dots,i_n} + \psi_{1,i_2,\dots,i_{n-1}}.$$

□

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